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THE RELATIONS BETWEEN BOREL'S AND CESÀRO'S METHODS OF SUMMATION

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I.

1. The results obtained in this paper are developments of an idea that has been prominent in much recent work on the theory of divergent series.

Consider the series

(1) $$a_0+a_1+a_2+\dots.$$

This series may converge, or may possess a "sum" according to one or other of a large variety of definitions of what is meant by the "sum" of a series: Hölder's, Cesàro's, or Riesz's definitions by mean values, Euler's definition as the limit of a power series, Borel's exponential definition, and so forth. To each of these definitions correspond certain limits of applicability. Thus Hölder's and Cesàro's definitions can never be successful unless

$$n^{-k}a_n \rightarrow 0$$

for some value of k; Euler's definition requires that $\Sigma a_n x^n$ should converge for $|x| < 1$; Borel's that

$$\Sigma \frac{a_n x^n}{n!}$$

should be an integral function of x. Roughly, we may say that any method of summation will fail if the series to which it is applied is *too divergent.** Or, in other words, to any method corresponds a certain upper limit of its power, the specification of which is a problem generally not difficult and often uninteresting.

It is only recently that it has been observed that the range of applicability of all methods of summation is limited from below, so to say, as well as from above. Methods fail, not only when the series to which it is attempted to apply them is *too divergent*, but also if it is *too nearly convergent:* not only is their *power* limited, but also their *delicacy*. The theorems which express this latter fact are more subtle than those which express the "limitation from above," and take a rather different form: they assert that, if a series is too nearly convergent, it cannot be summable unless it is *actually* convergent.

The theorems of this character which correspond to Cesàro's (or Hölder's) and to Euler's methods were discussed by us in two recent papers in these *Proceedings.*† It follows from a well known theorem of Tauber‡ that a series for which

(2) $$na_n \to 0$$

cannot be summable by any of these methods unless it is convergent. In these papers we showed that this condition may be replaced by the more general condition

(2′) $$|na_n| < K.$$

Similar results hold for Riesz's more general methods: these results will also be found in the papers referred to.

The primary object of this paper was to establish the analogue of Tauber's theorem for Borel's exponential method of summation. But this problem has led us on to a number of others, some of which are discussed here, while to others we hope to return later. There is one important point with regard to which our results are not complete. We

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* In these general explanations (as in the phrase "Theory of Divergent Series") it is convenient to use *divergent* as meaning simply non-convergent. In detailed work it is essential to distinguish between divergent and oscillatory series.

† Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 301; Littlewood, *ibid.*, Vol. 9, p. 434. The results of the first of these papers may be deduced as corollaries from those of the second: they have been extended, in a somewhat different direction, by Landau (*Prac Matematyczno-Fizycznych*, t. 21, p. 97), who shows that, when a_n is real, it is enough to suppose $na_n < K$ or $na_n > -K$, and makes an interesting application of the result to the Theory of Prime Numbers.

‡ If $\Sigma a_n x^n \to s$, as $x \to 1$, and $na_n \to 0$ as $n \to \infty$, then Σa_n is convergent (to sum s). For a proof see Littlewood, *l.c. supra*, and Bromwich, *Infinite Series*, p. 251.

prove (Theorem 1 below) that a series for which

(3) $$\sqrt{n}\,.\,a_n \to 0$$

cannot be summable by Borel's method, unless it is convergent. It can hardly be doubted that this result (the analogue of Tauber's theorem) is susceptible of the same generalisation: that is to say, that (3) may be replaced by

(3′) $$|\sqrt{n}\,.\,a_n| < K.$$

But this we have not at present succeeded in proving; and the difficulties attendant on the generalisation of Tauber's theorem suggest forcibly that the proof may not be at all easy to find.*

We shall use the symbols

$$K,\ \epsilon,\ \delta,\ O,\ o,\ \ldots,$$

in special senses for an explanation of which we must refer elsewhere.†

II.

2. Borel gave two definitions of the sum of the series

(1) $$a_0+a_1+a_2+\ldots.$$

According to his first definition the sum is

(2) $$s = \lim_{x\to\infty} e^{-x}\,\Sigma\,\frac{s_n x^n}{n!},$$

where $$s_n = a_0+a_1+\ldots+a_n.$$

According to the second it is

(2a) $$\int_0^\infty e^{-x}\,a(x)\,dx,$$

where

(2b) $$a(x) = \Sigma\,\frac{a_n x^n}{n!}.$$

These definitions are not exactly equivalent, the second being slightly

* In the abstract of this paper which appeared in the "Notes and Corrections" to Vol. 9 of the *Proceedings*, we implied that we were in possession of a proof. We have since discovered that our belief that we had found one was mistaken.

† See Hardy, "Orders of Infinity," *Camb. Math. Tracts*, No. 12. The only symbol whose use is not explained there is the small o, introduced by Landau (*Handbuch der Lehre von der Verteilung der Primzahlen*, p. 61). We write

$$f = o(\phi),$$

if ϕ is positive and $f/\phi \to 0$.

more general.* They are certainly equivalent whenever $a_n \to 0$. For the present we shall adopt the first definition as fundamental; if (2) holds, we shall say that the series (1) is *summable* (B) to sum s.

3. Our first object is to prove the following theorem:—

THEOREM 1.—*If Σa_n is summable (B), and*

$$\sqrt{n}\,.\,a_n \to 0,$$

then Σa_n is convergent.

This theorem, however, is only one of a hierarchy of theorems connecting Borel's with Cesàro's methods of summation. To establish these we shall require a number of lemmas.

We shall write

$$\begin{aligned} s_n &= a_0+a_1+\ldots+a_n, \\ s_n^1 &= s_0+s_1+\ldots+s_n, \\ s_n^2 &= s_0^1+s_1^1+\ldots+s_n^1, \\ &\ldots \quad \ldots \quad \ldots, \end{aligned}$$

so that Σa_n is summable (Ck) if

$$k!\, s_n^k / n^k \to s.\dagger$$

LEMMA 1.—*If $c_n \sim An^a$ as $n \to \infty$, then*

$$e^{-x} \Sigma c_n \frac{x^n}{n!} \sim Ax^a,$$

as $x \to \infty$.

This is, of course, well known.

LEMMA 2.—*We have identically*

$$e^{-x} \sum_0^\infty s_n^k \frac{x^{n+k}}{(n+k)!} = \left(\int_0^x dx\right)^k \left(e^{-x} \sum_0^\infty s_n \frac{x^n}{n!}\right).$$

We use the identity

$$s_n^k = s_n + \begin{bmatrix} k \\ 1 \end{bmatrix} s_{n-1} + \begin{bmatrix} k \\ 2 \end{bmatrix} s_{n-2} + \ldots + \begin{bmatrix} k \\ n \end{bmatrix} s_0,$$

where

$$\begin{bmatrix} p \\ q \end{bmatrix} = \frac{p(p+1)\ldots(p+q-1)}{q!}.$$

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* Bromwich, *Infinite Series*, p. 297.

† We need hardly point out that s_n^k does not stand for a power of s_n.

The coefficient of s_ν on the left-hand side is

$$e^{-x}\left\{\frac{x^{\nu+k}}{(\nu+k)!}+\begin{bmatrix}k\\1\end{bmatrix}\frac{x^{\nu+k+1}}{(\nu+k+1)!}+\begin{bmatrix}k\\2\end{bmatrix}\frac{x^{\nu+k+2}}{(\nu+k+2)!}+\dots\right\},$$

and on the right-hand side is

$$\left(\int_0^x dx\right)^k\left(e^{-x}\frac{x^\nu}{\nu!}\right).$$

That these two expressions are identical may be verified immediately by induction.

LEMMA 3.—*If* Σa_n *is summable* (*B*) *to sum* s, *then*

$$e^{-x}\sum^{\infty} s_n^k \frac{x^n}{(n+k)!}\to\frac{s}{k!}.$$

This is an immediate corollary from Lemma 2.

4. Now suppose Σa_n summable (*B*), so that

(3) $$e^{-x}\Sigma s_n^k\frac{x^n}{(n+k)!}\to\frac{s}{k!}$$

for any value of k. And let us suppose that there is a number a such that

(4) $$s_n^{k-1}=o(n^a).*$$

Then, if for shortness we write $s_n^k=t_n$, we have

(5) $$\Delta t_n\equiv t_n-t_{n+1}=-s_{n+1}^{k-1}=o(n^a).$$

If, in (3), we put $x=m$, we have

(6) $$S_1+S_2\to s/k!,$$

where $S_1=e^{-m}t_m\Sigma\dfrac{m^n}{(n+k)!}$, $\quad S_2=e^{-m}\Sigma(t_n-t_m)\dfrac{m^n}{(n+k)!}$.

By Lemma 1,

(7) $$S_1=m^{-k}t_m(1+\epsilon_m).$$

* If $k=0$, $s_n^{k-1}=a_n$.

We proceed to discuss S_2, which we write in the form

(8) $$e^{-m}\left(\sum_{0}^{(1-H)m} + \sum_{(1-H)m}^{(1+H)m} + \sum_{(1+H)m}^{\infty}\right) = S_3+S_4+S_5,$$

say. Here H is a positive constant less than unity, and $\sum_{\mu}^{\nu}$ denotes a summation extended to all integral values n such that $\alpha < n < \beta$: it is convenient to suppose H *irrational*, so that $(1-H)m$ and $(1+H)m$ cannot be integral, but the limits of summation may be left indefinite to the extent of a term or two without any effect on the argument.

To obtain upper limits for S_3, S_4, and S_5, we use the inequalities

(9) $$\begin{cases} |t_n - t_m| < Km^K & (in\ S_3), \\ |t_n - t_m| \leqslant \epsilon_m m^a |n-m| & (in\ S_4), \\ |t_n - t_m| < Kn^K & (in\ S_5), \end{cases}$$

which are immediate consequences of inequalities already established.

Let $u_n = m^n/n!$. Then we find at once, by an elementary application of Stirling's theorem, that

$$e^{-m}u_\mu < e^{-Km}, \qquad e^{-m}u_\nu < e^{-Km},$$

where u_μ and u_ν are the last u which occurs in S_3 and the first which occurs in S_5.

Hence $$|S_3| < Km^K e^{-m} \sum_{0}^{(1-H)m} n^{-k} u_n$$

$$< m^K e^{-m}(u_\mu + u_{\mu-1} + \ldots + u_0)$$

$$< m^K e^{-Km}\{1 + (1-H) + (1-H)^2 + \ldots\}$$

(10) $$< e^{-Km}.$$

Similarly, $$|S_5| < Ke^{-m} \sum_{(1+H)m}^{\infty} n^{K-k} u_n$$

$$< e^{-m} \sum_{(1+H)m}^{\infty} n^K u_n$$

$$< m^K e^{-Km}\left\{1 + \frac{2^K}{1+H} + \frac{3^K}{(1+H)^2} + \ldots\right\}$$

(11) $$< e^{-Km}.$$

Finally we have to consider S_4, which we write in the form

$$S_4 = \sum_{n>m} + \sum_{m>n} = S_6 + S_7.$$

We have
$$|S_6| = \left| e^{-m} \sum_{m}^{(1+H)m} (t_n - t_m) \frac{m^n}{(n+k)!} \right|$$
$$< Km^{-k} e^{-m} \sum_{m}^{(1+H)m} |t_n - t_m| \frac{m^n}{n!}$$
$$< m^{a-k} e^{-m} \epsilon_m \sum_{1}^{Hm} r \frac{m^{m+r}}{(m+r)!} \,*$$
$$< m^{a-k} e^{-m} \frac{m^m}{m!} \epsilon_m \sum \frac{rm^r}{(m+1)(m+2)\dots(m+r)}$$
$$< m^{a-k-\frac{1}{2}} \epsilon_m \sum \frac{r}{\left(1+\frac{1}{m}\right)\left(1+\frac{2}{m}\right)\dots\left(1+\frac{r}{m}\right)}$$

Now, for $1 \leqslant r < Hm$, we have
$$1\Big/\left(1+\frac{r}{m}\right) = \exp\left(-\frac{r}{m}+\frac{r^2}{2m^2}-\dots\right) < e^{-r/2m},$$
$$r\Big/\left\{\left(1+\frac{1}{m}\right)\left(1+\frac{2}{m}\right)\dots\left(1+\frac{r}{m}\right)\right\} < r\exp\left(-\frac{1}{2m}-\frac{2}{2m}-\dots-\frac{r}{2m}\right)$$
$$< re^{-r^2/4m}.$$

Hence
$$|S_6| < m^{a-k-\frac{1}{2}} \epsilon_m \sum_{0}^{\infty} re^{-r^2/4m}$$
$$< m^{a-k-\frac{1}{2}} \epsilon_m \int_0^\infty xe^{-x^2/4m} dx \,\dagger$$
$$< m^{a-k+\frac{1}{2}} \epsilon_m$$

(12)
$$= o(m^{a-k+\frac{1}{2}}).$$

Again,
$$|S_7| = \left| e^{-m} \sum_{(1-H)m}^{m} (t_n - t_m) \frac{m^n}{(n+k)!} \right|$$
$$< Km^{-k} e^{-m} \sum_{(1-H)m}^{m} |t_n - t_m| \frac{m^n}{n!}$$
$$< m^{a-k} e^{-m} \epsilon_m \sum_{1}^{Hm} r \frac{m^{m-r}}{(m-r)!}$$
$$< m^{a-k-\frac{1}{2}} \epsilon_m \sum r \left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\dots\left(1-\frac{r-1}{m}\right).$$

In this case we use the inequality
$$1-\frac{r}{m} < e^{-r/m},$$

* Using (5).

† When m is large, the integral is of order m. The difference between the integral and the series is less than a constant multiple of the maximum of $xe^{-x^2/4m}$, which is of order $\sqrt{m}$.

and the argument proceeds as before ; so that

(13) $$|S_7| = o(m^{a-k+\frac{1}{2}}).$$

From (4), (7), (10), (11), (12), and (13), we deduce

(14) $$s_m^k(1+\epsilon_m)/m^k + o(m^{a-k+\frac{1}{2}}) \to s/k!.$$

From this relation we can deduce Theorem 1 and the other theorems referred to in § 3.

5. Suppose first $a = k - \frac{1}{2}$. Then

(15) $$s_m^k/m^k \to s/k!.$$

Hence we obtain

THEOREM 2.—*If* Σa_n *is summable* (*B*), *and*

$$s_n^{k-1} = o(n^{k-\frac{1}{2}}),$$

then Σa_n *is summable* (*Ck*).

This reduces to Theorem 1 when $k = 0$ (when s_n^{-1} must be interpreted as meaning a_n).*

Next, suppose $a > k - \frac{1}{2}$. Then we obtain

THEOREM 3.—*If* Σa_n *is summable* (*B*) *and*

$$s_n^{k-1} = o(n^a)$$

where $a > k - \frac{1}{2}$, *then* $s_n^k = o(n^{a+\frac{1}{2}})$.†

THEOREM 4.—*If* Σa_n *is summable* (*B*), *and*

$$s_n^{k-1} = o(n^k),$$

then Σa_n *is summable* $(C, k+1)$.

For, by Theorem 3, we have $s_n = o(n^{k+\frac{1}{2}})$, and Theorem 4 accordingly follows from Theorem 2. The special case in which $k = 0$ is particularly interesting and deserves a separate statement.

THEOREM 4*a*.—*If* Σa_n *is summable* (*B*), *and* $a_n \to 0$, *then* Σa_n *is summable* (*C*1).

By a repeated application of the argument which led to Theorem 4, we deduce

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* The theorem was stated for the particular cases in which $k = 0, 1$ in the Abstract of this paper referred to above.

† That $s_n^k = o(n^{a+1})$ is trivial. The point of the theorem lies in the reduction of $a+1$ to $a + \frac{1}{2}$ as a result of the hypothesis of Borel summability.

THEOREM 5.—*If Σa_n is summable (B), and*

$$s_n^{k-1} = o(n^{k+\frac{1}{2}(r-1)}),$$

then Σa_n is summable $(C, k+r)$.

6. If $\Sigma a_n x^n$ is convergent for $|x| < 1$, and the function $f(x)$ represented by the sum of the series is regular for $x = 1$, the series Σa_n is summable (B). It therefore follows from Theorems 1 and 4*a* that

(i) *If $f(x)$ is regular for $x=1$, and $\sqrt{n}\,.\,a_n \to 0$, then Σa_n is convergent.*

(ii) *If $f(x)$ is regular for $x = 1$, and $a_n \to 0$, then Σa_n is summable $(C1)$.*

Each of these corollaries of our theorems is included in Fatou's theorem* that, *if $f(x)$ is regular for $x = 1$, and $a_n \to 0$, then Σa_n is convergent.* But we have, of course, assumed much less than regularity for $x = 1$.

If Σa_n is summable by Cesàro's means, or, more generally, if Abel's limit exists, we can only infer convergence if

$$a_n = O(1/n).$$

To assume that Σa_n is summable (B) is to assume *more* than that it is summable (C)† or by Abel's limit, but less than that $f(x)$ is regular. To this corresponds the fact that $\sqrt{n}\,.\,a_n \to 0$ asserts less than $a_n = O(1/n)$, but more than $a_n \to 0$.

7. The results of § 5 may be represented conveniently by means of a diagram.

If $a_n =$	$s_n =$	$s_n^1 =$	$s_n^2 =$	...	*then the series is*
$o(n^{-\frac{1}{2}})$					*convergent*
$o(1)$	$o(n^{\frac{1}{2}})$				*summable* $(C1)$
$o(n^{\frac{1}{2}})$	$o(n)$	$o(n^{\frac{3}{2}})$			,, $(C2)$
$o(n)$	$o(n^{\frac{3}{2}})$	$o(n^2)$	$o(n^{\frac{5}{2}})$		,, $(C3)$
...	...	...	...	...	...

* Fatou, *Thèse* (Paris, 1906) and *Acta Mathematica*, t. 30, p. 389. A simpler proof, and a series of important generalisations, have been given by Riesz, *Crelle's Journal*, Bd. 140, S. 89, and *Comptes Rendus*, Nov. 22, 1909; see also Landau, *Prac Matematyczno-Fizycznych*, t. 21, p. 151.

† See Theorem 6 below for a precise statement.

This diagram at once suggests that there should be further theorems corresponding to the spaces which we have not filled in, such as :—

(a) if $s_n = o(1)$, and Σa_n is summable (B), then Σa_n is convergent;

(b) if $s_n^1 = o(\surd n)$, and Σa_n is summable (B), then Σa_n is convergent;

and so on. These theorems are all trivial or false. Thus (a) is obviously trivial: the same is true of all the theorems which correspond to the vacant spaces which have two sides in common with those filled in in the diagram.

On the other hand, (b) is false. For, if $s_n^1 = o(\surd n)$, we have

$$(s_0+s_1+\ldots+s_n)/(n+1) = o(1/\surd n);$$

and so we can *deduce* from the condition $s_n^1 = o(\surd n)$ that Σa_n is summable (B) to sum 0.* Hence the theorem suggested would show that $s_n^1 = o(\surd n)$ by itself implies convergence to zero, and this is untrue, as s_n may well be of the form

$$\epsilon_n \surd n - \epsilon_{n-1} \surd(n-1)$$

without tending to zero.

A very interesting general conclusion may be drawn from the theorems comprised in our diagram, viz.,

THEOREM 6.—*If Σa_n is summable (B), and*

$$a_n = o(n^k)$$

for some value of k, then Σa_n is summable by Cesàro's means of sufficiently high order.

In the language of § 1, we may express this by saying that *Borel's method, although more powerful than Cesàro's, is never more delicate, and often less so.*

A particular case of Theorem 6 deserves special notice. It is well

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* If

$$\frac{s_0+s_1+\ldots+s_n}{n+1} = s + o\left(\frac{1}{\surd n}\right),$$

then Σa_n is summable (B) to sum s. See Hardy, *Quarterly Journal*, Vol. 35, p. 40; Bromwich, *Infinite Series*, pp. 319–322. It may be shown more generally (cf. Bromwich, *l.c.*) that

$$k!\, s_n^k/n^k = s + o(1/\surd n)$$

implies the same conclusion: we have thus a general condition which enables us to infer Borel from Cesàro summability. For some examples of series summable by Cesàro's, but not by Borel's, method, see Hardy, *l. c. supra* and *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 290, and §§ 10, 11 below.

known that a power series is summable (*B*) at any regular point on its circle of convergence. It therefore follows that, *if* $f(x)$ *is regular for* $x = 1$, *and* $a_n = o(n^k)$, *then* Σa_n *is summable* (*C*). This result has been found by Riesz.*

8. It is natural to enquire whether the preceding results may be extended to non-integral orders of Cesàro summation.† The necessary analysis is not difficult, but, as its conclusions are obvious generalisations of those already established, we shall be content to sketch the argument very shortly.

In the first place, Lemma 1 of § 3 is quite independent of any assumption as to the arithmetic nature of a.

Secondly, Lemma 2 may be replaced by the equation

$$e^{-x} \sum_0^\infty s_n^k \frac{x^{n+k}}{\Gamma(n+k+1)} = \frac{1}{\Gamma(k)} \int_0^x (x-t)^{k-1} e^{-t} \left(\sum_0^\infty s_n \frac{t^n}{n!}\right) dt,$$

where
$$s_n^k = s_n + \begin{bmatrix} k \\ 1 \end{bmatrix} s_{n-1} + \ldots + \begin{bmatrix} k \\ n \end{bmatrix} s_0,$$

$$\begin{bmatrix} p \\ q \end{bmatrix} = \frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q+1)}.$$

This equation may be shown to hold for any positive value of k. From it may be deduced the analogue of Lemma 3, viz., that

$$e^{-x} \sum_0^\infty s_n^k \frac{x^n}{(n+k)!} \rightarrow \frac{s}{\Gamma(k+1)},$$

if Σa_n is summable (*B*). We then deduce equation (14) of § 4,‡ precisely as in that section. We thus obtain Theorems 2, 3, 4, 5, freed from the restriction that k is an integer. The effect of this is to replace each set of theorems, corresponding to a set of spaces lying on a line parallel to the principal diagonal of the diagram of § 7, by a *continuous* series of theorems.

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* *L.c. supra* (p. 9). Riesz assumes more than we do, and so obtains a more precise result: in fact, he establishes summability (*Ck*), whereas all that can be deduced from our hypothesis is summability (C, $2k+1$).

† For the theory of such methods of summation, see Chapman, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 369; Hardy and Chapman, *Quarterly Journal*, Vol. 42, p. 181; and various writings of Bohr, Knopp, and Riesz quoted in the latter paper. A later note by Riesz appeared in the *Comptes Rendus*, June 12th, 1911.

‡ Of course with $\Gamma(k+1)$ for $k!$.

9. So far we have confined ourselves to Borel's definition (2). The question remains whether our results remain valid when this definition is replaced by the integral definition expressed by (2*a*) and (2*b*).

If $a_n \to 0$, the two definitions are certainly equivalent. For the necessary and sufficient condition for equivalence is that

$$e^{-x} a(x) \to 0.$$

It is, however, not difficult to see that all our results still hold when Borel's integral definition is adopted.

For, if
$$a_0 + a_1 + a_2 + \ldots$$
is summable by the integral definition, then
$$0 + a_0 + a_1 + a_2 + \ldots$$
is summable by the definition (2).* Moreover, if the first series satisfies one of our conditions

$$a_n = o(n^{-\frac{1}{2}}), \quad a_n = o(1), \quad s_n = o(n^{\frac{1}{2}}), \quad \ldots,$$

the second satisfies a corresponding condition, and is accordingly summable by the appropriate one of Cesàro's means. But the two series are completely equivalent in regard to the application of Cesàro's method. Thus all our results apply as well to one of Borel's definitions as to the other.

III.

10. A good deal of light may be thrown on the foregoing theorems by the study of a particular series, viz., the series

(15) $$\Sigma n^{-b} e^{in^a},$$

where a and b are real and $0 < a < 1$.†

This series is convergent if $a+b > 1$, summable $(C1)$ if $2a+b > 1$, summable $(C2)$ if $3a+b > 1$, and so on.‡ If $b = 1-a$, it is finitely oscillating. In this case, if $a < \frac{1}{2}$, we have

$$n^{-b} e^{in^a} = o(n^{-\frac{1}{2}});$$

* Hardy, *Quarterly Journal*, Vol. 35, p. 34.

† We might equally well consider the more general series

$$\Sigma n^{-b} e^{iAn^a}.$$

‡ Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 142.

and so, by Theorem 1, the series cannot be summable (B). We are thus led to expect, in regard to the summability (B) of the series, different results according as $a > \frac{1}{2}$ or $a < \frac{1}{2}$. We shall, in fact, prove that *the series* (15) *is summable* (B) *for all values of* b *if* $a > \frac{1}{2}$, *but is never summable* (B) *when* $a < \frac{1}{2}$, *except when it is convergent.*

The proof of this result is tedious rather than difficult, and we shall content ourselves with sketching its main features.

In the first place, we can easily verify that, if

$$a_n = n^{-b} e^{in^a},$$

then
$$s_n = -(i/a)\, n^{1-a-b} e^{in^a} + \sum_{(\nu)} K n^{\nu} e^{in^a} + C + o(1),$$

where the summation extends over a finite number of values of ν, all less than $1-a-b$, and C is a constant arising from the application of the Euler-Maclaurin sum-formula.

We can now prove that, *if* $a > \frac{1}{2}$,

$$(16) \qquad e^{-x} \sum_{0}^{\infty} n^{\nu} e^{in^a} \frac{x^n}{n!} \to 0$$

as $x \to \infty$, *for all values of* ν. It will then follow that the series (15) is summable to sum C. Let

$$m = [x].$$

Then it is easy to see that we may replace the left-hand side of (16) by

$$(17) \qquad e^{-x} \sum_{-\mu}^{\mu} (m+r)^{\nu} e^{i(m+r)^a} \frac{x^{m+r}}{(m+r)!},$$

where
$$\mu \sim m^{\frac{1}{2}+\delta}.$$

We then show, by a straightforward but tedious process of approximation, that, if we write $x = m+f$, and keep f constant as $m \to \infty$, we can write (17) in the form

$$(18) \qquad K m^{\nu-\frac{1}{2}} e^{im^a} \sum_{-\mu}^{\mu} \left[e^{iam^{a-1}r-(r^2/2m)} \left\{ 1 + \Sigma K r^A m^B + \Sigma O(r^{\alpha} m^{\beta}) \right\} \right],$$

where the number of terms in each sum is finite, A is integral, and

$$\nu + \tfrac{1}{2}\alpha + \beta < 0$$

for each pair of indices α, β.

Next, it is easy to see that

$$Km^{\nu-\frac{1}{2}}e^{im^a}\sum_{-\mu}^{\mu}\{e^{iam^{a-1}r-(r^2/2m)}O(r^a m^\beta)\} = O(m^{\nu+\beta-\frac{1}{2}})\int_{-\infty}^{\infty}e^{-r^2/2m}|r|^a\,dr$$

$$= O(m^{\nu+\frac{1}{2}a+\beta}) = o(1).$$

Thus these terms may be neglected, and everything is reduced to showing that a number of terms of the type

(19) $$m^p\sum_{-\mu}^{\mu}r^q e^{iam^{a-1}r-(r^2/2m)},$$

where q is an integer, tend to zero. It is easily proved that the limits of summation may be replaced by ∞ and $-\infty$.

First suppose $q=0$. Then

$$\sum_{-\infty}^{\infty}e^{iam^{a-1}r-r^2/2m} = 1+2\sum_{1}^{\infty}e^{-r^2/2m}\cos r\theta = \vartheta_3(v,\tau),^*$$

where $\theta = am^{a-1}, \quad v=\theta/2\pi, \quad \tau = i/2m\pi.$

Now† $$\vartheta_3(v,\tau) = \sqrt{\left(\frac{i}{\tau}\right)}\,e^{-\pi i v^2/\tau}\vartheta_3\left(\frac{v}{\tau},\,-\frac{1}{\tau}\right),$$

$$\vartheta_3\left(\frac{v}{\tau},\,-\frac{1}{\tau}\right) = 1+2\sum_{1}^{\infty}e^{-2mr^2\pi^2}\cosh 2mr\theta\pi = 1+o(1).$$

Hence $$\vartheta_3(v,\tau)\sim\sqrt{(2m\pi)}\,e^{-\frac{1}{2}m\theta^2} = \sqrt{(2m\pi)}\,e^{-\frac{1}{2}a^2m^{2a-1}},$$

which tends to zero more rapidly than any power of m.

When q is not zero the argument is a little more complicated, but in essence the same: in this case we use the q-th derivative of the theta-function with respect to v.

Thus the left-hand side of (16) tends to zero if $x=m+f$, where $0\leqslant f<1$, and $m\to\infty$. Moreover it does so uniformly with respect to f, and so our proof is completed.

11. When $0<a<\frac{1}{2}$, the discussion is somewhat similar, but rather simpler. The essential difference lies in the fact that we can choose δ so that $iam^{a-1}r$ is small throughout a range of values of r of magnitude $m^{\frac{1}{2}+\delta}$, so that, in approximating to $e^{i(m+r)^a}$, we can use e^{im^a} instead of the

* Tannery and Molk, *Fonctions Elliptiques*, t. II, p. 252.

† *Ibid.*, p. 263.

more accurate approximation

$$e^{im^a+iam^{a-1}r}.$$

The result is that the dominant term of our final result assumes the form

$$-\frac{i}{a\sqrt{(2\pi)}}\, m^{\frac{1}{2}-a-b}\, e^{im^a} \sum_{-\mu}^{\mu} e^{-r^2/2m} \sim -(i/a)\, m^{1-a-b} e^{im^a},$$

and the series (15) is summable if and only if $a+b>1$, *i.e.*, if and only if it is convergent.

12. If $a=\frac{1}{2}+\delta$, we require $b>\frac{1}{2}-\delta$ for convergence. Hence we can find a series of the type (15), summable (B), but not convergent, and such that $a_n = O(n^{-\frac{1}{2}+\delta})$. This affords a formal proof that the index $\frac{1}{2}$ of Theorem 1 cannot be replaced by any lower index. We can show similarly, by means of the series considered in §§ 10, 11, that the indices of the powers of n, which occur in Theorems 2–5, are as small as they can possibly be.

A much more difficult question remains. *Is Theorem 1 true if the condition* $\sqrt{n}.a_n \to 0$ *is replaced by* $|\sqrt{n}.a_n| < K$, *and can similar changes be made in the other theorems?* It has been proved recently* that a similar extension may be given to Tauber's converse of Abel's theorem, and it is natural to suppose that the extension is possible here also.

It is interesting to consider for a moment what light is thrown on this question by the series of §§ 10, 11. The crucial case is that in which the series oscillates finitely; *i.e.*, when

$$a_n = n^{a-1} e^{in^a}.$$

Then $\quad a_n = o(n^{-\frac{1}{2}}) \quad (a<\frac{1}{2}), \qquad a_n = O(n^{-\frac{1}{2}}) \quad (a=\frac{1}{2}).$

In the first case the series is certainly not summable, by Theorem 1 or by the results of §§ 10, 11. And the question of interest is *whether*

$$\Sigma \frac{e^{i\sqrt{n}}}{\sqrt{n}}$$

is summable (B).†

The answer to this question is (as analogy would lead us to expect) in

* Littlewood, *l.c.*, p. 434.

† If not, Theorem 1 shows that $\Sigma n^{-b} e^{i\sqrt{n}}$ is never summable unless convergent.

the negative. In fact,

$$s_n = C - 2ie^{i\sqrt{n}} + o(1),$$

and it may be shown that

$$e^{-x} \sum e^{i\sqrt{n}} \frac{x^n}{n!} \sim e^{-(1/8)+i\sqrt{x}}.\text{*}$$

Thus the evidence obtained from this series points to the truth of the suggested generalisations. But, as we stated in § 1, we have not been able to find a satisfactory proof of them.

13. Theorem 1 has an interesting application to the problem of the multiplication of series. It is easy to prove that *if* Σa_n *and* Σb_n *are summable* (B), *and*

$$a_n = O(1/n), \qquad b_n = O(1/n),$$

then the product series Σc_n, *formed in accordance with Cauchy's rule, is also summable* (B). But it is evident that

$$c_n = O(\log n/n) = o(1/\sqrt{n}),$$

and therefore, by Theorem 1, Σc_n is *convergent*. We thus obtain a simple proof of a known theorem.†

* If $a < \frac{1}{2}$,

$$e^{-x} \Sigma e^{in^a} \frac{x^n}{n!} \sim e^{ix^a},$$

while, if $a > \frac{1}{2}$, the left-hand side tends exponentially to zero (see §§ 10, 11). In the critical case we obtain a result resembling the first, but differing owing to the presence of the factor $e^{-1/8}$. The correspondence between the oscillations of the original series and of Borel's integral is not so precise in this case as it is shown to be, when $a < \frac{1}{2}$, by the formula at the end of § 11. For an illustration of the corresponding phenomenon in connection with Tauber's theorem, see Littlewood, *l.c. supra*, p. 436: in the formulæ there given a constant term $\zeta(1+ai)$ is omitted, but the insertion of this term does not affect the argument.

† Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 6, p. 410. We have obtained a number of further results on this subject, to which we hope to return shortly. In particular we have proved that any convergent series for which $a_n = O(1/n)$—and therefore any such series summable by any of Cesàro's means—is summable $(C, -1+\delta)$ for any positive δ. We are thus able, by the use of Mr. Chapman's negative indices of summability, to deduce the multiplication theorem referred to above from the theorem that a series of this type cannot be summable (C) without being convergent.