A THEOREM IN THE THEORY OF FONCTIONS OF A REAL VARIABLE.

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§ 1.

Among the many interesting results in DINI's Functions of a Real Variable not the least remarkable is the theorem [1]) that a function defined for an interval, and itself the differential coefficient of another function, possesses one of the most familiar properties of a continuous function, viz. that it assumes in every interval every value *between* its upper and lower limits in that interval.

BAIRE has shewn [2]) that the limit of a continuous function necessarily has points of continuity everywhere dense, and furthermore that the points of continuity with respect to each perfect set are everywhere dense in that set, in other words, the function so defined is pointwise discontinuous with respect to every perfect set. The particular limit of a continuous function constituted by a differential coefficient preserves therefore to a striking degree properties of the function of which it is a limit. The question naturally arises. Has the limit of a continuous function in the general case the first named of the properties in question? It is easy to construct examples shewing that this is not so [3]). We are led further to ask. What additional condition must hold in order that a function which is the limit of a continuous function may in every interval assume every value between its upper and lower limit in that interval?

This question is dealt with in the following note. It is found that *the necessary and sufficient condition* [4]) *that a function which is pointwise discontinuous with respect to every perfect set in the segment in which it is defined, should assume in every interval*

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[1]) See §§ 141, 147, 171, 172.

[2]) *Sur les fonctions de variables réelles* [Annali di Matematica pura ed applicata, serie III, tomo III (1899), pp. 1-123], §§ 22-30; other references in HOBSON's Functions of a Real Variable (1907), p. 525.

[3]) For instance the function which is zero at every irrational point of the segment (0, 1), and is $\frac{1}{q}$ at any rational point $\frac{p}{q}$ ($p < q$ and prime to it).

[4]) This condition may be compared with the well known *sufficient* condition that a function may assume its upper (lower) limiting value, viz. that at every point its value $\geqq$ ($\leqq$) every limit at the point.

every value between its upper and lower limits in that interval, is that the value of the function at each point is one of the limits of values in the neighbourhood of the point on the right and also one of the limits of values in the neighbourhood of the point on the left.

We thus get incidentally a new proof of DINI's theorem on quite different lines from those in his classical treatise.

§ 2.

THEOREM — *If $f(x)$ is a function which is defined for every point of an interval and has the following two properties.*

A) the value $f(x)$ at any point P is one of the limits of values in the neighbourhood of P on the right and also on the left [5]:

B) the function is pointwise discontinuous with respect to every perfect set (or, which is the same, the function is the limit of a continuous function); then

C) $f(x)$ assumes every value k between its upper and lower limits.

Let S_1 be the set of points for which $f > k$, and S_2 the set for which $f < k$; then we shall shew first that (omitting the trivial case in which the upper lower limits coincide) *both S_1 and S_2 are dense in themselves on both sides.*

For, by the definition of upper and lower limit, there will certainly be at least one point in each set. Let P_1 be a point of S_1, and let the value of f there be $k + e$; then, by A), there is on each side of P a sequence of points having P as limit, such that, passing along the sequence, f has the limit $k + e$, so that after a certain stage every point of the sequence must belong to S_1. Since this is true on both sides of P, P is a limit on both sides for the set S_1, in other words S_1 (and similarly S_2) is dense in itself on both sides.

If one of the sets, say S_1, is not dense everywhere, there is a definite set of non-abutting intervals free of points of S_1 (viz. the « black intervals » of the perfect set which is the derived of S_1). Not only do the internal points of these intervals not belong to S_1, this is true of the end-points also, since these could not be limiting points on both sides for the set S_1.

Similarly, if S_2 is not dense everywhere, we get a second set of non-abutting black intervals, no point of which, including the end-points, belongs to S_2.

1) It may be that both these sets of intervals exist, and that one of the first set abuts with one of the second set. In this case the common end-point belongs to neither S_1 nor S_2, so that the value of the function there is k.

2) If this is not the case, either neither set of intervals exists, or else all the intervals so determined from both S_1 and S_2 form a set of non-abutting black intervals, so

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5) At the end-points of the fundamental segment this is to be taken throughout to apply exceptionally to one side only.

that the points not internal to such intervals form a perfect set U [6]), which, if both the sets S_1 and S_2 are dense everywhere, is the continuum itself.

Let P be any point of this perfect set U, and d any interval containing P as internal point, so that d does not lie entirely inside one of the black intervals of U, and therefore all the points of d do not belong to one only of the sets S_1 and S_2. Then we can make the same statement of all the points of U in the interval d. For suppose, if possible, that all the points of U in d belonged to S_1. Then none of the end-points, and therefore none of the internal points, of any black intervals lying wholly, or in part, inside d, would belong to S_2; therefore no point of d would belong to S_2, which, since d is not part of a black interval, is not the case. Similarly all the points of U in d are not points of S_2. Thus inside d there is either a point belonging to neither S_1 nor S_2, and this would be a point at which $f = k$, or else there are inside d points of U which belong to S_1 and points of U which belong to S_2.

In this latter case it follows that P is a limiting point of points of U where the value of f is $> k$, and of points of U where the value of f is $< k$, so that, if P is a point of continuity with respect to the perfect set U, the value of f at P must be precisely k.

Now by B) there is such a point of continuity in the perfect set U, thus it follows in this case also that the function actually assumes the value k, which proves the theorem.

COROLLARY.—*If f is a function satisfying the condition A), but, in some particular interval, not assuming every value between its upper and lower limits, then there is a perfect set at every point of which the function is discontinuous with respect to that perfect set.*

The examples given below shew that the case contemplated in the Corollary is a possible one, and that the function may be either totally or only pointwise discontinuous.

§ 3.

Condition A) has been proved *sufficient* in the case of a function which is the limit of a continuous function in order that C) may hold, that is in order that in every interval the function may assume all values between its upper and lower limits in that interval. *That it is a necessary condition not only for such a function but for any function is easily seen.*

In fact if P be any point at which the value of the function is c, and we take with P as left-hand end-point any interval, as small as we please, the function assumes, by hypothesis, inside this interval one at least of the values $c + e$, $c - e$, where e is as small as we please. Taking as new interval the left-hand half of the first interval, the function assumes in like manner one at least of the values $c + \frac{1}{2} e, c - \frac{1}{2} e$, inside

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[6]) An end-point of the fundamental segment is to be considered as belonging to U if, and only if, it is not an end-point of a black interval.

this interval; whence, by repeated application of this process, we see that the value c at P is a limit of values in the neighbourhood of P on the right, and similarly, of course, on the left.

Here we have tacitly assumed that the point P is not internal to an interval in which the function is constant, in which case no discussion is required.

Thus we see finally that *the necessary and sufficient condition that a function which is pointwise discontinuous with respect to every perfect set should in every interval assume every value between its upper and lower limit in that interval, is that A) should hold.*

§ 4.

The following examples shew that condition A), though necessary is not sufficient to insure C) in the case of a discontinuous function which is not the limit of a continuous function.

EXAMPLE I. — Let the rational numbers between 0 and 1, both excluded, be arranged in countable order, and be $R_1, R_2, R_3, \ldots$.

Consider a function which has at every rational point $\frac{p}{q}$ with denominator $q=2^n$ the value R_n, for all integers n. The values at the remaining points (that is the irrational points and those rational points $\frac{p}{q}$ whose denominators are not powers of 2) may be any values we please between 0 and 1.

Then if x be any point in the segment (0, 1), and we describe with x as end-point any interval, inside this interval there will be rational points corresponding to denominators 2^n for all but a finite number of values of n. Therefore inside this interval the function assumes all but a finite number of values of the series $R_1, R_2, R_3, \ldots$. Therefore if y be any value, rational or irrational, between 0 and 1, both included, the function has the limit y at some point inside the interval. Since this is true however large m, and therefore however small the interval may be, it follows that one of the limits which we approach in moving towards x from whichever side we please is y; in other words, whether we approach from the right or the left, the limits which we obtain for the values of the function in the neighbourhood of any point x are all the numbers between 0 and 1 both included.

By suitably assigning the values at the remaining points, we can insure that any irrational values we please are not assumed by the function thus we see that condition A) alone does not insure the result C).

All these functions are totally discontinuous. The following special cases of the type may be noticed.

1) Let the value at the remaining points be 1. No irrational value is assumed by the function, but the function assumes its upper limit in every interval.

2) Let the value at the remaining points be $\frac{1}{2}$. The same is true as in 1), except that the upper limit 1 is not assumed as a value.

3) At each of the remaining points x let the function be x. In the whole segment (0, 1) every value between the upper and lower limits is assumed, but this is not true of any partial segment.

EXAMPLE II. — The preceding example shews that condition A) is, in the case of a totally discontinuous function, insufficient to secure C). Using the well-known correspondence between the continuum and a perfect set nowhere dense, we can deduce that the same is true when the function is pointwise discontinuous.

Taking, for instance, the typical ternary set of zero content, whose black intervals are got by dividing the segment (0, 1) into three equal parts and blackening the middle segment, and repeating this process in each white segment *and infinitum,* the correspondence is simply got by making both end-points of these black intervals in order correspond to the rational points whose denominators are powers of 2, obtained in order in like manner by binary instead of ternary division.

To be more precise, expressing the binary numbers in the scale of 2, the number

$$0.e_1 e_2 e_3 \dots e_n 1$$

(where each e is either 0 or 1), corresponds to the two numbers got by multiplying each e by 2, and replacing the terminal 1 by 2 and by $0\dot{2}$ respectively, these numbers being interpreted in the scale of 3.

Each point corresponds to the limit of the points corresponding to rational points in its neighbourhood, so that the correspondent of the non-terminating binary number

$$0.e_1 e_2 e_3 \dots$$

not ending in $\dot{1}$, is got by multiplying by 2 and interpreting in the scale of 3.

If now y be any of these ternary numbers and x the corresponding binary number, we take

$$F(y) = f(x)$$

where $f(x)$ is one of the functions constructed in the preceding example.

At any points of a black interval of the ternary perfect set however, let F have the same value as at the end-points of that interval.

This function F is continuous except at every point of the ternary perfect set, at each point of which it is discontinuous with oscillation unity. Since the ternary perfect set is dense nowhere, this function is a pointwise discontinuous function, and it satisfies condition A); only inside the black intervals of the perfect set, however, does C) hold true.

§ 5.

As above remarked the theorem of § 2 is a generalisation of the well-known property of a continuous function, viz. that it actually assumes in every interval every value between its upper and lower limits, in this case *included.*

It is perhaps worth while giving the proof of this property on the lines of that in § 2.

The points at which $f(x)$ is greater than the value k in question are easily seen to be infinite in number; they also form a set dense in itself on both sides [7]) and therefore cannot form a closed set, since they certainly do not fill up the whole segment. It follows that all the non-included limiting points of this set are points at which $f = k$. For, if L be one of these points, then, by hypothesis, $f(L)$ if not equal to k, must be less than k, say $f(L) = k - e$. But in every interval surrounding L there are points at which $f > k$, therefore there is a discontinuity at L with oscillation equal to e or greater, which is not true. Therefore $f(L) = k$. Q. E. D.

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7) This set is of course formed by the internal points of an interval or a set of intervals.