

XIII.—*On the Application of Hamilton's Characteristic Function to Special Cases of Constraint.* By Professor TAIT.

(Read 20th March 1865.)

1. One of the grandest steps which has ever been made in Dynamical Science is contained in two papers, "*On a General Method in Dynamics*," contributed to the Philosophical Transactions for 1834 and 1835 by Sir W. R. HAMILTON. It is there shown that the complete solution of any kinetical problem, involving the action of a given conservative system of forces, and constraint depending upon the reaction of smooth guiding curves or surfaces, also given, is reducible to the determination of a single quantity called the *Characteristic Function* of the motion. This quantity is to be found from a partial differential equation of the first order, and second degree; and it has been shown that, from any *complete* integral of this equation, all the circumstances of the motion may be deduced by differentiation. So far as I can discover, this method has not been applied to inverse problems, of the nature of the Brachistochrone for instance, where the object aimed at is essentially the determination of the constraint requisite to produce a given result. It is easy to see, however, that a large class of such questions may be treated successfully by a process perfectly analogous to that of HAMILTON; though the characteristic function in such cases is not the same function (of the quantities determining the motion) as that of the Method of *Varying Action*.

2. It is unnecessary to enter into any great detail with reference to the present subject; because any one who is familiar with HAMILTON'S beautiful investigations will have no difficulty in applying them, with the requisite slight modifications, to the subject of this paper. I shall therefore content myself with a brief explanation of the application of the method to the problem of the Brachistochrone, and a mere indication of some other curious problems which are easily solved in a similar manner.

3. The problem of the Brachistochrone for a single particle is, in its simplest form, as follows:—

Find the form of the (smooth) constraining curve along which a particle will pass, under the action of a given conservative system of forces, from one given point to another in the least possible time, the initial velocity being given.

The problem may easily be complicated by supposing, for instance, the terminal points not to be definitely assigned, but to lie each on a given surface:

still farther, by supposing the initial velocity to depend, according to some given law, upon the coördinates of the initial point, and so forth. But such complications introduce analytical difficulties of the quasi-arithmetical kind merely, not of a physical nature; and we leave them to those who are curious in such matters.

4. In symbols, if τ be the time of passing from x_0, y_0, z_0 to x, y, z , we must have

$$\tau = \int_{x_0, y_0, z_0}^{x, y, z} \frac{ds}{v}$$

a minimum: subject to the sole condition

$$v^2 = 2(H - V)$$

where H is the whole energy, and V the potential of the system of forces on unit mass at the point x, y, z .

Hence, taking the variation,

$$\delta\tau = \int \left(\frac{d\delta s}{v} - \frac{ds\delta v}{v^2} \right).$$

But

$$ds\delta s = dx\delta x + dy\delta y + dz\delta z;$$

and

$$v\delta v = \delta(H - V) = X\delta x + Y\delta y + Z\delta z + \delta H,$$

if X, Y, Z be the component forces on unit mass at x, y, z . Thus we have

$$\begin{aligned} \delta\tau = & \left[\frac{1}{v^2} \left(\frac{dx}{dt} \delta x + \frac{dy}{dt} \delta y + \frac{dz}{dt} \delta z \right) - \delta H \int \frac{ds}{v^3} \right] \\ & - \int \left\{ \delta x \left[d \left(\frac{dx}{v^2} \right) + \frac{Xdt}{v^2} \right] + \&c. \right\}; \end{aligned}$$

where the whole, integrated or not, is to be taken between the given limits.

If the limits and the initial velocity be fixed, the first part of the expression for $\delta\tau$ disappears; and, that the integral may vanish, we must have

$$d \left(\frac{dx}{v^2} \right) + \frac{Xdt}{v^2} = 0, \quad . \quad . \quad . \quad . \quad . \quad (A).$$

with similar equations in y and z . This is simply the ordinary result given in treatises on kinetics.

But if we consider the effect of the alteration of the limits, or of the initial energy, we have

$$\left. \begin{aligned} \frac{\delta \tau}{\delta x} &= \frac{1}{v^2} \frac{dx}{dt}, & \frac{\delta \tau}{\delta x_0} &= - \left(\frac{1}{v^2} \frac{dx}{dt} \right)_0, \\ &\&c. & &\&c. \end{aligned} \right\} \quad \text{and} \quad \frac{\delta \tau}{\delta H} = - \int_{x_0, y_0, z_0}^{x, y, z} \frac{ds}{v^3}. \quad (1).$$

5. Hence, if τ could be found as a function of x, y, z, x_0, y_0, z_0 , and H , it is obvious that its partial differential coefficients with respect to these quantities would give the motion completely.

But, neglecting altogether the initial limit, we see that

$$\begin{aligned} \left(\frac{d\tau}{dx} \right)^2 + \left(\frac{d\tau}{dy} \right)^2 + \left(\frac{d\tau}{dz} \right)^2 &= \frac{1}{v^4} \left(\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right) \\ &= \frac{1}{v^2} = \frac{1}{2(H-V)} \quad (2). \end{aligned}$$

6. It can be easily shown, by a process similar to that employed for *Varying Action*,* that, if any integral of this equation can be found, its partial differential coefficients with respect to x, y, z are respectively equal to the corresponding components of the velocity, in a curve which is a brachistochrone for the given forces, *each divided by the square of the whole velocity*.

A *complete* integral of (2) must of course contain, besides H , two arbitrary constants α, β . If, then, τ be a complete integral, the equations of the brachistochrone are easily shown to be

$$\frac{d\tau}{d\alpha} = \mathfrak{A}, \quad \frac{d\tau}{d\beta} = \mathfrak{B} \quad (3);$$

where \mathfrak{A} and \mathfrak{B} are two new arbitrary constants.

Also we have the relation

$$\frac{d\tau}{dH} = - \int \frac{dt}{v^2} = - \int \frac{ds}{v^3} \quad (4).$$

7. Before proceeding farther with the theory, we may apply the results already obtained to one or two well-known problems; commencing with the original case proposed by BERNOULLI.

8. *To find the brachistochrone, when gravity is the only impressed force, and the particle has the velocity due to a fall from a given horizontal plane.*

Taking the axis of y vertically downwards, we have

$$V = -gy.$$

Also, we may write

$$H = g\alpha.$$

* THOMSON and TAIT's *Natural Philosophy*, § 323, or TAIT and STEELE's *Dynamics of a Particle* (2d edition), §§ 252, 253.

Hence

$$\left(\frac{d\tau}{dx}\right)^2 + \left(\frac{d\tau}{dy}\right)^2 + \left(\frac{d\tau}{dz}\right)^2 = \frac{1}{2g(a+y)}.$$

This equation is obviously satisfied by

$$\left(\frac{d\tau}{dx}\right) = M, \quad \left(\frac{d\tau}{dz}\right) = N, \quad \left(\frac{d\tau}{dy}\right)^2 = \frac{1}{2g(a+y)} - M^2 - N^2.$$

But

$$\frac{\left(\frac{d\tau}{dx}\right)}{\left(\frac{d\tau}{dz}\right)} = \frac{\frac{dx}{dt}}{\frac{dz}{dt}} \text{ (by § 6) } = \frac{dx}{dz}.$$

Hence $\frac{dx}{dz} = \frac{M}{N}$, that is *the path is in a vertical plane*. We may take this as the plane of xy . Hence our equation becomes

$$\left(\frac{d\tau}{dx}\right)^2 + \left(\frac{d\tau}{dy}\right)^2 = \frac{1}{2g(a+y)}.$$

We may now write

$$\left. \begin{aligned} \frac{d\tau}{dx} &= \frac{1}{\sqrt{2gb}} \\ \left(\frac{d\tau}{dy}\right)^2 &= \frac{1}{2g} \left(\frac{1}{a+y} - \frac{1}{b}\right), \end{aligned} \right\} \dots \dots \dots (5).$$

where b is an arbitrary constant.

By (5) we have, at once,

$$\sqrt{2g} \tau = \frac{x}{\sqrt{b}} + \int dy \sqrt{\frac{1}{a+y} - \frac{1}{b}} \dots \dots \dots (6).$$

Hence the equation of the brachistochrone is (by § 6)

$$\frac{d\tau}{db} = \text{const.}$$

or

$$C = -\frac{x}{b^{\frac{3}{2}}} + \frac{1}{b^2} \int \frac{dy}{\sqrt{\frac{1}{a+y} - \frac{1}{b}}};$$

that is, changing the constant, and effecting the integration,

$$C_1 = -x - \sqrt{(b-a-y)(a+y)} + \frac{b}{2} \text{ vers. } \frac{2(a+y)}{b} \dots \dots \dots (7).$$

the common equation of the *Cycloid*, the velocity at any point being that due to a fall from the base.

In this case we have evidently

$$\begin{aligned}\frac{d\tau}{dH} &= -\int \frac{ds}{v^3} = \frac{1}{g} \frac{d\tau}{da} = -\frac{1}{2\sqrt{2g^3}} \int \frac{dy}{(a+y)^2 \sqrt{\frac{1}{a+y} - \frac{1}{b}}} \\ &= \frac{1}{\sqrt{2g^3}} \sqrt{\frac{1}{a+y} - \frac{1}{b}} + C_2.\end{aligned}$$

The above (at first sight apparently too limited) assumptions

$$\frac{d\tau}{dx} = M, \quad \frac{d\tau}{dz} = N,$$

and the consequent reduction of the question to a *plane* problem, may seem to require some justification. This is easily supplied, thus: In the equation

$$\left(\frac{d\tau}{dx}\right)^2 + \left(\frac{d\tau}{dy}\right)^2 + \left(\frac{d\tau}{dz}\right)^2 = F^2,$$

the direction-cosines of the tangent to the brachistochrone, at the point x, y, z , are, by (1.),

$$l = \frac{1}{F} \frac{d\tau}{dx}, \quad m = \frac{1}{F} \frac{d\tau}{dy}, \quad n = \frac{1}{F} \frac{d\tau}{dz}.$$

At the adjacent point $x + \delta x, y + \delta y, z + \delta z$, where we have, of course,

$$\frac{\delta x}{l} = \frac{\delta y}{m} = \frac{\delta z}{n} = \delta s,$$

the value of l becomes

$$\begin{aligned}l' &= \frac{\frac{d\tau}{dx} + \frac{d^2\tau}{dx^2} \delta x + \frac{d^2\tau}{dxdy} \delta y + \frac{d^2\tau}{dxdz} \delta z}{F + \delta F} \\ &= \frac{\frac{d\tau}{dx} + \frac{\delta s}{F} \left(\frac{d\tau}{dx} \frac{d^2\tau}{dx^2} + \frac{d\tau}{dy} \frac{d^2\tau}{dxdy} + \frac{d\tau}{dz} \frac{d^2\tau}{dxdz} \right)}{F + \delta F} \\ &= \frac{\frac{d\tau}{dx} + \left(\frac{dF}{dx}\right) \delta s}{F + \delta F}.\end{aligned}$$

But in the above problem F is a function of y only, and we must therefore have

$$\frac{l'}{n'} = \frac{l}{n},$$

which shows that the curve is in a plane parallel to the axis of y .

9. To find the Brachistochrone when the force is central, and proportional to a power of the distance; the velocity being also proportional to a power of the distance, that is, being the velocity from infinity if the force is attractive, from the centre if it is repulsive.

Here

$$v^2 = 2(H - V) = \frac{\mu}{r^n},$$

and the central force at distance r is evidently

$$-\frac{dV}{dr} = -\frac{n\mu}{2r^{n+1}}.$$

Thus (2) becomes

$$\left(\frac{d\tau}{dx}\right)^2 + \left(\frac{d\tau}{dy}\right)^2 + \left(\frac{d\tau}{dz}\right)^2 = \frac{r^n}{\mu}$$

or, changing to polar co-ordinates,

$$\left(\frac{d\tau}{dr}\right)^2 + \frac{1}{r^2} \left(\frac{d\tau}{d\theta}\right)^2 + \frac{1}{r^2 \sin^2 \theta} \left(\frac{d\tau}{d\phi}\right)^2 = \frac{r^n}{\mu}.$$

It is obvious that we must take

$$\frac{d\tau}{d\phi} = 0,$$

which shows that the path is in a plane passing through the centre of force. The above equation will then be satisfied by

$$\frac{d\tau}{d\theta} = a, \quad \frac{d\tau}{dr} = \sqrt{\frac{r^n}{\mu} - \frac{a^2}{r^2}}.$$

Hence we have

$$\begin{aligned} \tau &= a\theta + \int dr \sqrt{\frac{r^n}{\mu} - \frac{a^2}{r^2}}, \\ &= a\theta + \frac{2a}{n+2} \left\{ \sqrt{\frac{r^{n+2}}{\mu a^2} - 1} - \cos^{-1} \frac{\sqrt{\mu a}}{r^{\frac{n+2}{2}}} \right\} + C. \end{aligned}$$

And the equation of the brachistochrone, which is evidently a plane curve, is

$$\begin{aligned} \mathfrak{A} &= \theta + \frac{2}{n+2} \left\{ \sqrt{\frac{r^{n+2}}{\mu a^2} - 1} - \cos^{-1} \frac{\sqrt{\mu a}}{r^{\frac{n+2}{2}}} \right\} \\ &+ \frac{2a}{n+2} \left\{ -\frac{\frac{r^{n+2}}{\mu a^2}}{\sqrt{\frac{r^{n+2}}{\mu a^2} - 1}} + \frac{\sqrt{\mu}}{r^{\frac{n+2}{2}}} \frac{1}{\sqrt{1 - \frac{\mu a^2}{r^{n+2}}}} \right\} \\ &= \theta - \frac{2}{n+2} \cos^{-1} \frac{\sqrt{\mu a}}{r^{\frac{n+2}{2}}} : \end{aligned}$$

or

$$r^{\frac{n+2}{2}} = \sqrt{\mu a} \sec \frac{n+2}{2} (\theta - \mathfrak{A}),$$

while the equation of the *free* path is

$$\left(\frac{r}{\alpha}\right)^{\frac{n-2}{2}} = \cos \frac{n-2}{2} (\theta + \beta).$$

The above integration fails in the case of $n = -2$; that is, when the force is repulsive and directly as the distance, the velocity vanishing at the centre of force. But in this case

$$\tau = \alpha\theta + \sqrt{\frac{1}{\mu} - \alpha^2} \log Cr,$$

and the equation of the brachistochrone is

$$\mathfrak{A} = \theta - \frac{\alpha}{\sqrt{\frac{1}{\mu} - \alpha^2}} \log Cr,$$

the logarithmic spiral. Eliminating r between these equations, we see that the time is proportional to the polar angle.

Since a definite form has been assigned to the expression for the velocity in this problem, it is obvious that H is given, and therefore that there is no $\frac{d\tau}{dH}$.

The assumption

$$\frac{d\tau}{d\phi} = 0$$

is easily justified, in the case of any equation of the form

$$\left(\frac{d\tau}{dr}\right)^2 + \frac{1}{r^2} \left(\frac{d\tau}{d\theta}\right)^2 + \frac{1}{r^2 \sin^2 \theta} \left(\frac{d\tau}{d\phi}\right)^2 = F^2,$$

if F be a function of r only. For

$$\delta \left(\frac{d\tau}{d\phi}\right) = \frac{d^2\tau}{drd\phi} \delta r + \frac{d^2\tau}{d\theta d\phi} \delta \theta + \frac{d^2\tau}{d\phi^2} \delta \phi.$$

But

$$\frac{d\tau}{dr} = F^2 \frac{dr}{dt}, \quad \frac{d\tau}{rd\theta} = F^2 \frac{rd\theta}{dt}, \quad \frac{d\tau}{r \sin \theta d\phi} = F^2 \frac{r \sin \theta d\phi}{dt}.$$

Hence

$$\delta \left(\frac{d\tau}{d\phi}\right) = \frac{\delta t}{F^2} \left\{ \frac{d\tau}{dr} \frac{d^2\tau}{drd\phi} + \frac{1}{r^2} \frac{d\tau}{d\theta} \frac{d^2\tau}{d\theta d\phi} + \frac{1}{r^2 \sin^2 \theta} \frac{d\tau}{d\phi} \frac{d^2\tau}{d\phi^2} \right\} = \frac{\delta t}{F} \left(\frac{dF}{d\phi}\right) = 0.$$

That is, unless F contains ϕ , $\frac{d\tau}{d\phi}$ is necessarily a constant, β suppose.

But, in the present case, if we give this constant any value but zero, we introduce a problem much more general than that proposed, for the expression for the reciprocal of the square of the velocity becomes

$$\frac{r^n}{\mu} - \frac{\beta^2}{r^2 \sin^2 \theta}.$$

10. As an example of a tortuous curve we take the following :

Determine the form of the brachistochrone when the velocity at any point of space is proportional to the distance from a given line.

Taking the line as the axis of z , our equation obviously becomes

$$\left(\frac{d\tau}{dx}\right)^2 + \left(\frac{d\tau}{dy}\right)^2 + \left(\frac{d\tau}{dz}\right)^2 = \frac{a^2}{x^2 + y^2}.$$

Hence

$$\frac{d\tau}{dz} = a,$$

and, substituting this, and changing to polar co-ordinates in a plane parallel to xy ,

$$\left(\frac{d\tau}{dr}\right)^2 + \frac{1}{r^2} \left(\frac{d\tau}{d\theta}\right)^2 = \frac{a^2}{r^2} - a^2.$$

Hence we may take

$$\frac{d\tau}{d\theta} = \beta,$$

and there remains

$$\frac{d\tau}{dr} = \frac{1}{r} \sqrt{a^2 - \beta^2 - a^2 r^2}.$$

Integrating, we have

$$\tau = az + \beta\theta - \sqrt{a^2 - \beta^2} \log. \left[\frac{\sqrt{a^2 - \beta^2}}{r} + \sqrt{\frac{a^2 - \beta^2}{r^2} - a^2} \right] + \sqrt{a^2 - \beta^2 - a^2 r^2}.$$

By equating to constants the partial differential coefficients of τ with respect to a and β , we obtain the two equations of the brachistochrone

$$\mathfrak{A} = z - \frac{ar^2}{\sqrt{a^2 - \beta^2} + \sqrt{a^2 - \beta^2 - a^2 r^2}},$$

and

$$\mathfrak{B} = \theta + \frac{\beta}{\sqrt{a^2 - \beta^2}} \log. \left[\frac{\sqrt{a^2 - \beta^2}}{r} + \sqrt{\frac{a^2 - \beta^2}{r^2} - a^2} \right].$$

The former of these is the equation of a sphere, as may be seen at once by putting it in the form

$$a(z - \mathfrak{A}) = \sqrt{a^2 - \beta^2} - \sqrt{a^2 - \beta^2 - a^2 r^2}.$$

The remaining equation, by altering the value of \mathfrak{B} , may be reduced to the form

$$2 \frac{\sqrt{a^2 - \beta^2}}{a} = r \left(\epsilon^{\frac{\sqrt{a^2 - \beta^2}}{\beta} (\theta - \mathfrak{B})} + \epsilon^{-\frac{\sqrt{a^2 - \beta^2}}{\beta} (\theta - \mathfrak{B})} \right)$$

which is at once recognised as a cylinder, whose base is one of COTES' Spirals.

Also, if we remark that, by (1),

$$r \frac{d\theta}{dt} = v^2 \frac{d\tau}{r d\theta} = \frac{r^2}{a^2} \cdot \frac{\beta}{r} = \frac{\beta v}{a}$$

we see that

$$\cos \psi = \frac{r \frac{d\theta}{dt}}{v} = \frac{\beta}{\alpha} = \text{const.}$$

where ψ is the inclination of the element $r\delta\theta$ to the corresponding element δs of the brachistochrone. That is, the brachistochrone cuts all circles on the above sphere, whose planes are parallel to xy , at a constant angle. (*Loxodrome*.)

11. It is easily seen that

$$\tau = C$$

is the equation of an *Isochronous* surface.

Also, since

$$\frac{\left(\frac{d\tau}{dx}\right)}{\frac{dx}{dt}} = \frac{\left(\frac{d\tau}{dy}\right)}{\frac{dy}{dt}} = \frac{\left(\frac{d\tau}{dz}\right)}{\frac{dz}{dt}}$$

the brachistochrone cuts all such surfaces at right angles.

And the normal distance between two consecutive isochronous surfaces is proportional to the velocity in the brachistochrone of which it forms an element. For, of course,

$$\delta s = v \delta \tau.$$

12. Generally, putting

$$\mathfrak{T} = \left(\frac{d\tau}{dx}\right)^2 + \left(\frac{d\tau}{dy}\right)^2 + \left(\frac{d\tau}{dz}\right)^2 = \frac{1}{2(H-V)} \quad (7),$$

we have

$$2(H-V) = \frac{1}{\mathfrak{T}},$$

and

$$X = - \left(\frac{dV}{dx}\right) = - \frac{1}{2\mathfrak{T}^2} \frac{d\mathfrak{T}}{dx} \quad (8),$$

with similar expressions for Y and Z .

Also, by (1), we have

$$\left. \begin{aligned} \frac{d\tau}{dx} &= \mathfrak{T} \frac{dx}{dt}, \text{ \&c.} \\ \frac{d\tau}{dH} &= - \int \mathfrak{T} dt \end{aligned} \right\} \quad (9),$$

and

Hence

$$\begin{aligned} \frac{d^2x}{dt^2} &= \frac{d}{dt} \left(\frac{1}{\mathfrak{T}} \frac{d\tau}{dx} \right) \\ &= \frac{1}{\mathfrak{T}} \frac{d}{dt} \left(\frac{d\tau}{dx} \right) - \frac{1}{\mathfrak{T}^2} \frac{d\tau}{dx} \frac{d\mathfrak{T}}{dt} \quad (10). \end{aligned}$$

are the rectangular components of the component of the impressed force perpendicular to the path.

But, if R be the force of constraint, λ, μ, ν , its direction-cosines, we have by ordinary kinetics

$$\frac{d^2x}{dt^2} = X - R\lambda, \quad \&c.$$

Hence
$$R\lambda = 2 \left(X - \frac{dx}{ds} \left(X \frac{dx}{ds} + Y \frac{dy}{ds} + Z \frac{dz}{ds} \right) \right), \quad \&c., \quad \&c.,$$

and therefore the whole pressure is *double* that due to the impressed forces.

From the above follows also the well-known theorem, that *the osculating plane of the brachistochrone contains, at each point, the resultant of the impressed forces.* For it has been shown that this resultant coincides in direction with the centrifugal force, and the latter of course lies in the osculating plane.

14. Another, and perhaps simpler proof of the theorem above is furnished directly by (10). Thus, squaring and adding the three equations of that form, after substituting in them from (11), we have

$$\begin{aligned} \left(\frac{d^2x}{dt^2} \right)^2 + \left(\frac{d^2y}{dt^2} \right)^2 + \left(\frac{d^2z}{dt^2} \right)^2 &= \frac{1}{4\mathfrak{T}^4} \left\{ \left(\frac{d\mathfrak{T}}{dx} \right)^2 + \left(\frac{d\mathfrak{T}}{dy} \right)^2 + \left(\frac{d\mathfrak{T}}{dz} \right)^2 \right\} \\ &\quad - \frac{1}{\mathfrak{T}^4} \frac{d\mathfrak{T}}{dt} \left\{ \frac{d\tau}{dx} \frac{d\mathfrak{T}}{dx} + \frac{d\tau}{dy} \frac{d\mathfrak{T}}{dy} + \frac{d\tau}{dz} \frac{d\mathfrak{T}}{dz} \right\} \\ &\quad + \frac{1}{\mathfrak{T}^4} \left(\frac{d\mathfrak{T}}{dt} \right)^2 \left\{ \left(\frac{d\tau}{dx} \right)^2 + \left(\frac{d\tau}{dy} \right)^2 + \left(\frac{d\tau}{dz} \right)^2 \right\} \\ &= \frac{1}{4\mathfrak{T}^4} \left\{ \left(\frac{d\mathfrak{T}}{dx} \right)^2 + \left(\frac{d\mathfrak{T}}{dy} \right)^2 + \left(\frac{d\mathfrak{T}}{dz} \right)^2 \right\} - \frac{1}{\mathfrak{T}^4} \frac{d\mathfrak{T}}{dt} \left(\mathfrak{T} \frac{d\mathfrak{T}}{dt} \right) + \frac{1}{\mathfrak{T}^4} \left(\frac{d\mathfrak{T}}{dt} \right)^2 (\mathfrak{T}) \end{aligned}$$

[by (12) and (7)]

$$= \frac{1}{4\mathfrak{T}^4} \left\{ \left(\frac{d\mathfrak{T}}{dx} \right)^2 + \left(\frac{d\mathfrak{T}}{dy} \right)^2 + \left(\frac{d\mathfrak{T}}{dz} \right)^2 \right\} = X^2 + Y^2 + Z^2, \quad \text{by (8).}$$

Hence *the whole acceleration is equal to the resultant of the impressed forces*; and therefore the component of the acceleration, normal to the curve, must be equal to that of the resultant of the impressed forces; from which the theorem follows at once if we can show independently that the resultant of the impressed forces lies in the osculating plane. This is easily done as follows. We have

$$\delta x = \frac{\delta t}{\mathfrak{T}} \frac{d\tau}{dx}, \quad \&c., \quad \text{by (9).}$$

Hence
$$\delta^2 x = \frac{\delta t}{\mathfrak{T}} \delta \left(\frac{d\tau}{dx} \right) - \frac{\delta x}{\mathfrak{T}} \delta \mathfrak{T}, \quad \&c.$$

Now, by (8) and (11), $\delta\left(\frac{d\tau}{dx}\right)$ &c., are proportional to the direction-cosines of the resultant force, which therefore lies in the common plane of two consecutive elements of the curve.

15. The equation of the surfaces which are orthogonal to the path is

$$\tau = C;$$

and that of equipotential surfaces

$$V = C_1.$$

That these may coincide we must have

$$\tau = \phi(V),$$

where ϕ is any function whatever.

Hence

$$\left\{\phi'(V)\right\}^2 \left\{\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right\} = \frac{1}{2(H-V)}.$$

If we write

$$V = \int \sqrt{2(H-V)} \phi'(V) dV = \psi(V), \quad (15).$$

this becomes

$$\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 = 1, \quad (16).$$

A complete primitive of this equation is, of course,

$$V = lx + my + nz - p,$$

where p is any function of l, m, n , and

$$l^2 + m^2 + n^2 = 1.$$

The general primitive, equated to a constant, is therefore obviously the equation of a series of surfaces such that the normal distance between any two consecutive members of the series is everywhere the same. It is evident from (15) that the surfaces thus found are identical with the isochronous and equipotential surfaces, when these coincide. The equations of their orthogonal trajectory, that is, of the free path which is also a brachistochrone, are therefore,

$$\frac{\delta x}{\left(\frac{dV}{dx}\right)} = \frac{\delta y}{\left(\frac{dV}{dy}\right)} = \frac{\delta z}{\left(\frac{dV}{dz}\right)} = \frac{\left(\frac{dV}{dx}\right)\delta x + \left(\frac{dV}{dy}\right)\delta y + \left(\frac{dV}{dz}\right)\delta z}{\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2} = \delta V = \delta C, \quad (17).$$

Hence,

$$\delta x = \delta C \left(\frac{dV}{dx}\right), \text{ \&c.,}$$

and, therefore,

$$\delta^2 x = \delta C \left\{ \left(\frac{d^2 V}{dx^2} \right) \delta x + \left(\frac{d^2 V}{dx dy} \right) \delta y + \left(\frac{d^2 V}{dx dz} \right) \delta z \right\} + \delta^2 C \left(\frac{dV}{dx} \right).$$

But, substituting the values of δx , &c., from (17), this becomes

$$\delta^2 x = (\delta C)^2 \left\{ \left(\frac{dV}{dx} \right) \left(\frac{d^2 V}{dx^2} \right) + \left(\frac{dV}{dy} \right) \left(\frac{d^2 V}{dx dy} \right) + \left(\frac{dV}{dz} \right) \left(\frac{d^2 V}{dx dz} \right) \right\} + \delta^2 C \left(\frac{dV}{dx} \right),$$

and the first part vanishes, by (16).

Hence

$$\frac{\delta^2 x}{\delta x} = \frac{\delta^2 y}{\delta y} = \frac{\delta^2 z}{\delta z} = \frac{\delta^2 C}{\delta C},$$

which show that when the path is simultaneously a free path and a brachistochrone, it is necessarily rectilinear.

This might have been inferred at once, from the theorem of § 13, which shows that if the free path be a brachistochrone, there can be no pressure due to the motion, *i.e.*, no curvature. But the above investigation is given as containing curious additional information. It shows, for instance, that if the force be the same at all points of each of a series of equipotential surfaces, the lines of force are rectilinear. Also, that if the flux of heat be constant per unit of area over each one of a series of isothermal surfaces, though not necessarily the same for all, the propagation of heat takes place in straight lines. And, as particular cases of these theorems, if the force or the flux of heat be the same throughout a given space, the attraction, or the flux, therein takes place in parallel lines.

16. HAMILTON'S equation for the determination of the Characteristic Function (A) in the case of the free motion of a single particle is

$$\left(\frac{dA}{dx} \right)^2 + \left(\frac{dA}{dy} \right)^2 + \left(\frac{dA}{dz} \right)^2 = 2(H - V) \quad (18).$$

The comparison of this with (2) suggests a useful transformation. Introducing in that equation a factor θ^2 , an undetermined function of x, y, z , we have

$$\left(\theta \frac{d\tau}{dx} \right)^2 + \left(\theta \frac{d\tau}{dy} \right)^2 + \left(\theta \frac{d\tau}{dz} \right)^2 = \frac{\theta^2}{2(H - V)} \quad (19).$$

If we make

$$\theta = \phi'(\tau) \quad (20),$$

and

$$\frac{\theta^2}{2(H - V)} = 2(H_1 - V_1) \quad (21),$$

(19) becomes

$$\left(\frac{d\phi(\tau)}{dx} \right)^2 + \left(\frac{d\phi(\tau)}{dy} \right)^2 + \left(\frac{d\phi(\tau)}{dz} \right)^2 = 2(H_1 - V_1) \quad (22).$$

Here it is obvious, by (18), that $\phi(\tau)$ is the action in a *free* path coinciding with the brachistochrone, and that $2(H_1 - V_1)$ is the square of the velocity in this path.

Hence the curious result that, *if τ be the time through any arc of a given brachistochrone, the same path will be described freely under the action of forces whose potential is V_1 , where*

$$2(H_1 - V_1) = \frac{(\phi'(\tau))^2}{2(H - V)},$$

ϕ' being any function whatever ; and $\phi(\tau)$ representing the action in the free path.

17. The simplest supposition we can make is that $\phi'(\tau)$ is constant. In this case the velocity in the free path is inversely proportional to that in the brachistochrone at the same point ; and the action in the one is proportional to the time in the other. In fact, as Professor W. THOMSON has pointed out to me, in this case the investigation may be made with extreme simplicity, thus—

In the brachistochrone we have

$$\int \frac{ds}{v} \text{ a minimum.}$$

Putting $\nu = \frac{1}{v}$, and considering ν as the velocity in the same path due to another (easily determinable) potential ; we must have

$$\int \nu ds \text{ a minimum.}$$

This is the ordinary condition of *Least Action*, and belongs, therefore, to a free path.

Hence, since the cycloid is the brachistochrone for gravity, and since in it $v^2 = 2gy$, it will be a free path if $\nu^2 = \frac{1}{2gy}$, that is for a system of force where the potential is found from

$$H_1 - V_1 = \frac{1}{4gy}.$$

This gives

$$-\frac{dV_1}{dx} = 0, \quad -\frac{dV_1}{dy} = -\frac{1}{4gy^2}.$$

In other words, a cycloid may be described freely under the action of a force towards, and inversely as the square of the distance from, the base ; and the velocity at any point will be the reciprocal of that in the same cycloid when it is the common brachistochrone.

This result is easily verified by a direct process.

18. But we have, by § 16, an infinite number of other systems of forces under which this cycloid will be described freely.

For by § 8 we have, putting $a=0$, since the base is now the axis of x ,

$$\begin{aligned}\sqrt{2g}\tau &= \frac{x}{\sqrt{b}} + \int dy \sqrt{\frac{1}{y} - \frac{1}{b}} \\ &= \frac{x}{\sqrt{b}} - \sqrt{b} \cos^{-1} \sqrt{\frac{y}{b}} + \sqrt{\frac{y}{b}} \sqrt{b-y} + C.\end{aligned}$$

Hence, whatever be ϕ' , the cycloid is a free path for the system

$$v^2 = 2(H_1 - V_1) = \frac{\left\{ \phi' \left(\frac{x}{\sqrt{b}} - \sqrt{b} \cos^{-1} \sqrt{\frac{y}{b}} + \sqrt{\frac{y}{b}} \sqrt{b-y} + C \right) \right\}^2}{2gy}.$$

19. The converse of the proposition in § 16 is also curious. Taking HAMILTON'S equation (18), we have,

$$(\phi'(A))^2 \left\{ \left(\frac{dA}{dx} \right)^2 + \left(\frac{dA}{dy} \right)^2 + \left(\frac{dA}{dz} \right)^2 \right\} = 2(H - V)(\phi'(A))^2 \quad . \quad . \quad (23).$$

Comparing this with (2), we see that $\tau = \phi(A)$ is the brachistochronic expression for the time in a path which is a free path for potential V . The requisite potential is now found from

$$\frac{1}{2(H_1 - V_1)} = 2(H - V)(\phi'(A))^2 \quad . \quad . \quad . \quad . \quad (24).$$

Hence, if A be the action in a given free path, the same path will be a brachistochrone for forces whose potential is V_1 , determined by (24), V being the potential in the free path.

Thus, the parabola

$$(x - \alpha)^2 = 4\alpha(y - \alpha)$$

is the free path for $v^2 = 2gy$. And the action is given by

$$\frac{1}{\sqrt{2g}} A = x\sqrt{\alpha} + \frac{2}{3}(y - \alpha)^{\frac{3}{2}}.$$

Hence this parabola is the brachistochrone for

$$2(H_1 - V_1) = \frac{1}{2gy(\phi'(A))^2}.$$

In the simplest case $\phi'(A) = 1$, and we have

$$-\frac{dV_1}{dx} = 0, \quad -\frac{dV_1}{dy} = -\frac{1}{4gy^2}.$$

Hence, by § 17, the parabola is a brachistochrone when a cycloid is the free path.

20. Again, if

$$v^2 = 2 \left(\frac{\mu}{r} - H \right), \quad . \quad . \quad . \quad . \quad (25).$$

where H and μ are essentially positive, the free path is an ellipse of which the origin (the centre of force) is a focus.

This ellipse is the brachistochrone for the potential V_1 , and whole energy H_1 , where

$$\frac{C}{2(H_1 - V_1)} = 2 \left(\frac{\mu}{r} - H \right),$$

or

$$V_1 = H_1 - \frac{Cr}{4(\mu - Hr)}.$$

This corresponds to a central force

$$\begin{aligned} -\frac{dV_1}{dr} &= \frac{C}{4(\mu - Hr)} + \frac{CHr}{4(\mu - Hr)^2} \\ &= \frac{C\mu}{4(\mu - Hr)^2}. \end{aligned}$$

The velocity at any point is

$$\sqrt{\frac{Cr}{2(\mu - Hr)}}.$$

In the ellipse, we know by ordinary kinetics that

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right).$$

Comparing this with the above formula (25) we have

$$\frac{\mu}{H} = 2a.$$

Hence the velocity in the free ellipse is

$$v = \sqrt{\frac{\mu}{a}} \sqrt{\frac{2a - r}{r}}. \quad (26).$$

That in the same ellipse, when it is a brachistochrone, is, as above,

$$v_1 = \sqrt{\frac{Cr}{2(\mu - Hr)}} = \sqrt{\frac{Ca}{\mu}} \sqrt{\frac{r}{2a - r}}.$$

But if we refer it to the other focus of the ellipse we have

$$r_1 = 2a - r.$$

Hence

$$v_1 = \sqrt{\frac{Ca}{\mu}} \sqrt{\frac{2a - r_1}{r_1}}. \quad (27).$$

Comparing (26) and (27), we have the singular result that *a planet moving freely about a centre of force in the focus of its elliptic orbit is describing a brachistochrone (for the same law of velocity as regards position) about the other focus.* The reason of this remarkable property, as well as of the connected one that

while the time in an elliptic orbit is (of course) measured by the area described about one focus, the action is measured by that described about the other,* is easily traced to the fact that the rectangle under the perpendiculars from the foci on any tangent is constant.

21. It follows from HAMILTON'S investigations, that in the free ellipse we have

$$A = \int \frac{2 \left(\frac{\mu}{r} - H \right) dr}{\sqrt{2 \left(\frac{\mu}{r} - H \right) - \frac{\alpha^2}{r^2}}},$$

where α depends upon the excentricity of the ellipse by the formula

$$\alpha^2 = \frac{\mu^2}{2H} (1 - e^2).$$

The theorem may therefore be generalized as follows:—The free ellipse will be a brachistochrone, if the velocity be given by

$$v^2 = 2(H_1 - V_1) = \frac{1}{2 \left(\frac{\mu}{r} - H \right) \left\{ \phi'(A) \right\}^2},$$

where ϕ' is any function, and A is the integral last written. By differentiation with respect to r , we get the law of central force requisite.

But results of this nature may be deduced to any desired extent, without more trouble than the requisite integrations involve.

22. The examples immediately preceding are but particular cases of the following general theorem, which is easily seen to be involved in the results of §§ 16, 19. *If we have two curves, P and Q , of which P is a free path, and Q a brachistochrone, for a given conservative system of forces; P will be a brachistochrone for a system of forces for which Q is a free path—and the action and time in any arc of either, when it is described freely, are functions of the time and action respectively, in the same arc, when it is a brachistochrone.*

23. It is easy to see, that there exists a very singular analogy between the processes we have just given, and those suggested by certain problems in optics.

Assuming, for an instant, the exploded corpuscular theory of Light, Varying Action is at once applicable to the determination of the path of a corpuscle. On the other hand, if we assume, as our fundamental hypothesis, that light takes the least possible time to pass from one point of its path to another, the foregoing investigations would be directly applicable to find the path in a medium whose refractive index (on which the velocity depends), at any point, is a given function of the co-ordinates; in other words, in a heterogeneous singly refracting medium.

In the beautiful investigations of HAMILTON, on the *Theory of Systems of Rays*

* Proc. R.S.E. March 1865, or TAIT and STEELE'S Dynamics of a Particle (2d edition) § 258.

(Trans. R.I.A., 1824-32), the path of a ray is assumed to be a straight line in any one medium. Here the velocity depends only upon the *direction* of the ray, as in homogeneous doubly refracting media, and the problem has no analogy with the conservative case which is treated above.

24. As an instance of an optical problem I take the following, due I believe to MAXWELL.* *If the refractive index of a medium be such a function of the distance from a given point that the path of any one ray is a circle, the path of every other ray is a circle; and all rays diverging from any one point converge accurately in another.* Or, in another form, find the relation between the velocity and the distance from the centre of force that the brachistochrone may always be a circle.

The symmetry shows that our investigations need only involve two dimensions. Taking the centre of force as pole, the equation of a circle is

$$r^2 - 2ar \cos(\theta - \mathfrak{A}) = \rho^2 - a^2 = b^2 \text{ suppose.}$$

Hence

$$\mathfrak{A} = \theta - \cos^{-1} \frac{b^2 - r^2}{2ar}.$$

This is obviously the equation before written (3) in the form

$$\frac{d\tau}{da} = \mathfrak{A}.$$

Hence

$$\tau = a\theta - \int da \cos^{-1} \frac{b^2 - r^2}{2ar}.$$

But, if v be the velocity (the reciprocal of the refractive index in the optical problem),

$$\left(\frac{d\tau}{dr}\right)^2 + \frac{1}{r^2} \left(\frac{d\tau}{d\theta}\right)^2 = \frac{1}{v^2}.$$

Hence

$$\frac{d\tau}{dr} = \sqrt{\frac{1}{v^2} - \frac{a^2}{r^2}} = -\frac{d}{dr} \int da \cos^{-1} \frac{b^2 - r^2}{2ar} = -\int da \frac{b^2 + r^2}{r \sqrt{(4a^2 r^2 - (b^2 - r^2)^2)}}.$$

But v is not a function of a , so that we get by differentiation with respect to that quantity

$$\frac{\frac{a}{r^2}}{\sqrt{\frac{1}{v^2} - \frac{a^2}{r^2}}} = \frac{b^2 + r^2}{r \sqrt{(4a^2 r^2 - (b^2 - r^2)^2)}}.$$

This is easily reduced to

$$v^2 a^2 = \frac{(b^2 + r^2)^2}{4(a^2 + b^2)} = \frac{(b^2 + r^2)^2}{4\rho^2}.$$

The condition, that v is a function of r and absolute constants only, thus leads

* Cambridge and Dublin Math. Journal, IX., p. 9.

at once to two conclusions: b is an absolute constant; and so is $2\rho a$, for which we may write c . a is therefore inversely as the diameter of the circle; and

$$v = \frac{b^2 + r^2}{c}.$$

From the form of the equation of the path it is obvious that $-b^2$ is the rectangle under the segments of any chord drawn through the centre of force.

Hence, in the optical problem, if a ray leave, in any direction, a point distant r from the origin, it will pass through another point in the prolongation of r , distant $\frac{b^2}{r}$ from the origin; and, in the kinetic problem, there is an infinite number of brachistochrones (circles all, and the time being the same for all) when two points thus related are taken as the initial and final points.

25. Such examples might be multiplied indefinitely. For instance, if the refractive index of a medium be inversely proportional to the square root of the distance from a given point, the path is a parabola about the point as focus; that every ray may be a cardioid whose cusp is at the point, the square of the refractive index must be inversely as the cube of the distance: and so on.

26. The processes of § 4 may of course be applied to innumerable problems besides the determination of the form and properties of brachistochrones, but I shall content myself with an example or two. Thus, if we take

$$\Phi = \int f(v) ds$$

as the characteristic function, we have

$$\frac{d\Phi}{dx} = \frac{f(v)}{v} \frac{dx}{dt}, \text{ \&c., and } \frac{d\Phi}{dH} = \int f'(v) dt.$$

Of this, besides the cases $f(v)=v$, and $f(v)=\frac{1}{v}$, which we have already considered, the most curious is that where

$$f(v) = \frac{v^2}{2};$$

that is, when *the space average of the kinetic energy is a minimum*. In this case.

$$\left(\frac{d\Phi}{dx}\right)^2 + \left(\frac{d\Phi}{dy}\right)^2 + \left(\frac{d\Phi}{dz}\right)^2 = \frac{v^4}{4} = (H - V)^2,$$

and

$$\frac{d\Phi}{dH} = s.$$

Again, if we take

$$\Phi = \int F(x, y, z) f(v) ds$$

$$\frac{d\Phi}{dx} = \frac{Ff}{v} \frac{dx}{dt}, \text{ \&c., and } \frac{d\Phi}{dH} = \int F f'(v) dt.$$

Hence, if
$$F(x, y, z) = \frac{\text{Constant}}{f'(v)},$$

we have
$$\frac{d\Phi}{dH} = Ct,$$

so that there is an infinite number of values of the characteristic function, besides that of HAMILTON, which give the time through any arc of the orbit by their differential coefficients with respect to H .

27. Enough of this; I conclude with the remark that various investigations in Statics supply us with excellent examples in our subject.* Take the common catenary, for instance, its equation is found by the conditions

$$\int y ds = \text{minimum}, \quad \text{and} \quad \int ds = \text{constant},$$

the axis of y being directed vertically upwards.

This gives

$$\delta \int (y + a) ds = 0.$$

Hence the catenary is the free path of a particle whose velocity is given by

$$v = C(y + a);$$

that is, if the force be in the direction of, and proportional to, the ordinate, and repulsive from the axis of x . In the same way we see that the catenary is the brachistochrone if the velocity be inversely as the distance from the axis; that is, if the force be attractive, and inversely as the cube of the distance from the axis.

* Compare THOMSON and TAIT's Natural Philosophy, §§ 581, 582.