



LII. On stationary waves in flowing water.—Part II

Sir William Thomson F.R.S.

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to make, for the surface of the soft carbon might not be in the same state of cleanliness as that of the hard carbon.

But even suppose we allow that there was better contact with the mercury in the case of the soft carbon than in that of the hard carbon, it does not necessarily follow that the observed decrease of resistance consequent on increase of pressure occurred in the *carbon* of the button. The buttons are, I believe, formed by compressing lampblack *mixed with gum-water*. Must we not suppose, then, that the particles of lampblack are bound to each other by the gum, and separated from each other by the gum, to a greater or less extent? Would not the diminution of resistance experienced in the body of the button when pressure was applied be due to one or both of the two following causes:—

1. Diminution of the resistance of the thin coating of gum between the particles of carbon?

2. Better surface-contact between one particle of carbon and another?

The influence of *time* on the change of resistance might be quite accounted for by supposing cause 1 to be at work*.

LII. On Stationary Waves in Flowing Water.—Part II.

By Sir WILLIAM THOMSON, F.R.S. &c.

[Continued from p. 357.]

Correction in Part II.—In lines 4–7 of paragraph 3 on page 357, delete the words “, because” ... to ... “canal”; and add the sentence “The explanation of this will be more fully developed in Part III., to be published in December.”

TO find, as promised in Part I., the sum of horizontal pressures on an inequality of the bottom, or on a bar, or on a series of inequalities or bars, consider the horizontal components of momentum of different portions of the water in the following manner. Because the motion is steady, the momentum of the matter at any instant within any fixed volume of space *S* remains constant; and therefore the rate of delivery of momentum from *S* by water flowing out on one side above gain of momentum by water flowing into *S* on the other side must be equal to the total amount of horizontal force acting on the water which at any instant is within *S*; the direction of this force being that of the flow when the momentum of the leaving water exceeds that of the entering water. Now let *S* be the space bounded by the bottom, the

* See the account of my own experiments on the *viscous* metals zinc and tin, quoted above.

free surface of the water, and four vertical planes, two of them, called A_0 , A , perpendicular to the stream, and two of them parallel to the stream and at unit distance from one another. Let \mathfrak{P} P B , and \mathfrak{P}_0 B_0 be vertical lines on the two transverse ends A and A_0 of the space S ; \mathfrak{P} , \mathfrak{P}_0 being points of the surface, and B , B_0 points of the bottom. Let

$$\mathfrak{P}B = D \text{ and } \mathfrak{P}P = y,$$

and let u be the horizontal component velocity at P . The rate of delivery of momentum (per unit of time understood) from S by water flowing across A is equal to

$$\int_0^D u^2 dy \quad . \quad . \quad . \quad . \quad . \quad (1);$$

and the excess of delivery of momentum from S across A above receipt of momentum across A_0 is equal to

$$\int_0^D u^2 dy - \left\{ \int_0^D u^2 dy \right\}_0 \quad . \quad . \quad . \quad . \quad . \quad (2).$$

When this is positive, the water between A_0 and A must experience, on the whole, a pressure in the direction from A_0 towards A , made up of difference of fluid-pressures on the end sections A_0 and A , and pressures upon the water by fixed inequalities, if there are any, between A_0 and A . Hence if X , X_0 denote the integral fluid-pressures on the ideal planes A , A_0 , and F the sum of horizontal pressures of the inequalities on the fluid, regarded as positive when the direction of the total is from A towards A_0 , (2) must be equal to

$$X_0 - X - F \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Hence we have

$$F = \left\{ X + \int_0^D u^2 dy \right\}_0 - \left(X + \int_0^D u^2 dy \right) \quad . \quad . \quad . \quad (4).$$

Now the fluid-pressure at P is equal to $gy + \frac{1}{2}(q^2 - q'^2)$, by the elementary formula for pressure in steady motion (the pressure at the free surface being taken as zero), q and q' denoting the velocity of the fluid at \mathfrak{P} and P respectively. Hence

$$X = \int_0^D [gy + \frac{1}{2}(q^2 - q'^2)] dy = \frac{1}{2}(gD + q^2)D - \frac{1}{2} \int_0^D q'^2 dy \quad . \quad (5).$$

Hence

$$X + \int_0^D u^2 dy = \frac{1}{2}(gD + q^2)D + \frac{1}{2} \int_0^D (u^2 - v^2) dy \quad . \quad . \quad (6),$$

if v be the vertical component velocity at P .

This, and the corresponding expression relatively to A_0 , gives, by (3), the sum of horizontal pressure on all inequalities between A_0 and A , when the problem of the fluid motion in the circumstances is so far solved as to give D , q , and $u^2 - v^2$ for each of the end sections A_0 , A .

Suppose, now, A_0 to be so far on the up-stream side of the inequalities that the motion of the water across it is sensibly uniform and horizontal, with velocity which we shall denote by U_0 ; so that, for A_0 , (6) becomes

$$\left\{ X + \int_0^D u^2 dy \right\}_0 = \frac{1}{2}gD_0^2 + U_0^2D_0 \quad . \quad . \quad . \quad (7).$$

Hence, and by (6) and (4),

$$F = \frac{1}{2}g(D_0^2 - D^2) + U_0^2D_0 - \frac{1}{2}q^2D - \frac{1}{2} \int_0^D (u^2 - v^2) dy \quad . \quad (8).$$

Now, by the law of velocity at the free surface in steady motion, we have

$$\frac{1}{2}q^2 = \frac{1}{2}U_0^2 + g(D_0 - D) \quad . \quad . \quad . \quad (9);$$

because, the points B_0 , B of the bottom being on the same level, $D_0 - D$ is the difference of levels between the surface-points \mathfrak{A}_0 and \mathfrak{A} . Hence (8) becomes

$$F = \frac{1}{2}g(D_0 - D)^2 + U_0^2(D_0 - D) - \frac{1}{2}(U^2 - U_0^2)D + \frac{1}{2} \int_0^D (v^2 + U^2 - u^2) dy \quad . \quad (10),$$

where U denotes a constant which may have any value. It is convenient to make it the mean horizontal component velocity across $\mathfrak{A}B$: we therefore take

$$U = \frac{1}{D} \int_0^D u dx \quad . \quad . \quad . \quad (11):$$

and, because the quantities flowing in across A_0 and out across A are equal, as the motion is steady, we have

$$UD = U_0D_0 \quad . \quad . \quad . \quad (12).$$

Using this to eliminate U_0 from (10), we find

$$F = \frac{1}{2} \left(g - \frac{U^2D}{D_0^2} \right) (D_0 - D)^2 + \frac{1}{2} \int_0^D (v^2 + U^2 - u^2) dy \quad . \quad (13).$$

To evaluate $D_0 - D$ when we know enough about the motion, and to see how its value is related to other characteristic quantities, let us look back to (9), and in it take

$$q^2 = U^2 + v^2 \quad . \quad . \quad . \quad (14).$$

Thus, if \mathfrak{P} be chosen at a point of the water-surface where the horizontal component velocity is rigorously or approximately equal to U , then v is rigorously or approximately the vertical component velocity at \mathfrak{P} . Using now (14) in (9), with UD/D_0 for U_0 , we find

$$D_0 - D = \frac{\frac{1}{2} v^2}{g - \frac{\frac{1}{2} (D_0 + D) U^2}{D_0^2}} \quad \dots \quad (15);$$

which, used in (13), gives

$$F = \frac{v^4}{8} \frac{g - \frac{U^2 D}{D_0^2}}{\left[g - \frac{\frac{1}{2} (D_0 + D) U^2}{D_0^2} \right]^2} + \frac{1}{2} \int_0^D (v^2 + U^2 - u^2) dy \quad (16).$$

Hence, when the change of level, $D_0 - D$, is but small, in comparison with D or D_0 , we have

$$F \doteq \frac{1}{8} \frac{v^4}{U^2} + \frac{1}{2} \int_0^D (v^2 + U^2 - u^2) dy \quad \dots \quad (17),$$

where \doteq denotes approximate equality. Going back to (16), let \mathfrak{P} be so chosen on the water-surface that

$$\int_0^D u^2 dy = U^2 D \quad \dots \quad (18),$$

which it is clear we can do, because at a crest the first member is less than the second, and at a hollow greater. When the motion is infinitely nearly simple harmonic (the stream-lines curves of lines), the position of \mathfrak{P} thus chosen will be exactly the middle between crest and hollow. When the motion is anything, however great, up to Stokes's highest possible wave, the chosen place of \mathfrak{P} is a less or more rough approximation to the mid-level point of a wave: it is always rigorously determinate. For brevity we shall call it, that is to say a point defined by (18), a nodal point. Thus, when \mathfrak{P} is taken as a nodal point, (16) becomes simplified to

$$F = \frac{v^4}{8} \frac{g - \frac{U^2 D}{D_0^2}}{\left[g - \frac{\frac{1}{2} (D_0 + D) U^2}{D_0^2} \right]^2} + \frac{1}{2} \int_0^D v^2 dy \quad \dots \quad (19).$$

This expression is rigorous. In it v , which is given rigorously by (14), is approximately (not rigorously) equal to the vertical component velocity at \mathfrak{P} : and if we suppose D given,

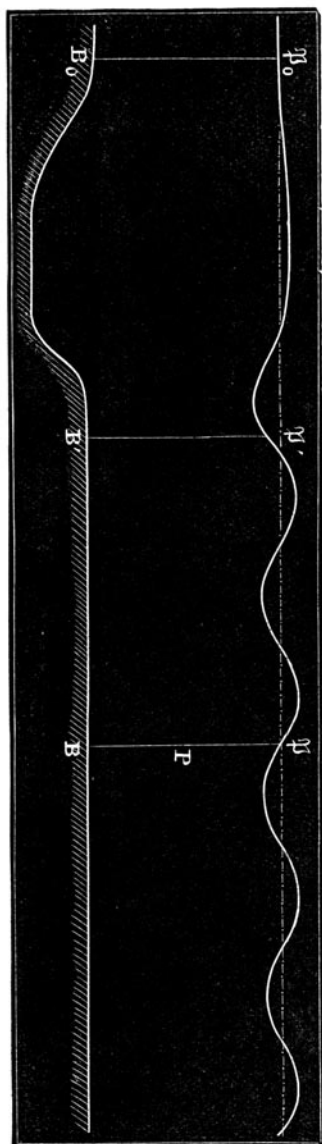


Fig. 2.

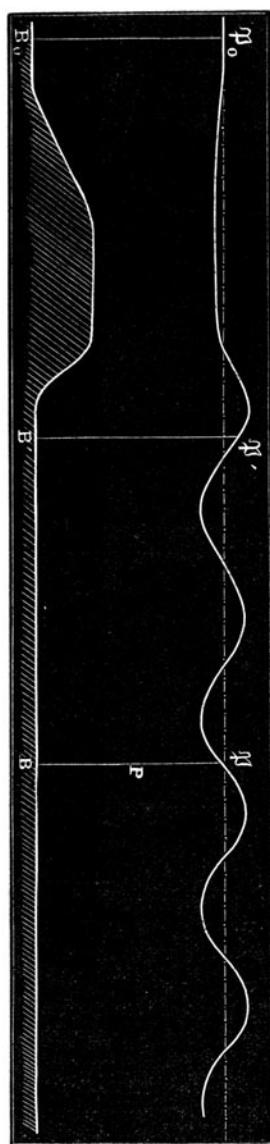


Fig. 1.

D_0 is found by (15), which is a cubic equation in D_0 , most easily solved by successive approximations according to the process obviously indicated by the form in which the equation appears in (15). (As a first approximation take D for D_0 in the second member, and so on.)

To work out the formula (19) for the case of infinitesimal displacement, we may take \mathfrak{P} at a great enough distance from inequalities to let the surface in its neighbourhood be sensibly a curve of sines, and the motion simple harmonic. The investigation is facilitated by also taking \mathfrak{P} at a node, as in the diagrams. If we take

$$y = h \sin mx \quad . \quad . \quad . \quad . \quad . \quad (20)$$

as the equation of the free surface, the known solution for simple harmonic waves in water of depth D gives,

$$\left. \begin{aligned} u &= U \left\{ 1 + mh \frac{e^{m(D-y)} + e^{-m(D-y)}}{e^{mD} - e^{-mD}} \sin mx \right\}, \\ v &= Umh \frac{e^{m(D-y)} - e^{-m(D-y)}}{e^{mD} - e^{-mD}} \cos mx, \end{aligned} \right\} \quad (21).$$

where $U = \sqrt{\left\{ \frac{g}{m} \frac{e^{mD} - e^{-mD}}{e^{mD} + e^{-mD}} \right\}}.$

Hence, where $x=0$, as in the nodal section $\mathfrak{P} P B$,

$$u = U, \text{ and } v = Umh \frac{e^{m(D-y)} - e^{-m(D-y)}}{e^{mD} - e^{-mD}} \quad . \quad . \quad (22);$$

also

$$\int_0^D v^2 dy = \frac{1}{2} U^2 m h^2 \frac{e^{2mD} - e^{-2mD} - 4mD}{(e^{mD} - e^{-mD})^2} \quad . \quad . \quad . \quad (23),$$

$$= \frac{1}{2} g h^2 \left\{ 1 - \frac{4mD}{e^{2mD} - e^{-2mD}} \right\} \quad . \quad . \quad . \quad (24).$$

Now going back to (19) we see that when U approaches the critical velocity $\sqrt{gD \frac{D_0^2}{\frac{1}{2}(D_0 + D)D}}$, the first term might become important, even though the corrugations at a great distance down-stream from the inequalities were infinitesimal. Reserving consideration of this case, and supposing for the present U to be considerably smaller than the critical value, we may neglect the first term in comparison with the second, remembering that in fact quantities comparable with the first

term are neglected in the approximation (24) to the value of the second ; and we have, as our final approximate result,

$$F = \frac{1}{4}gh^2 \left(1 - \frac{4mD}{e^{2mD} - e^{-2mD}} \right) \dots \dots \dots (25).$$

There is no difficulty in understanding the permanent steadiness of the motion which we have now been considering: to any finite distance, however great, on either the up-stream or down-stream side of the inequalities, if the water in the finite space considered is given in this state of motion, and if water is admitted on the one side and carried away on the the other side conformably. But it is very interesting and instructive to consider the initiation of such a state of things from an antecedent condition of uniform flow over a plane bottom. Suppose, as the primary condition, an inequality, whether elevation or depression, to exist in the bottom, but to be carried along with the water, so that the flow of the water is everywhere uniform and in parallel lines. If the inequality is an elevation above the bottom, our supposition is that the whole projecting piece, moving with the water, slips along the bottom. If the inequality be a depression in the bottom, the more awkward supposition must be made of a plasticity of the bottom, and the form of the inequality carried along, while the bottom is kept rigidly plane before and after this depression.

Suppose, now, the inequality is gradually or suddenly brought to rest, what will be the resulting motion of the water? The question is identical with that of finding the motion of water in a canal, when by an external force, such as that of a towing-rope, a boat is gradually or suddenly set in motion through it ; or, rather, it would be identical if the boat were a beam filling the whole breadth across the canal, so that the motion of the water shall be purely two-dimensional. I hope in a later article (Part III. or Part IV. of the present series) to investigate the formation of the procession of standing waves in the wake of the obstacle, and its gradual extension farther and farther down-stream from the obstacle, the motion having become sensibly steady in the its neighbourhood, and becoming so to greater and greater distances down-stream by the completion of the growth of fresh waves. The disturbance sent up-stream from the initiating irregularity must also be considered. Equation (15) shows that whether the irregularity be an elevation, as in our first diagram (fig. 1), or a depression, as in fig. 2, a rising of level must travel up-stream, at a velocity relatively to the water which we know must be $\sqrt{gD_0'}$, where D_0' is inter-

mediate between D_0 and the smaller depth, which we shall call D' , in the undisturbed stream above. But however gradually the initiating irregularity may have been instituted, this travelling of an elevation up-stream must develop a bore; because the velocity of propagation is, as it were, different in different parts of the slope, being $\sqrt{gD'}$ at the commencement of the slope, and ranging from this, through $\sqrt{gD_0'}$, to $\sqrt{gD_0}$ as the depth rises from D' to D_0 ; so that, as it were, the brow of the plateau in its advance up-stream overtakes the talus, till the slope becomes too steep for our approximation. The inevitable bore and "broken" water (inevitable without viscosity of the water, or some surface-action preventing the excessive steepness) would modify affairs down-stream in a manner which it is difficult to imagine. It becomes, therefore, interesting to see how it may be avoided, whether by surface-action, or by giving some viscosity to the water. It is more interesting to do this by surface-action, and to allow the water to be perfectly inviscid, so that our standing waves down-stream may be perfectly unimpaired. And we may do it very simply by covering the free surface all over (up-stream and down-stream) with an infinitely thin viscously elastic flexible membrane, stiffened transversely (after the manner of the sail of a Chinese junk) by rigid massless bars with ends travelling up and down in vertical guides on the sides of the canal. If we suppose the motion of these ends to be resisted by forces proportional to their velocities, and the membrane to exercise (positive or negative) contractile tensional force in simple proportion to the velocity of the change of its length in each infinitely small part; we have a mechanical arrangement by which is realized the mathematical condition of a surface normal pressure varying according to normal component velocity of the otherwise free surface, and in simple proportion to this normal velocity when the slope is infinitesimal. By making the viscous forces sufficiently great, we may make the progress of the rise of level up-stream as gradual as we please, and perfectly avoid the bore. We may also make the progress of the procession of stationary waves down-stream as slow as we please. The form of the water-surface over the inequality or inequalities, and to any distance from them, both up-stream and down-stream, is not ultimately affected at all by the viscous covering; and it becomes, as time advances, more and more nearly that of the mathematical solution for steady motion, which I hope to give, with graphic illustrations drawn according to calculation from the solution, in Part III.