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VIII. On some formulæ which serve to indicate the limits of the application of indeterminate methods to the solution of certain problems

James Cockle Esq. M.A.

VIII. *On some Formulæ which serve to indicate the limits of the application of Indeterminate Methods to the solution of certain Problems.* By JAMES COCKLE, *Esq., M.A., of Trinity College, Cambridge; Barrister-at-Law, of the Middle Temple. Second and concluding Part*.*

[The first part will be found at pp. 181–183 of the preceding volume†.]

LET $x_1, x_2, .., x_n$ denote the roots of

$$x^n + x'x^{n-1} + x''x^{n-2} + . + x^{(n)} = 0,$$

the general equation of the nth degree, and, by a similar notation, let $y_1, y_2, . y_n$ be the roots, and $y', y'', . y^{(n)}$ the coefficients of the equation in y, when

$$y = \xi^{(0)} + \xi' x + \xi'' x^2 + \ . \ + \xi^{\text{xxii}} x^{22},$$

it is required to point out how to determine $\xi^{(0)}, \xi', .., ..$, so that we may have

$$y' = 0, \ \ y'' = 0, \ \ y''' = 0, \ \text{ and } y'''' = 0.$$

By means of $y' = 0$, eliminate $\xi^{(0)}$ from the three succeeding equations, and, conformably with the notation adopted in one of my previous communications to this work ‡, represent the respective results by

$${}_2Y'_n = 0, \ \ {}_3Y'_n = 0, \ \text{ and } {}_4Y'_n = 0.$$

Next, by processes which I have already referred to in the present periodical §, let ${}_2Y'_n = 0$ be put under the form

$$h_1^2 + h_2^2 + . + h_{22}^2 = 0,$$

where $$h_1 = \varkappa'_1 \xi' + \varkappa''_1 \xi'' + . + \varkappa_1^{\text{xxii}} \xi^{\text{xxii}},$$

and, in general,

$$h_r = \varkappa_r^{(r)} \xi^{(r)} + \varkappa_r^{(r+1)} \xi^{(r+1)} + \ . \ + \varkappa_r^{\text{xxii}} \xi^{\text{xxii}}.$$

We have twenty-two quantities (ξ) at our disposal,—make

$$h_1^2 + h_2^2 = 0, \ . \ (\text{A.}), \quad h_3^2 + h_4^2 = 0, \ . \ (\text{B.}),$$

and $$h_5^2 + h_6^2 = 0, \ .., .., \ h_{21}^2 + h_{22}^2 = 0;$$

with the aid of the last nine of these equations (previously put under the form

$$h_{r-1} \pm \sqrt{-1}\, h_r = 0),$$

let nine of the ξ's be eliminated from the first two; after the elimination there will remain 22−9 or 13 of the quantities ξ', ξ'', . ξ^{xxii} still undetermined.

Now it is known ‖ that, whatever be the number of the quantities ξ, the above expression for y is equivalent to

$$y = \Xi^{(0)} + \Xi' x + \Xi'' x^2 + . + \Xi^{(n-1)} x^{n-1},$$

* Communicated by Sir George Cayley, Bart.

† Phil. Mag. S. 3. vol. xxix. ‡ Ibid. vol. xxviii. p. 191.

§ Ibid. vol. xxvii. pp. 126, 292, 293.

‖ Sixth Report of the British Association, p. 301 *et seq.*

where

$$\Xi^{(r)} = s_r^{(0)}\xi^{(0)} + s_r^{(1)}\xi' + s_r^{(2)}\xi'' + . + s_r^{(22)}\xi^{\text{xxii}};$$

so that the number of disposable and independent quantities (Ξ) contained in the expression for y can in no case exceed n; neither, after the elimination of the nine ξ's, can it exceed 13, as is seen on referring to the last of the above equations. The elimination of those quantities does not, however, diminish the number of disposable quantities (Ξ) except when n is greater than 13.

If then

$$h_1 = K_1'\Xi' + K_1''\Xi'' + K_1'''\Xi''' + K_1^{\text{iv}}\Xi^{\text{iv}} + B_1$$
$$h_2 = K_2''\Xi'' + K_2'''\Xi''' + K_2^{\text{iv}}\Xi^{\text{iv}} + B_2$$
$$h_3 = K_3'''\Xi''' + K_3^{\text{iv}}\Xi^{\text{iv}} + B_3$$
$$h_4 = K_4^{\text{iv}}\Xi^{\text{iv}} + B_4,$$

where $B_1, ., ., B_4$ are functions of the $n-5$ quantities Ξ^{v}, $\Xi^{\text{vi}}, . \Xi^{(n-1)}$, we see, by what precedes, that nine of the quantities ξ may be so determined as to enable us to decompose ${}_2Y_n = 0$ into

$$h_1^2 + h_2^2 = 0, \ . \ \text{(A.)} \quad \text{and} \quad h_3^2 + h_4^2 = 0, . \ \text{(B.)}$$

where $h_1, ., ., h_4$ have the forms last above given, and the $n-5$ quantities $\Xi^{\text{v}}, \Xi^{\text{vi}}, ., \Xi^{(n-1)}$ are undetermined, and perfectly at our disposal; at least when n is not greater than 13, and when n exceeds 13, we have eight of them undetermined and disposable. But it will be seen below that, for our present purpose, this last case does not require consideration. $\Xi', \Xi'', . \Xi^{\text{iv}}$, have as yet no other conditions than (A.) and (B.) to satisfy.

Depress (A.) and (B.) to linear equations, and eliminate Ξ''', Ξ^{iv}, from ${}_3Y'_n$ by their means. Then, on referring to my paper at pages 190–191 of the last volume but one of this work*, it will be seen that, without determining Ξ', Ξ'', it will be possible to reduce the resulting equation to the form

$$(\text{K}'_1, \Xi' + \text{K}''_1\Xi'' + \text{B}_1)^3 + (\text{K}''_2\Xi'' + \text{B}_2)^3 = 0;$$

or, $$\text{h}_1^3 + \text{h}_2^3 = 0; \quad . \ . \ . \ . \ . \ . \quad \text{(C.)}$$

and also that B_1 and B_2 will not give the illusory results which (under a different notation) I have before† pointed out, provided the number of disposable quantities $\Xi^{\text{v}}, ., ., \Xi^{(n-1)}$ be more than three in number; this gives the condition

$$n-5 > 3, \text{ or } n > 8 \quad . \ . \ . \ . \ . \ . \ . \quad (\text{y.}')‡$$

With the aid of (C.) reduced to a linear form, eliminate Ξ' or Ξ'' from ${}_4Y'_n = 0$, and solve the resulting equation.

* Phil. Mag. S. 3. vol. xxviii. † Ibid. pp. 190, 191, 395.

‡ This corresponds to the equation (y.) of p. 191 of Phil. Mag. S. 3. vol. xxviii.

In effect we solve (A.), (B.), (C.) and (D.), by means of Ξ', Ξ'', Ξ''', Ξ^{iv}; the remaining Ξ's (excepting $\Xi^{(0)}$) are determined in effecting the reduction of ${}_3Y'_n$ to the form of (C.); and $\Xi^{(0)}$ will be obtained from $y'=0$, after substituting in it the values of the other Ξ's.

The above investigations give the formula

$$n(1, 1, 1, 1) \text{ (or, } n\,(1^4))=9; \quad . \quad . \quad . \quad (317''.)$$

and shows that the *general equations of the* NINTH *and higher degrees may be transformed into others of the same degrees, from which the second, third, fourth and fifth terms disappear*: the corresponding formula for m (see the first part of this discussion*) is

$$m(1, 1, 1, 1) = 11 \text{ (or } 10). \quad . \quad . \quad . \quad (317.)$$

2 Church Yard Court, Temple,
November 28, 1846.

Postscript, Dec. 14, 1846.—On looking over the proof-slip, I observe that, in this paper, I have not had occasion to use the foregoing notation for the *roots* of equations. But the above may be considered to suggest the following permanent notation; viz. that x_r should represent a root, and $x^{(r)}$ the coefficient of the $(r+1)$th term of an equation in X; ${}_rX$ a quantity composed of symmetric functions of, and homogeneous and of the rth degree with respect to, x_1, x_2, . x_n; that ξ should denote the disposable quantities which enter (explicitly) into the equation for y and Ξ, the disposable quantities implicitly contained in that equation. When r is given as a number, we may, however, as above, express the coefficients by accents if r be small, or by Roman numerals if it be large. For facility of reference I have termed 'last' volume what was in fact the current one at the time of writing this paper.

IX. *On the Solvent Action of Drainage-Water on Soils.* *By* John Wilson, *Esq.*†

In the autumn of 1844, being a resident in East Lothian, where the system of *thorough draining* is very extensively carried out, it occurred to me that the drainage-water during its percolation of the soil must necessarily dissolve out and carry away a great portion of the soluble constituents of it, which, by the practice as at present followed, are carried off the land and entirely lost to the farmer. I therefore took advantage of the first fall of rain sufficient to set the drains

* Phil. Mag. S. 3. vol. xxix. pp. 181–183.

† Communicated by the Chemical Society; having been read May 4, 1846.