MISCELLANEA.

The Principal Averages and the Laws of Error which lead to them.

By J. M. Keynes.

§1. The following is a common type of statistical problem. We are given a series of measurements, or observations, or estimates of the true value of a given quantity; and we wish to determine what function of these measurements will yield us the *most probable* value of the quantity, on the basis of this evidence. The problem is not determinate unless we have some good ground for making an assumption as to how likely we are in each case to make errors of given magnitudes. But such an assumption, with or without justification, is frequently made.

The functions of the original measurements which we commonly employ, in order to yield us approximations to the most probable value of the quantity measured, are the various kinds of means or averages—the arithmetic average, for example, or the median. The relation, which we assume, between errors of different magnitudes, and the probabilities that we have made errors of those magnitudes, is called a *law of error*. Corresponding to each law of error which we might assume, there is some function of the measurements which represents the most probable value of the quantity. The object of the following investigation is to discover what laws of error, if we assume them, correspond to each of the simple types of average, and to discover this by means of a systematic method. It is also an object to make the preliminary proof very precise, in order to bring out with clearness exactly what assumptions are involved in it.

§2. It is necessary for this purpose to introduce the fundamental symbol of probability.* The probability of any proposition varies according to the evidence upon which we base it.† In this case the probability, that the quantity has a certain value, depends upon the magnitudes of the measurements which we have made, since these measurements constitute our evidence. Let H represent the evidence, and A the conclusion into whose probability we are

* I hope to publish shortly a treatise on Probability, in which the properties and the use of this symbol, of which the following is an example, are fully developed. But the following example, although it can be expounded most precisely by means of this symbol, does not depend for its validity upon controversial assumptions. The only formulæ which are employed, namely, the principles of inverse probability and of the multiplication of independent probabilities, are generally accepted, although different interpretations can be given to the philosophical significance of the probabilities which they enable us to manipulate. The reader is not to suppose that an unfamiliar symbolism cloaks, in this case, any unusual assumption.

† Some might prefer to put this as follows:—"The probability of any event varies according to the series to which we refer it, and we select this series by reference to such relevant evidence as we happen to possess."

inquiring relatively to this evidence. Then the probability of A, when H is the evidence, may be represented by the symbol A/H.

§3. Let us assume that the real value of the quantity is either $a_1, \ldots a_r \ldots$ or a_n, and let A_r represent the conclusion that the value is, in fact, a_r. Further let X_r represent the evidence that a measurement has been made of magnitude x_r.

If a measurement x_p has been made, what is the probability that the real value is a_s? The application of the ordinary rule of inverse probability yields the following result:—

$$A_s/X_pH = \frac{X_p/A_sH. \quad A_s/H}{\sum\limits_{r=1}^{r=n} X_p/A_rH. \quad A_r/H},$$

(the number of possible values of the quantity being n), where H stands for any other relevant evidence which we may have, in addition to the fact that a measurement x_p has been made. In the above A_s/X_pH represents the probability that the true value is a_s, given the measurement x_p; X_p/A_sH represents the probability of our making the measurement x_p, when it is given that the true value is a_s; and A_s/H represents the *a priori* probability, before any measurement has been made, of the true value's being a_s. The terms in the denominator correspond to the other possible values of the quantity.

Next, let us suppose that a number of measurements $x_1, \ldots x_m$, have been made; what is now the probability that the real value is a_s? We require the value of $A_s/X_1X_2 \ldots X_mH$. As before,

$$A_s/X_1X_2 \ldots X_mH = \frac{X_1 \ldots . X_m/A_sH. \quad A_s/H}{\sum\limits_{r=1}^{r=x} X_1 \ldots X_m/A_rH. \quad A_r/H}.$$

§4. At this point we must introduce the simplifying assumption that, if we knew the real value of the quantity, the different measurements of it would be *independent*, in the sense that a knowledge of what errors have actually been made in some of the measurements would not affect in any way our estimate of what errors are likely to be made in the others. We assume, in fact, that $X_r/X_p \ldots X_sA_rH = X_r/A_rH$. This assumption is exceedingly important. It is tantamount to the assumption that our law of error is unchanged throughout the series of observations in question. The general evidence H, that is to say, which justifies our assumption of the particular law of error which we do assume, is of such a character that a knowledge of the actual errors made in a number of measurements, not more numerous than those in question, are absolutely or approximately irrelevant to the problem as to what form of the law we ought to assume. The law of error which we assume will be based, presumably, on an experience of the relative frequency with which errors of different magnitudes have been made under analogous circumstances in the past. The above assumption will *not* be justified if the additional experience, which a knowledge of the errors in the new measurements would supply, is sufficiently comprehensive, relatively to our former experience, to be capable of modifying our assumption as to the shape of the law of error, or if it suggests that the circumstances, in which the measurements are being carried out, are not so closely analogous as was originally supposed.

§5. With this assumption, *i.e.*, that X_1, &c., are independent of one another relatively to evidence A_rH, &c., it follows from the ordinary rule for the multiplication of independent probabilities that

$$X_1 \dots X_m/A_sH = \prod_{q=1}^{q=m} X_q/A_sH.$$

$$\text{Hence, } A_s/X_1X_2 \dots X_mH = \frac{A_s/H. \prod_{q=1}^{q=m} X_q/A_sH}{\sum_{r=1}^{r=n}\left[\prod_{q=1}^{q=m} X_q/A_rH. \quad A_r/H\right]}.$$

The *most probable* value of the quantity under measurement, given the m measurements x_1 &c.,—which is our *quæsitum*—is therefore that value which makes the above expression a maximum. Since the denominator is the same for all values of a_s, we must find the value which makes the numerator a maximum. Let us assume that $A_1/H = A_2/H = \dots = A_n/H$. We assume, that is to say, that we have no reason *à priori* (*i.e.*, before any measurements have been made), for thinking any one of the possible values of the quantity more likely than any other. We require, therefore, the value of a_s, which makes the expression $\prod_{q=1}^{q=m} X_q/A_sH$ a maximum. Let us denote this value by x.

§6. We can make no further progress without a further assumption. Let us assume that X_q/A_sH—namely, the probability of a measurement x_q assuming the real value to be a_s—is an algebraic function f of x_q and a_s, the same function for all values of x_q and a_s within the limits of the problem.* We assume, that is to say, $X_q/A_sH = f(x_q, a_s)$, and we have to find the value of a_s, namely x, which makes $\prod_{q=1}^{q=m} f(x_q, x)$ a maximum. Equating to zero the differential coefficient of this expression with respect to x, we have $\sum_{q=1}^{q=m} \frac{f'(x_q, x)}{f(x_q, x)} = 0$,† where $f' = \frac{df}{dx}$. This equation may be written for brevity in the form $\Sigma \frac{f'_q}{f_q} = 0$.

If we solve this equation for x, the result gives us the value of the quantity under observation, which is most probable relatively to the measurements we have made.

* Gauss made, in effect, the more special assumption that X_q/A_sH is a function of e_q only, where e_q is the error and $e_q = a_s - x_q$. We shall find in the sequel that all symmetrical laws of error, such that positive and negative errors of the same absolute magnitude are equally likely, satisfy this condition—the normal law, for example, and the simplest median law. But other laws, such as those which lead to the geometric mean, do not satisfy it.

† Since none of the measurements actually made can be impossible, none of the expressions $f(x_q, x)$ can vanish.

The act of differentiation assumes that the possible values of x are so numerous and so uniformly distributed within the range in question, that we may, without sensible error, regard them as continuous.

§7. This completes the *prolegomena* of the inquiry. We are now in a position to discover what laws of error correspond to given assumptions respecting the algebraic relation between the measurements and the most probable value of the quantity, and *vice versa*. For the law of error determines the form of $f(x_q, x)$. And the form of $f(x_q, x)$ determines the algebraic relation $\Sigma \frac{f'_q}{f_q} = 0$ between the measurements and the most probable value. It may be well to repeat that $f(x_q, x)$ denotes the probability to us that an observer will make a measurement x_q in observing a quantity whose true value we know to be x. A law of error tells us what this probability is for all possible values of x_q and x within the limits of the problem.

(i) If the most probable value of the quantity is equal to the arithmetic mean of the measurements, what law of error does this imply?

$\Sigma \frac{f'_q}{f_q} = 0$ must be equivalent to $\Sigma(x - x_q) = 0$, since the most probable value x must equal $\frac{1}{m} \sum_{q=1}^{q=m} x_q$.

$\therefore \frac{f'_q}{f_q} = \phi''(x)(x - x_q)$ where $\phi''(x)$ is some function which is not zero and is independent of x_q.

Integrating

$\log f_q = \int \phi''(x)(x - x_q)d_x + \psi(x_q)$ where $\psi(x_q)$ is some function independent of x.

$$= \phi'(x)(x - x_q) - \phi(x) + \psi(x_q).$$

So that $\quad f_q = e^{\phi'(x)(x - x_q) - \phi(x) + \psi(x_q)}.$

Any law of error of this type, therefore, leads to the arithmetic mean of the measurements as the most probable value of the quantity measured.

If we put $\phi(x) = -k^2x^2$ and $\psi(x_q) = -k^2x_q^2 + \log A$, we obtain $f_q = Ae^{-k^2(x - x_q)^2}$, the form normally assumed.

$= Ae^{-k^2y_q^2}$, where y_q is the absolute magnitude of the error in the measurement x_q.

This is, clearly, only one amongst a number of possible solutions. But with one additional assumption we can prove that this is the only law of error which leads to the arithmetic mean. Let us assume that negative and positive errors of the same absolute amount are equally likely.

In this case f_q must be of the form $Be^{\theta(x - x_q)^2}$

$$\therefore \phi'(x)(x - x_q) - \phi(x) + \psi(x_q) = \theta(x - x_q)^2$$

Differentiating with respect to x

$$\phi''(x) = 2\frac{d}{d(x-x_q)^2}\theta(x-x_q)^2$$

But $\phi''(x)$ is, by hypothesis, independent of x_q.

$\therefore \quad \frac{d}{d(x-x_q)^2}\theta(x-x_q)^2 = -k^2$ where k is constant; integrating $\theta(x-x_q)^2 = -k^2(x-x_q)^2 + \log C$ and we have $f_q = Ae^{-k^2(x-x_q)^2}$ (where $A = BC$).

(ii) What is the law of error, if the geometric mean of the measurements leads to the most probable value of the quantity? In this case $\Sigma\frac{f'_q}{f_q} = 0$ must be equivalent to $\prod_{q=1}^{q=m} x_q = x^m$, *i.e.*, to

$$\Sigma \log \frac{x_q}{x} = 0.$$

Proceeding as before, we find that the law of error is

$$f_q = A\ e^{\phi'(x)\log\frac{x_q}{x} + \int\frac{\phi'(x)}{x}d\,x + \psi(x_q)}$$

There is no solution of this which satisfies the condition that negative and positive errors of the same absolute magnitude are equally likely. For we must have

$$\phi'(x)\ \log\frac{x_q}{x} + \int\frac{\phi'(x)}{x}d\,x + \psi(x_q) = \phi(x-x_q)^2$$

$$\text{or } \phi''(x)\ \log\frac{x_q}{x} = \frac{d}{d\,x}\,\phi(x-x_q)^2$$

which is impossible.

The simplest law of error, which leads to the geometric mean, seems to be obtained by putting $\phi'(x) = -k\,x$, $\psi(x_q) = 0$. This gives $f_q = A\left(\frac{x}{x_q}\right)^{kx} e^{-kx}$

A law of error, which leads to the geometric mean of the observations as the most probable value of the quantity, has been previously discussed by Sir Donald McAlister (*Proceedings of the Royal Society*, Vol. xxix (1879), p. 365). His investigation depends upon the obvious fact that, if the geometric mean of the observations yields the most probable value of the quantity, the arithmetic mean of the logarithms of the observations must yield the most probable value of the logarithm of the quantity. Hence, if we suppose that the logarithms of the observations obey the normal law of error (which leads to their arithmetic mean as the most probable value of the logarithms of the quantity), we can by substitution find a law of error for the observations themselves which must lead to the geometric mean of them as the most probable value of the quantity itself.

If, as before, the observations are denoted by x_q, &c., and the quantity by x, let their logarithms be denoted by z_q, &c., and by z. Then, if z_q, &c., obey the normal law of error, $f(z_q, z) = Ae^{-k^2(z_q-z)^2}$. Hence the law of error for x_q, &c., is determined by

$$\begin{aligned} f\,(x_q, x) &= Ae^{-k^2(\log x_q - \log x)^2} \\ &= Ae^{-k^2\left(\log\frac{x_q}{x}\right)^2}, \end{aligned}$$

and the most probable value of x must, clearly, be the geometric mean of x_q, &c.

This is the law of error which was arrived at by Sir Donald McAlister. It can easily be shown that it is a special case of the generalised form which I have given above of all laws of error leading to the geometric mean. For if we put $\psi(x_q) = -k^2(\log x_q)^2$, and $\phi'(x) = 2k^2 \log x$, we have

$$\begin{aligned} f_q &= \mathrm{A}e^{2k^2 \log x \log \frac{x_q}{x} + \int 2k^2 \frac{\log x}{x} dx - k^2(\log x_q)^2} \\ &= \mathrm{A}e^{2k^2 \log x \log x_q - 2k^2 (\log x)^2 + k^2 (\log x)^2 - k^2 (\log x_q)^2} \\ &= \mathrm{A}e^{-k^2 \left(\log \frac{x_q}{x}\right)^2}. \end{aligned}$$

A similar result has been obtained by Professor J. C. Kapteyn [*Skew Frequency Curves*, p. 22, published by the Astronomical Laboratory at Groningen (1903)]. But he is investigating frequency curves, not laws of error, and this result is merely incidental to his main discussion. His method, however, is not unlike a more generalised form of Sir Donald McAlister's. In order to discover the frequency curve of certain quantities x, he supposes that there are certain other quantities z, functions of the quantities x, which are given by $z = \mathrm{F}(x)$, and that the frequency curve of these quantities z is *normal*. By this device he is enabled in the investigation of a type of skew frequency curve, which is likely to be met with often, to utilise certain statistical constants corresponding to those which have been already calculated for the normal curve.

In fact the main advantage both of Sir Donald McAlister's law of error and of Professor Kapteyn's frequency curves lies in the possibility of adapting without much trouble to unsymmetrical phenomena numerous expressions which have been already calculated for the normal law of error and the normal curve of frequency.*

This method of proceeding from arithmetic to geometric laws of error is clearly capable of generalisation. We have dealt with the geometric law which can be derived from the normal arithmetic law. Similarly if we start from the simplest geometric law of error, namely, $f_q = \mathrm{A}\left(\frac{x}{x_q}\right)^{k^2x} e^{-k^2x}$, we can easily find, by writing $\log x = y$ and $\log x_q = y_q$, the corresponding arithmetic law, namely, $f_q = \mathrm{A}e^{k^2e^y(y - y_q) - k^2e^y}$, which is obtained from the generalised arithmetic law by putting $\phi(y) = k^2e^y$ and $\psi(y_q) = 0$. And, in general, corresponding to the arithmetic law

$$f_q = \mathrm{A}e^{\phi'(x)(x - x_q) - \phi(x) + \psi(x_q)},$$

we have the geometric law

$$f_q = \mathrm{A}e^{\phi'_1(z) \log \frac{z_q}{z} + \int \frac{\phi_1'(z)}{z} dz + \psi_1(z_q)}$$

where

$$x = \log z, x_q = \log z_q, \int \frac{\phi_1'(z)}{z} dz = \phi(\log z) \text{ and } \psi_1(z_q) = \psi(\log z_q).$$

* It may be added that Professor Kapteyn's monograph brings forward considerations which would be extremely valuable in determining the types of phenomena to which geometric laws of error are likely to be specially applicable.

(iii.) What law of error does the harmonic mean imply?

In this case, $\Sigma \frac{f'_q}{f_q} = 0$ must be equivalent to $\Sigma\left(\frac{1}{x_q} - \frac{1}{x}\right) = 0$.

Proceeding as before, we find that $f_q = Ae^{\phi'(x)\left[\frac{1}{x_q} - \frac{1}{x}\right] - \int \frac{\phi'(x)}{x^2} dx + \psi(x_q)}$. A simple form of this is obtained by putting $\phi'(x) = -k^2x^2$ and $\psi(x_q) = -k^2x_q$. Then $f_q = Ae^{-\frac{k^2}{x_q}(x-x_q)^2} = Ae^{-k^2\frac{y_q^2}{x_q}}$. With this law, positive and negative errors of the same absolute magnitude are not equally likely.

(iv) If the most probable value of the quantity is equal to the median of the measurements, what is the law of error?

The median is usually defined as the measurement which occupies the middle position when the measurements are ranged in order of magnitude. If the number of measurements m is odd, the most probable value of the quantity is the $\frac{m+1}{2}$th, and, if the number is even, all values between the $\frac{m}{2}$th and the $\left(\frac{m}{2}+1\right)$th are equally probable amongst themselves and more probable than any other. For the present purpose, however, it is necessary to make use of another property of the median, which was known to Fechner (who first introduced the median into use) but which seldom receives as much attention as it deserves. *If x is the median of a number of magnitudes, the sum of the absolute differences (i.e., the difference always reckoned positive) between x and each of the magnitudes is a minimum.* The median x of $x_1\ x_2\ .\ .\ .\ x_m$ is found, that is to say, by making $\sum_1^m |x_q - x|$ a minimum where $|x_q - x|$ is the difference always reckoned positive between x_q and x.

We can now return to the investigation of the law of error corresponding to the median.

Write $|x - x_q| = y_q$. Then since $\sum_1^m y_q$ is to be a minimum we must have $\sum_1^m \frac{x - x_q}{y_q} = 0$. Whence, proceeding as before, we have

$$f_q = Ae^{\int \frac{x-x_q}{y_q}\phi''(x)dx + \psi(x_q)}.$$

The simplest case of this is obtained by putting

$$\phi''(x) = -k^2$$

$$\psi(x_q) = \frac{x - x_q}{y_q}k^2x_q$$

whence $$f_q = Ae^{-k^2|x-x_q|} = Ae^{-k^2y_q}.$$

This satisfies the additional condition that positive and negative errors of equal magnitude are equally likely. Thus in this important respect the median is as satisfactory as the arithmetic mean, and the law of error which leads to it is as simple. It also resembles the normal law in that it is a function of the error *only*, and not of the magnitude of the measurement as well.

The median law of error, $f_q = \mathrm{A}e^{-k^2 y_q}$, where y_q is the absolute amount of the error always reckoned positive, is of some historical interest, because it was the earliest law of error to be formulated. The first attempt to bring the doctrine of averages into definite relation with the theory of probability and with laws of error was published by Laplace in 1774 in a memoir "sur la probabilité des causes par les événemens."* This memoir was not subsequently incorporated in his *Théorie Analytique*, and does not represent his more mature view. In the *Théorie* he drops altogether the law tentatively adopted in the memoir, and lays down the main lines of investigation for the next hundred years by the introduction of the *normal* law of error. The popularity of the normal law, with the arithmetic mean and the method of least squares as its corollaries, has been very largely due to its overwhelming advantages, in comparison with all other laws of error, for the purposes of mathematical development and manipulation. And in addition to these technical advantages, it is probably applicable as a first approximation to a larger and more manageable group of phenomena than any other single law. So powerful a hold indeed did the normal law obtain on the minds of statisticians, that until quite recent times only a few pioneers have seriously considered the possibility of preferring in certain circumstances other means to the arithmetic and other laws of error to the normal. Laplace's earlier memoir fell, therefore, out of remembrance. But it remains interesting, if only for the fact that a law of error there makes its appearance for the first time.

Laplace sets himself the problem in a somewhat simplified form:— "Déterminer le milieu que l'on doit prendre entre trois observations données d'un même phénomène." He begins by assuming a law of error $y = \phi(x)$, where y is the probability of an error x; and finally, by means of a number of somewhat arbitrary assumptions, arrives at the result $\phi(x) = \frac{m}{2} e^{-mx}$. If this formula is to follow from his arguments, x must denote the *absolute* error, always taken positive. It is not unlikely that Laplace was led to this result by considerations other than those by which he attempts to justify it.

Laplace, however, did not notice that his law of error led to the median. For, instead of finding the most probable value, which would have led him straight to it, he seeks the "mean of error"— the value, that is to say, which the true value is as likely to fall short of as to exceed. This value is, for the median law, laborious to find and awkward in the result. Laplace works it out correctly for the case where the observations are no more than three.

§8. I do not think that it is possible to find by this method a law of error which leads to the mode. But the following general formulæ are easily obtained:—

(v) If $\Sigma\theta(x_q, x) = 0$, is the law of relation between the measurements and the most probable value of the quantity, then the law of error $f_q(x_q, x)$ is given by $f_q = \mathrm{A}e^{\int \theta(x_q x)\phi''(x)dx + \psi(x_q)}$. Since f_q lies

* *Mémoirs présentés à l'Académie des Sciences*, vol. vi.

between 0 and 1, $\int\theta(x_q x)\phi''(x)dx + \psi(x_q) + \log A$ must be negative for all values of x_q and x that are physically possible; and, since the values of x_q are between them exhaustive,

$$\Sigma A e^{\int\theta(x_q x)\phi''(x)dx + \psi(x_q)} = 1$$

where the summation is for all terms that can be formed by giving x_q every value *à priori* possible.

(vi) The most general form of the law of error, when it is assumed that positive and negative errors of the same magnitude are equally probable, is $Ae^{-k^2 f(x-x_q)^2}$, where the most probable value of the quantity is given by the equation

$$\Sigma(x - x_q)f'(x - x_q)^2 = 0, \text{ where } f'(x - x_q)^2 = \frac{d}{d(x - x_q)^2} f(x-x_q)^2.$$

The arithmetic mean is a special case of this obtained by putting $f(x-x_q)^2 = (x-x_q)^2$; and the median is a special case obtained by putting $f(x - x_q)^2 = +\sqrt{(x - x_q)^2}$.

We can obtain other special cases by putting

$$f(x - x_q)^2 = (x - x_q)^4,$$

when the law of error is $Ae^{-k^2(x-x_q)^4}$ and the most probable values are the roots of $mx^3 - 3x^2\Sigma x_q + 3x\Sigma x^2_q - \Sigma x_q^3 = 0$; and by putting $f(x - x_q)^2 = \log (x - x_q)^2$, when the law of error is $\frac{A}{(x - x_q)^{2k^2}}$ and the most probable values the roots of $\Sigma \frac{1}{x - x_q} = 0$. In all these cases the law is a function of the error only.*

§ 9. In conclusion the preceding results may be summarised. We have assumed :—

(*a.*) That we have no reason, before making measurements, for supposing that the quantity we measure is more likely to have any one of its possible values than any other.

(*b.*) That the errors are independent, in the sense that a *knowledge* of how great an error has been made in one case does not affect our expectation of the probable magnitude of the error in the next.

(*c.*) That the probability of a measurement of given magnitude, when in addition to the *à priori* evidence the real value of the quantity is supposed known, is an algebraic function of this given magnitude of the measurement and of the real value of the quantity.

(*d.*) That we may regard the series of possible values as continuous, without sensible error.

(*e.*) That the *à priori* evidence permits us to assume a law of error of the type specified in (*c*); *i.e.*, that the algebraic function referred to in (*c*) is known to us *à priori.*

Subject to these assumptions, we reached the following conclusions :—

(1.) The most general form of the law of error is

$$f_q = Ae^{\int\phi''(x)\,\theta(x_q x)\,dx + \psi(x_q)},$$

* *cf.* § 6 (footnote).

leading to the equation $\Sigma\theta(x_q x) = 0$, connecting the most probable value and the actual measurements, where x is the most probable value and x_q, etc., the measurements.

(2.) Assuming that positive and negative errors of the same absolute magnitude are equally likely, the most general form is $f_q = Ae^{-k^2 f(x-x_q)^2}$, leading to the equation $\Sigma(x-x_q)f'(x-x_q)^2 = 0$, where $f'y = \frac{d}{dy}fy$. Of the special cases to which this form gives rise, the most interesting were

(3.) $f_q = Ae^{-k^2(x-x_q)^2} = Ae^{-k^2 y_q^2}$, where $y_q = |x - x_q|$, leading to the arithmetic mean of the measurements as the most probable value of the quantity; and

(4.) $f_q = Ae^{-k^2 y_q}$, leading to the median.

(5.) The most general form leading to the arithmetic mean is $f_q = Ae^{\phi'(x)(x-x_q) - \phi(x) + \psi(x_q)}$, with the special cases (3), and

(6.) $f_q = Ae^{k^2 e^x (x-x_q) - k^2 e^x}$.

(7.) The most general form leading to the geometric mean is $f_q = Ae^{\phi'(x)\log\frac{x_q}{x} + \int\frac{\phi'(x)}{x}dx + \psi(x_q)}$, with the special cases:—

(8.) $f_q = A\left(\frac{x}{x_q}\right)^{k^2 x} e^{-k^2 x}$, and

(9.) $f_q = Ae^{-k^2\left(\log\frac{x_q}{x}\right)^2}$.

(10.) The most general form leading to the harmonic mean is $f_q = A\,e^{\phi'(x)\left[\frac{1}{x_q} - \frac{1}{x}\right] - \int\frac{\phi'(x)}{x^2} + \psi(x_q)}$, with the special case

(11.) $f_q = Ae^{-k^2\frac{(x-x_q)^2}{x_q}} = Ae^{-k^2\frac{y_q^2}{x_q}}$.

(12.) The most general form leading to the median is

$$f_q = Ae^{\phi'(x)\frac{x-x_q}{y_q} + \psi(x_q)},$$

with the special case (4).

In each of these expressions, f_q is the probability of a measurement x_q, given that the true value is x.