



LII. The electrical resistance of thin metallic films, and a theory of the mechanism of conduction in such films

W.F.G. Swann D.Sc. A.R.C.S.

To cite this article: W.F.G. Swann D.Sc. A.R.C.S. (1914) LII. The electrical resistance of thin metallic films, and a theory of the mechanism of conduction in such films , Philosophical Magazine Series 6, 28:166, 467-496, DOI: [10.1080/14786441008635230](https://doi.org/10.1080/14786441008635230)

To link to this article: <http://dx.doi.org/10.1080/14786441008635230>



Published online: 08 Apr 2009.



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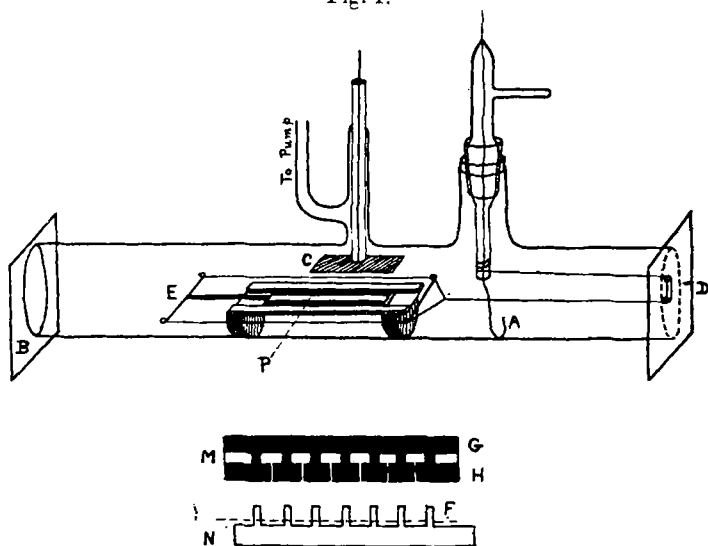


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θ^{-2} , on cooling the film down to liquid-air temperature (-180°C.) we should have $\lambda_{-180}/\lambda_0 = (273/93)^2 = 8.58$, so that the distance of the bend from the origin of thickness should be, at liquid-air temperature, 8.58 times its value at 0°C.

It was found that no such displacement of the bend took place, and apart from certain relatively small variations of the resistance of the films with temperature, which will be discussed later, the curves at the two temperatures were practically coincident, at any rate in the neighbourhood of the bend. The experiments are not taken as necessarily proving that λ does not vary as θ^{-2} , but rather as showing that the explanation of the sharp bend is to be found from another standpoint than that which explains it as due to the thickness of the film becoming comparable with the mean free path. It will be more convenient to discuss this point after the apparatus and experiments have been described.

Fig. 1.



Apparatus and Experiments.—The films were deposited by sputtering from a platinum cathode in vacuo, and all the films corresponding to a single set were deposited in the same vacuum. In this way a greater uniformity of the rate of deposition can be obtained than in the case where the apparatus is taken down after the deposition of each film, as in the experiments of Patterson. The apparatus is represented in fig. 1.

The cathode C, which consisted of a piece of platinum foil

2.5 cm. square, was fixed in a wide glass tube 4.5 cm. in diameter, which could be exhausted by means of an automatic pump. The top of the cathode was covered with mica. The anode A passed down through the centre of a tap (the purpose of which will presently be described) and was fixed permanently in position in the wide tube. The ends of the apparatus could be closed with glass plates B and D cemented on with sealing-wax.

A plate 1.5 cm. wide and 3.5 cm. long was taken, and a piece of tinfoil 2.5 mm. wide having been placed along the centre line parallel to the length of the plate, the whole of the remainder was covered with a thick deposit of platinum. On removing the tinfoil a clear space was left across which the films to be examined were afterwards deposited in different positions. To this end the plate was arranged to slide parallel to the axis of the tube, in a frame provided with two glass guides. One end of the plate was attached to a thread which passed through two eyes cemented to the plate D. The other end of the thread was attached to the tap, so that by turning the latter the plate could be slid along under the cathode. A piece of mica was fixed to the framework so as to completely cover the plate without touching it, and the films were deposited through a slit cut in this mica perpendicular to the axis of the plate, and extending right across the thick platinum strips.

About 10 films of different thicknesses were deposited. The plate was then removed, and after slits had been cut in the platinized strip on one side of the plate, so as to separate the films, the latter presented the appearance shown in fig. 1, M. As a precaution against accidental short-circuiting along the edge of the plate, the whole of the platinum was cleared away to a distance of 1 mm. from the edge. In order to allow of connexion being made to the various films, a set of tongues corresponding to the separate sections of the strip was cut in a piece of copper foil (see N, fig. 1). These were gripped firmly in position by means of a spring clip provided with cork-lined jaws to ensure a tight contact, and after snipping the foil along the position indicated by the dotted line, wires were soldered to each of the tongues. A piece of foil with a wire attached was also gripped to the strip G. The whole arrangement was put in a glass tube 3 cm. in diameter and plugged in with cotton wool, the tube being afterwards corked at the top while the wires passed out through the cork.

The films were deposited by the discharge from a 4-inch coil, the interrupter being adjusted to run very steadily.

By turning the tap the plate could be slid along the tube, and films could be deposited through the mica slit in various positions on the plate. The discharge was allowed to pass for about 1 hour before the deposition of the first film, and before the deposition of each successive film the vacuum was adjusted to the same value, as indicated by the appearance of a subsidiary discharge tube. In the cases of the longer times of deposit, the deposition was carried on in several stages, the vacuum being adjusted before each stage. After the films were made they were placed in a tube and heated for about 20 hours to a temperature of 115°C . in order to anneal them.

The resistances of the films of low resistance were measured by a post-office box in the ordinary way. Those of high resistance were measured by tapping off a certain fraction of the e.m.f. of a cell from a potentiometer, and causing it to send a current through the film and through a galvanometer, the apparatus being afterwards standardized by means of a known high resistance.

The resistances of the films were measured at 100°C . (the tube containing them being in steam), at 14°C . in water, and at -180°C . in liquid air.

It was noticed, especially in the case of the thinnest films, that if a film had been allowed to rest for some time and its resistance was measured at room temperature (14°C . for example), then at 100°C ., and afterwards at 14°C ., there was a fairly large difference between the two measurements at 14°C . If, however, the film was again raised to 100°C . and then cooled to 14°C ., the last two measurements at 14°C . agreed very well. In fact, after a film has been raised to 100°C . its behaviour with regard to lower temperatures remains regular for a considerable time. This is well shown by the following set of numbers given in Table 1. The table gives the resistances in ohms of the films corresponding to the temperatures given in the top line, and to the times of the deposit given in the second column, the first column giving the number of the film for convenience in reference. All these films were of course made in the same vacuum on the same plate.

In estimating the concordance of these results, no weight must be placed on the results for film (1), since the galvanometer deflexion corresponding to this film of very high resistance amounted only to about 6 scale-divisions. It will be seen that after the first heating to $99^{\circ}\cdot 8\text{C}$. the observations become fairly concordant, and the mean results corresponding to the readings taken after the first heating to

TABLE I.—Films Set 1.

No. of Film.	Time, secs.	13°·5 C.	—180° C.	13°·4 C.	99°·8 C.	13°·8 C.	13°·4 C.	99°·9 C.	14°·5 C.	—180° C.	14°·4 C.
1	50	3170×10 ⁵	2760×10 ⁶	2950×10 ⁵	1450×10 ³	2110×10 ⁵	2190×10 ⁵	1570×10 ⁵	2380×10 ⁵	2120×10 ⁶	2360×10 ⁵
2	80	5887×10 ³	1840×10 ³	5894×10 ³	4179×10 ³	4459×10 ³	4594×10 ³	4186×10 ³	4628×10 ³	1380×10 ⁴	4484×10 ³
3	110	1398×10 ³	2519×10 ³	1402×10 ³	1094×10 ³	1047×10 ³	1082×10 ³	1095×10 ³	1098×10 ³	1904×10 ³	1066×10 ³
4	160	8574×10 ²	1249×10 ³	8465×10 ²	7119×10 ²	7119×10 ²	7276×10 ²	7177×10 ²	7308×10 ²	1032×10 ³	7065×10 ²
5	190	8877	8963	8874	8825	8764	8773	8818	8773	8862	8750
6	220	2469·5	2462	2469·3	2467	2450·5	2451·7	2465·5	2451·5	2444·7	2448·2
7	250	1235	1223·5	1234·5	1236·7	1229·7	1230·3	1237·7	1230	1217·5	1228·3

TABLE II.—Films Set 1.

Temperature.		—180° C.	14° C.	99°·8 C.	—180° C.	14° C.	99°·8 C.
No. of Film.	Time, secs.	Resistance of Films, Mean values.			10 ^{—2} ×Resistance×Time of deposit.		
		1	2	3	1	2	3
1	50	2120×10 ⁶	2260×10 ⁵	1500×10 ⁵	1060×10 ⁶	1130×10 ⁵	7500×10 ⁴
2	80	1380×10 ⁴	4541×10 ³	4182×10 ³	1104×10 ⁴	3632×10 ³	3345×10 ³
3	110	1904×10 ³	1073×10 ³	1094×10 ³	2094×10 ³	1180×10 ³	1204×10 ³
4	160	1032×10 ³	7167×10 ²	7148×10 ²	1651×10 ³	1147×10 ³	1144×10 ³
5	190	8862	8765	8821	1684×10	1665×10	1676×10
6	220	2444·7	2450·5	2466·3	5378	5391	5426
7	250	1217·5	1229·6	1237·2	3044	3074	3063

99°·8 C. are given in Table II., the last three columns giving relative values of the apparent specific resistance obtained by multiplying the resistances of the films by the corresponding times of deposit.

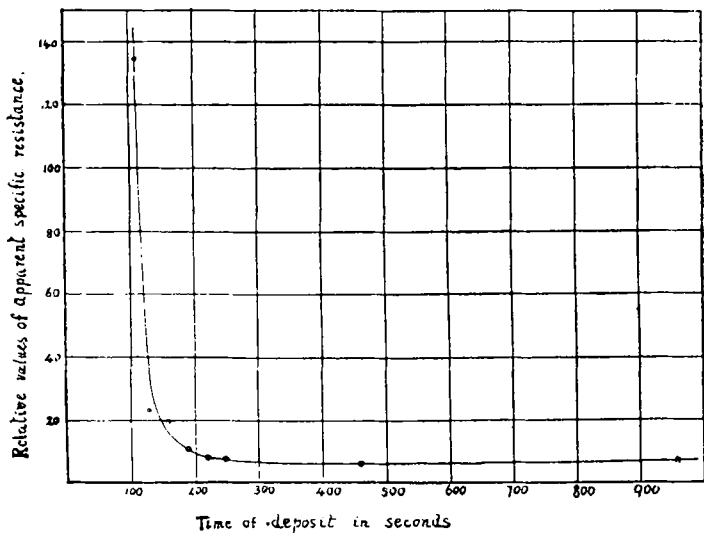
The results in Table II. present various interesting features which will be discussed later, since they refer to very thin films, practically all of them corresponding to films of a thickness below the critical value referred to on page 467.

The results for a set of thicker films are given in Table III., in which the first six columns labelled resistance represent the results of the actual observation, and the last three columns represent relative values of the specific resistance. In deducing the latter, the mean observations for each temperature are used.

Although the times of deposit are the same as in Table I. a greater rate of deposit was obtained by using a greater discharge current.

The curve corresponding to Table III. is shown in fig. 2.

Fig. 2.



The fact that the points do not lie perfectly on a smooth curve is not to be taken as an indication that the observations of the resistance are unreliable to the extent of the apparent error, since it is merely due to the fact that the time of deposit is not necessarily proportional to the amount deposited. For instance, a change of 7 per cent. in the rate of

TABLE III.—Films Set 2.

Temperature.		99° 8 C.	13° 4 C.	-180° C.	13° 4 C.	99° 8 C.	14° 8 C.	-180° C.	13° 9 C.	99° 8 C.
No. of Film.	Time of deposit.	Resistances of Films.						Resist. \times Time of deposit $\times 10^{-2}$. Mean values.		
		Not measured.	Not measured.	∞	∞	∞	∞	∞	∞	∞
1	50	Not measured.	Not measured.	1200×10^6	281×10^6	224×10^6	286×10^6	9600×10^6	2264×10^6	1792×10^6
2	80	Not measured.	Not measured.	28400	24400	24520	24465	31240	26860	26940
3	110	24460	24390	3483	3500	3526	3506	4528	4553	4580
4	130	3520	3499	2394	2408	2427	2414	3880	3856	3878
5	160	2422	2407	1129	1145	1158	1150	2145	2179	2200
6	190	1158	1147	687.3	698.5	706.4	700.5	1512	1589	1554
7	230	706.6	699.7	601.4	613.8	620.4	614.4	1503	1535	1551
8	250	620.3	613.8	262.1	268.7	272.2	268.8	1206	1236	1253
9	460	272.4	268.5	142.4	146.5	146.3	142.8	1367	1397	1413
10	960	148.1	147.2							

deposit during the 130 second period would account for the departure of the corresponding point shown from the smooth curve*, but such changes have no effect on conclusions which we may draw with regard to the variation of the resistance of any one film with temperature. For such purposes the observations may be looked upon as accurate to the extent of the agreement shown in columns 3 to 8 of Table III.

If the curves for -180°C. and $99^{\circ}\cdot 8\text{C.}$ are plotted, it turns out that instead of the distance of the bends from the origin of thickness being in the ratio of $8\cdot 58$ to unity as would be required by the simple theory, the two curves are practically coincident in this region, both with each other and with the curve for $13^{\circ}\cdot 9\text{C.}$ In fact, such temperature variations as the films show are, except in the case of very thin films, extremely small. Further, the rate of increase of the resistance at the critical thickness is enormously greater than would be indicated by the simple theory, which in virtue of the comparatively small variation of $\log \lambda/t$ with t , (equation 2), practically gives the curve relating to the specific resistance and t as an hyperbola. The fact that the experimental bend is sharper than the theoretical one has been noted by Patterson, and it has been suggested that it is due to a want of uniformity in the densities of the free electrons at the surface and inside, but while such a fact might be invoked to explain even a 100 per cent. difference, it is difficult to see how it can be the explanation of the enormous discrepancy found, a discrepancy amounting to thousands of per cent. on the steep part of the curve.

We can obtain an explanation of the extremely sharp bend if we assume that in the deposition of the film the molecules come down in groups, and the phenomenon with which we are concerned is not one of a film of a gradually increasing thickness, but in the early stages is somewhat more analogous to the condition of affairs on a pavement when rain-drops begin to fall, and before enough drops have fallen to cover the pavement. Taking the results of Patterson, who measured the thickness of some of his films, we find that the amount

* In the experiments which were first performed with a different apparatus, a film was deposited for, say, 50 seconds, and measured *in situ*. Another film was then deposited on the top, and the measurements were repeated, and so on. This method has the advantage of reducing considerably the disadvantages arising from fluctuations in the coil, but difficulties arose in carrying out the experiment in this way, which rendered the method impracticable.

which was deposited up to the thickness corresponding to the bend was sufficient to cover the whole plate to a depth of about 60 molecules. If, however, we imagine that the groups which come down contain, say, more than 10^6 molecules, the groups would not at this stage join up so as to form a continuous film. The molecules would be piled up to a greater height in each group than if they had come down singly, but this greater height in places is only obtained at the cost of the existence of gaps between the groups*. The fact that the resistance does not become absolutely infinite for a definite time of deposit would be explained by the fact that though all the groups would not be in contact at the critical time, certain of them would be, and the current would be able to find paths through the film, the number of such paths diminishing rapidly for smaller times of deposit.

Though the above theory would explain the sharp bend, a modification of it is more fruitful in explaining the peculiar behaviour which these films show with regard to change of temperature, and it is this modified view which forms the basis of the theory about to be developed. We shall assume that the molecules come down in groups, which are, however, not so large as those cited above. We shall imagine that they cover the plate for a time of deposit less than that corresponding to the critical value. We shall imagine, however, that they are not pressed into very intimate contact, and that it is only those electrons which have more than a certain minimum velocity, depending on the closeness of packing, which can travel from one group to the next. We shall suppose that this necessary velocity gets less as the time of deposit increases and the groups are pressed more intimately into contact by the greater molecular forces brought into play. The effect of the electric field has now to be considered as that of accelerating the electrons which are travelling against the field and retarding those travelling in the opposite direction, so that the number of electrons crossing the gaps against the field is greater than the number travelling in the opposite direction. The details of the analysis connected with the problem will be found in the appendix to this paper. It will be sufficient here to refer to the main steps.

* Since this paragraph was written a paper has come to my notice by M. Houlléviq (Comptes Rendus, t. 150. p. 1237), in which the author suggests that the molecules come down in groups. He assumes that the resistance remains infinite until the groups join up, and on this basis calculates the number of molecules in the groups.

If we fix our attention on the groups of molecules we can imagine each group as separated from its neighbours by a kind of equipotential surface (represented by the dotted lines in fig. 3), these surfaces being such that if an electron leaves one of the groups with sufficient velocity to reach a surface, it will escape to the next group. Since one of these surfaces must surround each group, it is obvious that they will be polygonal surfaces with approximately plane faces, and will be of such a nature as to fit together perfectly, nowhere enclosing a space unless that space contains one of the groups. They will divide up the whole region into a kind of honeycombed cell structure. We shall imagine that in the absence of the electric field, the electrons in these cells obey Maxwell's law, the number dn having a velocity between c and $c+dc$ being given by

$$dn = A e^{-\frac{1}{2} m c^2} c^2 dc, \quad \dots \quad (3)$$

where

$$A = \frac{4n(hm)^{3/2}}{\sqrt{\pi}} \quad h = \frac{3}{4\alpha\theta},$$

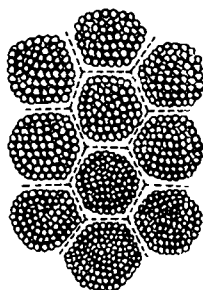
$\alpha\theta$ is the kinetic energy of a gas molecule at a temperature θ , n is the number of electrons per c.c., and m is the mass of an electron.

We shall suppose that it is only in the very small regions or gaps* separating the groups that the molecular forces come into play. These regions we shall suppose to be very narrow compared with the width of the group as a whole. We shall further suppose that those electrons whose velocity resolved perpendicular to the gap is greater than u , when the electron leaves the group in the absence of the field, will be able to get across.

The problem which we shall first consider is one where the metal is divided as it were into a large number of slabs of width l by gaps crossing the metal with their planes perpendicular to the direction of the field. We shall afterwards give reasons for assuming (see Appendix, page 493) that the result so obtained for the conductivity is practically true for the more complicated cell structure cited above.

* It is not necessary to imagine an actual separation between the groups. All that is necessary is that there shall be regions (typified by the above gaps), on crossing which the molecular forces change with considerable abruptness.

Fig. 3.



When the electric field is absent the number of electrons crossing one of the gaps in any given direction will of course be balanced by the number crossing in the opposite direction. When the field which we shall call X is on, however, the symmetry of the flow will be disturbed, and we shall obtain a resultant current. The current for a given field will be less than when the gaps are absent, and since the conductivity of the metal in the slabs separating the gaps has to remain the same, the result will be that a displacement of electricity will take place in the slabs, so that there will be a finite drop of potential V across each gap, of such an amount as to insure that the field $X - \frac{V}{l}$ in the slabs is such as will drive a current through them equal to that which crosses the gaps.

If we imagine the field $X - \frac{V}{l}$, which we shall call X_1 , to urge the electrons from left to right, it must be noted that owing to the potential drop across the gaps, the energy corresponding to the x velocity which an electron travelling from right to left must have by the time it reaches the gap, in order to cross, is less than $\frac{1}{2}mu^2$ by a quantity of the order of half the energy produced in the electron in a drop through a potential V . The actual value of x necessary is such that

$$\dot{x}^2 > u^2 + \frac{2\eta e}{m} - \frac{Ve}{m}$$

(see Appendix, Problem 1), where η is a function of V , and is a quantity such that $\eta e/mu^2$ is of the order $(Ve/mu^2)^2$ compared with unity. On the other hand, for an electron travelling in the opposite direction, the necessary value of \dot{x} is (see Appendix) such that

$$\dot{x}^2 > u^2 + \frac{2\eta e}{m} + \frac{Ve}{m}.$$

The reason for the existence of the term Ve/m in the above is obvious, and it will not be advisable here to break in upon the general train of reasoning in order to discuss the significance of the quantity η , which is of the nature of a correction, and is fully explained in the Appendix.

The current density across the gap is to be obtained by calculating the rate of flow of electrons per square centimetre across the gap from left to right, subtracting the total rate of flow from right to left, and multiplying the result by e .

The current density so obtained (see Appendix, Problem 2) is

$$i = \frac{Ae}{2(hm)^2} \left\{ \frac{\epsilon^{-a^2}}{2} \sinh(heV) + \frac{2}{3} X_1 he\lambda \left[(1+a^2) \epsilon^{-a^2} - 2a^3 I + \frac{3}{4} (heV)^2 \left\{ \epsilon^{-a^2} - \frac{I}{a} \right\} \right] \right\}, \quad \dots \quad (4)$$

where

$$a^2 = hmu^2 + 2he\eta, \quad I = \int_a^\infty \epsilon^{-\zeta^2} d\zeta,$$

and λ is the mean free path, and where X_1 is the same thing as $X - \frac{V}{l}$.

In the above equation quantities smaller than $(heV)^3$ compared with unity are neglected. The maximum value of V is Xl , so that these quantities are at most only of the order $(helX)^3$. For a field of 1 volt per cm. we have $heX = 14$, since $h = 1.4 \times 10^{13}$ at 0° C. and $e = 10^{-20}$. l is in general probably a quantity of the order 10^{-5} cm., so that these terms are quite negligible. We have not neglected terms of the order $(heV)^2$ because we wish at a later stage of the paper to discuss the question of the deviations from Ohm's law.

When there are no gaps, $a = 0$, $V = 0^*$, and consequently

$$\frac{Ae^2 h \lambda}{3(hm)^2}, \quad i. e. \quad \sqrt{\frac{2}{3\pi}} \frac{ne^2 \lambda v}{\alpha \theta}$$

represents the conductivity for the metal in bulk, the difference between the numerical factor and that usually given being due to a circumstance already discussed by the author in a previous paper †. Thus in the metal between the gaps,

the current density will be $\frac{Ae^2 h \lambda X_1}{3(hm)^2}$. Equating this to i as given by (4), we obtain after a little reduction

$$X - \frac{V}{l} = X_1 = \frac{3V}{4\lambda F(a)} \left\{ 1 + (heV)^2 f(a) \right\}, \quad \dots \quad (5)$$

* In so far as I/a is infinite when a is zero, it may be thought that the term in (4) involving this quantity becomes infinite under these conditions. It turns out, however, that when a is infinitesimal, V^2 becomes an infinitesimal of a sufficiently low order of magnitude to insure the vanishing of $V^2 I/a$. It has not been thought necessary to give the formal analytical proof of this, especially as we shall only have occasion to use the expression for cases where a is of the order unity. It is of course obvious, without analytical proof, that $V^2 I/a$ must be zero when a is zero, from consideration of the fact that in this case our results must reduce to those for the case of the metal without gaps.

† Phil. Mag. March 1914, p. 441.

where

$$F(a) = \epsilon^{a^2} - (1 + a^2) + 2a^3 \epsilon^{a^2} I, \quad (6)$$

and

$$f(a) = \frac{\frac{1}{6} + \frac{7}{12} \epsilon^{-a^2} - \frac{a^2}{6} \epsilon^{-a^2} + \left(\frac{a^3}{3} - \frac{3}{4a} \right) I}{1 - (1 + a^2) \epsilon^{-a^2} + 2a^3 I}.$$

If for the moment we neglect the term $(heV)^2 f(a)$ in order to obtain an approximate value of V which will be sufficiently accurate to use in the small term we find

$$X - \frac{V}{l} = X_1 = \frac{3V}{4\lambda F(a)},$$

so that

$$V = \frac{4\lambda l F(a)}{4\lambda F(a) + 3l} X_1.$$

Substituting this value in the *small* term $(heV)^2 f(a)$ of equation (5), we obtain, after a little reduction, from equation (5) itself

$$X_1 = \left[\frac{X}{\frac{4\lambda}{3l} F(a) + 1} \right] \left[1 + \frac{(helX)^2 f(a)}{\left\{ 1 + \frac{3l}{4\lambda F(a)} \right\}^3} \right].$$

Thus since

$$i = \sqrt{\frac{2}{3\pi}} \cdot \frac{ne^2 \lambda v}{\alpha \theta} X_1,$$

we have

$$i = \sqrt{\frac{2}{3\pi}} \cdot \frac{ne^2 \lambda v}{\alpha \theta} \left[\frac{X}{\frac{4\lambda}{3l} F(a) + 1} \right] \left[1 + \frac{(helX)^2 f(a)}{\left\{ 1 + \frac{3l}{4\lambda F(a)} \right\}^3} \right]. \quad (7)$$

a involves V , and so X (see Appendix, Problem 1) in the form of a small term of the second order of small quantities. It will be noted that the equation indicates a departure from Ohm's law. For a field of 1 volt per cm. $heX = 14$, and for values of l such as we are dealing with in these films, *i. e.* values probably of the order 10^{-5} cm., the departure from Ohm's law would be quite inappreciable. The formula (7) is, however, not without interest, and we shall return to it later. For the present we shall assume terms of the orders $(helX)^2$ and $\left(\frac{Ve}{mu^2} \right)^2$ to be negligible, so that we are left with

$$i = \sqrt{\frac{2}{3\pi}} \frac{ne^2 \lambda v}{\alpha \theta} \left\{ \frac{X}{\frac{4\lambda}{3l} F(a) + 1} \right\},$$

in which a^2 may now be taken as hmu^2 .

The conductivity is obtained by dividing by X , and the specific resistance s is given by

$$s = \sqrt{\frac{3\pi}{2}} \cdot \frac{\alpha\theta}{ne^2\lambda v} \left\{ 1 + \frac{4\lambda}{3l} F(a) \right\}, \quad \dots (9)$$

or writing $F(a)$ in full, and substituting for hmu^2 its value $3mu^2/4\alpha\theta$, we have

$$s = \sqrt{\frac{3\pi}{2}} \frac{\alpha\theta}{ne^2\lambda v} \left\{ 1 + \frac{4\lambda}{3l} \left[\epsilon^{\frac{3mu^2}{4\alpha\theta}} - \left(1 + \frac{3mu^2}{4\alpha\theta} \right) + 2 \left(\frac{3mu^2}{4\alpha\theta} \right)^{3/2} \int_{\odot}^{\infty} \epsilon^{-\xi^2} d\xi \right] \right\}, \quad \dots (10)$$

where \odot is written for $(3mu^2/4\alpha\theta)^{1/2}$.

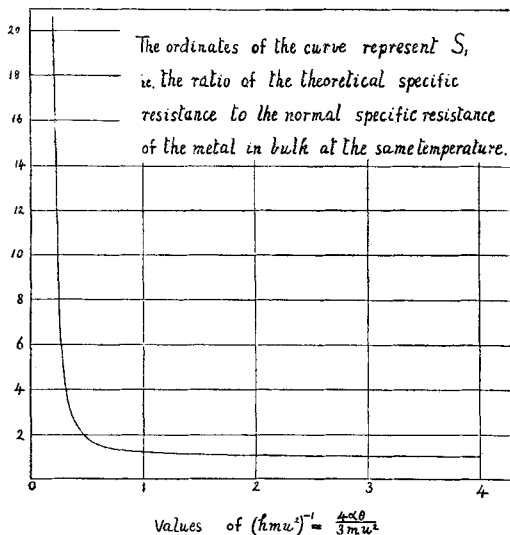
Sharpness of the change of specific resistance at the critical value.

In the first place we notice that the exponential factor and the integral vary so rapidly with u as u increases beyond a certain value, that a very sharp change in conductivity with the change in closeness of packing of the groups may be expected. The value of u will depend upon the time of deposit. In the first stages when the groups are very loosely packed, u may be expected to diminish very rapidly as t increases, but the rate of variation of u with t must tend to zero as t tends to ∞ . It is impossible for us to attempt to deduce the accurate theoretical relation between s and t unless we know the relation between u and t , but we can readily see that the general characteristics of the curve will be the same as that of fig 2. Suppose, for example, we take such a relation between u and t as $u^2 = k/t$, where k is a constant. Such a relation would represent qualitatively the variation of u with t referred to above, and the curve representing s plotted against t would be of a similar shape to that representing s plotted against $1/u^2$. The curve in fig. 4 represents the curly bracket of equation (10), which we shall

call s_1 , plotted against $\frac{4\alpha\theta}{3mu^2}$, for a case where $\lambda/l = \frac{1}{10}$, θ being constant. s_1 of course here represents the ratio of s to the normal value for the same temperature, and we see that the bend in the curve is a very sharp one, and the curve is in fact very similar in its general characteristics to that of fig. 2. If u varied as a smaller inverse power of t than the first, the bend would of course be less sharp, but the nature

of the exponential functions occurring in (10) are such as to retain the sharpness of the bend in the curve for s_1 plotted

Fig. 4.



against t , even though u^2 may be made to vary very slowly with t . Apart from this, however, we should not expect a slow variation of u^2 with t in the neighbourhood of the bend, i. e. under the conditions in which the looseness of the packing is pronounced.

The variations of the resistances of the films with temperature.

An examination of Tables II. and III. will show :

(1) that for thick films, *e. g.* Table III. films 4–10, also Table II. films 6 and 7, the specific resistance increases slightly with rise in temperature.

(2) For thin films, *e. g.* Table II. films 1–4, s decreases rapidly with rise in temperature.

(3) In the case of intermediate films, such as film 5, Table II. and film 3, Table III., the temperature coefficient shows a reversal of sign over the range of temperature employed.

The fact that the temperature coefficient is of opposite sign for thick films to what it is for thin films is already known, but as far as I know, the reversal of the temperature coefficient with alteration of temperature in the same film has not been recorded. The explanation of these curious

facts with regard to the temperature coefficient are contained in the theory which I have put forward to account for the behaviour of these films. If λ/l were independent of the temperature, since s_1 would then be a function only of $\alpha\theta/mu^2$, fig. 4 would be the shape of the curve s_1 plotted against θ , u^2 being constant. It would of course correspond to a rapid decrease of s_1 with the temperature for small values of θ , and a slow decrease for large values. The only effect of a diminution of λ with increase of temperature, such as is necessary to insure agreement with the Thomson effect, would be to accentuate the decrease of s_1 with increase of temperature. We know that the factor $\alpha\theta/ne^2\lambda v$ in 10 increases on the whole with the temperature, so that s will show a minimum for some temperature, and for that temperature the temperature coefficient will be zero. Above this temperature s will increase with rise in temperature owing to the preponderance of the effect of the factor $\alpha\theta/ne^2\lambda v$; below it s will decrease with rise in temperature. The greater u^2 , the greater the value of θ for which the reversal occurs.

If we turn to Tables II. and III. we see that these are just the phenomena which we find experimentally. Films 4-10, Table III. and films 6 and 7, Table II. are so thick (*i. e.* u is so small) that over each of the ranges tested the temperature coefficient is positive. A sufficiently low temperature has not been reached in the case of these films to enable them to show a decrease, or at any rate a net decrease, of resistance with rise of temperature over the ranges tested. On the other hand, film 3, Table III. and film 5, Table II. show a slight increase in resistance from 0° C. to 100° C., and a decrease in resistance from -180° C. to 14° C. The value of θ/u^2 for these particular films is thus such that the reversal occurs over the range tested. Again, films 2, Table III. and films 1-4, Table II., which correspond to very small times of deposit (large values of u) show only negative temperature coefficients. At sufficiently high temperatures they would, however, according to the theory, show an increase of resistance with temperature.

At ordinary temperatures the exponential functions in (10) result in s diminishing very rapidly with increase of θ for large values of u^2 (thin films), and this is just what is found experimentally. The very thin films (films 1-4, Table II. film 2, Table III.) show enormous negative temperature coefficients.

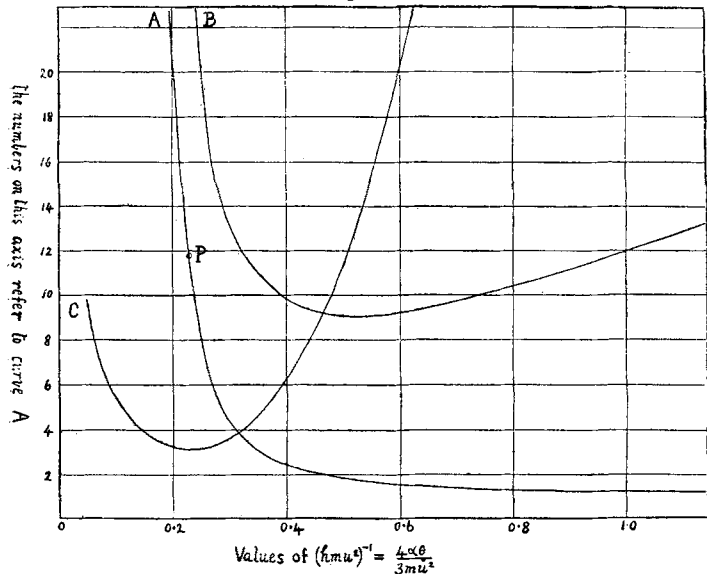
The present theory thus qualitatively accounts for the facts concerned with the temperature coefficients of the films, and it is to be noted that our arguments in this connexion

are quite independent of any assumptions with regard to the relation between u^2 and the time of deposit. Before leaving the question of the temperature variations of the films we must, however, look carefully into one point further.

Referring to Table III., we see that film 3, which shows a reversal of the temperature coefficient in the neighbourhood of the room temperature, has a resistance about 20 times the normal value, *i. e.* the value for the thickest films. Thus s_1 is about 20. Now if we knew λ/l as a function of θ it would be possible to calculate the value of s_1 corresponding to a reversal at any temperature, by substituting in the expression for s_1 the value of u^2 obtained by differentiating s with respect to θ and equating to zero. Suppose, for example, λ/l were independent of the temperature and equal to 1/10. We find for this case $4\alpha\theta/3mu^2 = 0.52$ and $s_1 = 1.75$, *i. e.* the value is far less than 20, and a consideration of the reasons which have been responsible for this small value of s_1 will show that they are not such as would be modified in such a direction as to give a larger value of s_1 by a modification of the above assumption as to the constancy of λ/l .

In fig. 5, curve A represents s_1 plotted against $4\alpha\theta/3mu^2$,

Fig. 5.



$\frac{\lambda}{l}$ being independent of θ , and B represents s plotted against θ (the scale of the ordinates being arbitrary). Now B is obtained

from A by multiplying all the ordinates of the latter by a factor

proportional to $\sqrt{\frac{3\pi}{2}} \cdot \frac{\alpha\theta}{ne^2\lambda v}$, which as a whole is propor-

tional to θ^* . In virtue of the comparatively slow variation of this factor with θ , and of the very rapid variation of s_1 with θ in those regions where s_1 has a value appreciably greater than its minimum value unity, it will be obvious that the minimum in B cannot occur in a vertical line with any point in A which is far up the steep part of the curve. Our theory in its present form thus predicts that the value of the specific resistance for which a temperature coefficient reversal is obtained, cannot be a value which is much greater than the normal value. This does not prevent a film of which the resistance is far above the normal at any given temperature showing a reversal at some other temperature, in fact it must do so. The point is, however, that this would have to be a temperature which was so high that the resistance was little above the normal for that temperature.

A little consideration will show that a modification of our conditions in the direction of assuming either a smaller value of λ/l or a diminution of λ/l with the temperature will only enhance the above difficulty. An assumption of a larger value of λ/l than the value $\frac{1}{10}$ would make matters better, but not appreciably so. λ/l must of course never be greater than unity, and even when it approaches this value our calculation of (10) would require modification.

The fact is that our theory requires too sharp a variation of s_1 with θ for small values of θ , and, indeed, apart from the above considerations, though variation of the resistance of the very thin films with temperature is very great, it is not as great as our theory in its present form would require. The difficulties in these respects can be surmounted if we imagine that u^2 is a function of the temperature of such a kind as to insure that θ/u^2 varies according to a smaller power of the temperature than the first†. Suppose, for example, that u^2 were proportional to $\theta^{3/4}$, so that θ/u^2 was proportional to $\theta^{1/4}$. s_1 would then vary very much less rapidly with temperature, while at constant temperature it would vary as rapidly with u^2 as before. The curve B would now represent s_1 plotted against a quantity proportional to $\theta^{1/4}$. If we now multiply the ordinates by a quantity proportional to θ to obtain relative values of s , we obtain the

* It must be remembered that n is a function of θ .

† There are many ways in which such a state of affairs can be imagined.

curve C. It will be seen that the minimum now occurs for a smaller value of $\frac{4\alpha\theta}{3\mu u^2}$, and it occurs in a vertical line with the point P, *i. e.* a point in B which corresponds to a value $s_1 = 11.8$, *i. e.* to a case where the resistance of the film has 11.8 times the normal value. While it is not suggested that the variation of u^2 with θ is of any such simple nature as that which we have cited above, this method of looking at the matter is perhaps useful as indicating the kind of physical assumption which it is necessary to make in order to bring the present theory into a more strict agreement with the facts than it would show in its most elementary form.

It will be noticed that the films for very large times of deposit all show very small temperature coefficients from 0° C. to 100° C. We might expect, for example, that since film 4, Table III., which corresponds to a time of deposit of 130 secs., shows a positive temperature coefficient, film 10, which corresponds to a time of deposit of 960 secs., would show a temperature coefficient equal to that shown by the metal in bulk; we must remember, however, that the value of u for film 10 is probably not far different from that for film 4. Once several layers of molecules have become grouped one on top of the other, a further increase of the quantity of deposit would have very little effect on u . It seems probable that the continuation of the deposition only diminishes u down to a certain limit, after which some totally different process becomes necessary to reduce it much farther. In favour of this view we have the fact that the specific resistances of the thickest films obtained by deposition are much greater than that for the metal in bulk.

It is interesting to notice that the phenomena resulting from the collection of the molecules into groups might be expected to occur even in the metal in bulk if the metal partakes at all of a crystalline structure, and the bounding planes between the crystals would function as what we have called the gaps. The normal high value of the temperature coefficient for ordinary metals suggests that the values of u and of l are such that these phenomena are not appreciable at ordinary temperatures, but there should always be some temperature sufficiently low at which the temperature coefficient does attain a zero value, below which value it would of course be negative, and it is a significant fact in favour of this view that experiments have shown that at low temperatures the temperature coefficient becomes abnormally small, which seems like an approach to the zero value referred to above.

The Departure from Ohm's Law.

Referring to equation (7) we first observe that in the coefficient of $(Xelh)^2$, a^2 may be looked upon as hmu^2 , since η is negligible in these terms. We next notice that it is only films which have specific resistances well above the normal value in which we may hope to find deviations from Ohm's

law, for so long as $\frac{X}{\frac{4}{3} \frac{\lambda}{l} F(a) + 1}$ in equation (7) is very

nearly equal to X , $\frac{4}{3} \frac{\lambda}{l} F(a)$ must be small, and the deviation from Ohm's law due to the fact that a involves the quantity η , which itself involves V , is consequently small.

Also, when $\frac{4}{3} \frac{\lambda}{l} F(a)$ is small, $\frac{3l}{4\lambda F(a)}$ is large, and the coefficient of the term involving $(Xelh)^2$ is small. The maximum departure from Ohm's law will be attained when s , and consequently $F(a)$, is very large. Indeed, for this case we can obtain an expression for i in a more convenient form.

If we substitute the value of X_1 given by (5) in (4) we see that when ϵ^{-a^2} is very small (*i. e.* for films of abnormally high resistance), the first member on the right-hand side of (4) becomes all important, and we have

$$i = \frac{Al}{4(hm)^2} \epsilon^{-a^2} \sinh(helV). \quad \dots \quad (11)$$

Since for this case (5) gives $V = Xl$ and since $a^2 = hmu^2 + 2he\eta$ we have, on substituting for A ,

$$i = \sqrt{\frac{2}{3\pi}} \cdot \frac{ne^2\lambda v}{\alpha\theta} \cdot \frac{3}{4} \frac{l}{\lambda} \epsilon^{-hmu^2} \epsilon^{-2he\eta} \frac{\sinh(helX)}{helX}.$$

We cannot express η as a function of V unless we know the nature of the intermolecular forces in the gaps, but if we make the very reasonable assumption made on page 489 we have

$$\frac{2he\eta}{mu^2} = \frac{1}{4} \left(\frac{heV}{mu^2} \right)^2 = \frac{1}{4} \left(\frac{helX}{mu^2} \right)^2.$$

Hence, on expanding $\epsilon^{-2he\eta}$ and the \sinh function, we have

$$i = \sqrt{\frac{2}{3\pi}} \cdot \frac{ne^2\lambda v}{\alpha\theta} \cdot \frac{3}{4} \frac{l}{\lambda} \epsilon^{-hmu^2} \left\{ 1 + \left(\frac{1}{6} - \frac{1}{4hmu^2} \right) (helX)^2 \right\}$$

If hmu^2 is greater than, say, 3 or 4, the term representing the deviation from Ohm's law approximates to $(helX)^2/6$, and the term arising from the more doubtful assumption with regard to η becomes less important. The formula then indicates an increase of the conductivity with the field. If X is 1 volt per cm. $heX=14$, and we readily see that in order to obtain a one per cent. deviation from Ohm's law on doubling the field, it is necessary that $l=0.017$. Of course if $X=100$ volts per cm. a value of l equal to 0.00017 will be sufficient. We thus see that unless the molecules are collected into large aggregates containing many molecules, the deviations from Ohm's law to be expected are small. It might happen, however, that a film which had been ruptured in several places by being subjected to a large current, or by being otherwise drastically treated, would act as though the molecules were collected into large groups, and would show deviations from Ohm's law. In fact this possibility is a very likely one.

It may be noted that by cutting narrow slits in a film, similar to those referred to by Professor R. W. Wood in his paper on "Electron Atmospheres"*, it should be possible to produce artificial conditions of the above kind, and to obtain films showing large deviations from Ohm's law, and generally exhibiting the characteristic properties of films of high resistance, such properties as large negative temperature-coefficients, &c.

The expression (11) of course applies to a single slit when the slit is sufficiently wide to cause a large increase in the resistance of the film. V is in this case practically equal to the applied e.m.f. Measurements made on a single slit should enable us to calculate h , and so test the theory by obtaining the mean kinetic energy of a gas molecule at the temperature of the experiment, for by differentiating (11) with respect to V and dividing the resulting equation by equation (11) we obtain

$$\frac{1}{i} \left(\frac{di}{dV} \right) = \frac{he}{\tanh heV} - \frac{(he)^2 V}{2hmu^2}.$$

If measurements of i and $\frac{di}{dV}$ are made for a value of V so small that heV is small compared with hmu^2 , we can neglect the second term on the right-hand side, and so determine h .

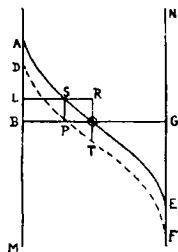
* Phil. Mag. [6] xxiv. p. 316 (1912).

APPENDIX.

Problem (1) in reference to the equations involving the quantity η on page 477.

Let the ordinates of the curve AOE (fig. 6) represent the pull towards the left, on an electron, due to the molecular forces. AM and NF represent the faces of the gap. Different positions across the gap are represented by different points on BG, O being the centre of the gap. Let the dotted curve DPF, which is obtained by subtracting the constant pull due to the electric field from the curve AOE, represent the resultant pull on the electron when the field is acting. The place of zero pull is no longer O but P when the field is acting.

Fig. 6.



In order that an electron shall be able to cross from B to G it is necessary that it shall have enough energy to get to the point of zero force. When there is no field this energy is $\frac{1}{2}mu^2$, and is represented by the area AOB. When the field acts the energy necessary is represented by the area DPB. We shall denote an area by putting a line over the top of the letters. Thus the limiting value of $\frac{1}{2}m\dot{x}^2$ is

$$\begin{aligned}\frac{1}{2}m\dot{x}^2 &= \overline{DPB} = \overline{AOB} - \overline{AOTD} + \overline{POT} = \frac{1}{2}mu^2 - \overline{LROB} + \overline{POT} \\ &= \frac{1}{2}mu^2 - \frac{Ve}{2} + \overline{POT},\end{aligned}$$

for it is to be noted that $\overline{AOTD} = \overline{LROB}$, since by drawing vertical lines through these two areas we divide them up into a number of corresponding elementary parallelograms, each pair of which, being on the equal bases and between the same parallels, are equal. Writing ηe for the area POT we have

$$\dot{x}^2 = u^2 - \frac{Ve}{m} + \frac{2\eta e}{m}.$$

An exactly similar argument applied to the electrons flowing from right to left leads to the result

$$\dot{x}_1^2 = u^2 + \frac{Ve}{m} + \frac{2\eta e}{m}.$$

It will be noted from symmetry that the curve AOE must have either a single point of inflexion at O, or two points of inflexion, one on each side of O, and it is thus probable that

Hence

$$\frac{SP}{AB} = \frac{PO}{OB} = \frac{V_e}{2mu^2},$$

so that

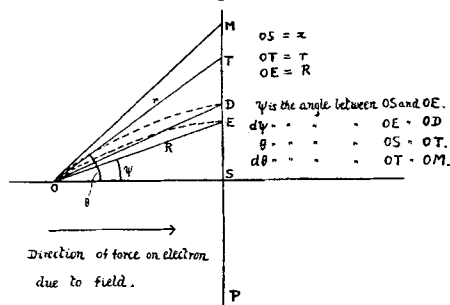
$$\frac{2\eta e}{mu^2} = \frac{(\text{SP})(\text{PO})}{(\text{AB})(\text{OB})} = \frac{1}{4} \left(\frac{Ve}{mu^2} \right)^2,$$

and, in any case, this will be the order of magnitude of $2\eta/mv^2$.

Problem 2. Deduction of the expression for i given on page 478.

We shall at first limit ourselves to the case where the gaps are perpendicular to the x axis. Let PM (fig. 7) represent

Fig. 7.



the position of one of the gaps, and let the electric force be in such a direction as to urge the electrons from left to right.

If ϕ is measured about OS in a plane perpendicular to the paper, and if δn is the number of electrons per c.c. which in the absence of the field are in the velocity range c to $c+dc$, the number in this range starting out from any element $\delta\omega$ at O per second, and contained within the solid angle $\sin\theta\,d\theta\,d\phi$, is

$$\frac{c\delta n}{4\pi\lambda} \sin\theta d\theta d\phi \delta\omega,$$

where θ is the angle between OS and the radius vector OT drawn from O, and λ is the mean free path. When the field is off, those of the above electrons which succeed in getting far enough without collisions will strike the plane on the area represented by TM, but when the field is on they will be bent round, and will travel along paths contained within the dotted lines OD and OE, and those which get far

enough will strike the plane on the area represented by DE. In the notation of fig. 7 this area amounts to

$$R^2 \frac{\sin \psi}{\cos \psi} d\phi d\psi.$$

We shall, in the present problem, make the assumption that when the field is on, the number of electrons starting off from an element with any given velocity is the same as if the field were absent, and that the initial directions of ejection are symmetrically arranged. This assumption is analogous to the assumption made in Drude's calculation of the conductivity of the metal in bulk. We shall also assume that the thickness of the metal slabs which the gaps separate is sufficiently large compared with the mean free path to enable us, in treating the flow across any gap, to assume that the metal extends on each side of the gap in a continuous condition to infinity. This assumption is very approximately true unless λ/l is as small as one or two, in virtue of the very small number of electrons which travel distances appreciably greater than the mean free path. Lastly, we shall make the assumption that the mean free path for any given velocity is independent of the velocity. It is not absolutely necessary to make this assumption; I have considered the problem when λ is a function of the velocity c , and as the general conclusions are not materially modified thereby, it is perhaps as well to avoid introducing this complication.

The number of electrons which pass through one square centimetre per second due to the group referred to above is

$$\Delta n = \frac{c \delta n}{4\pi \lambda R^2} \cdot \frac{\sin \theta \cdot \cos \psi d\theta}{\sin \psi d\psi} \epsilon^{-R/\lambda} \delta \omega^*. \quad (12)$$

Now if t is the time taken for an electron to go from O to E, and if $ES=y$ we have

$$x = ct \cos \theta + \frac{X_1 e}{2m} t^2, \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$y = ct \sin \theta, \quad . \quad . \quad . \quad . \quad . \quad (14)$$

also

$$\cos \psi = \frac{x}{(x^2 + y^2)^{1/2}}. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Substituting in (15) from (13) and (14), and neglecting in

* In the expression $\epsilon^{-R/\lambda}$, R should strictly speaking be replaced by the length of the *dotted* path, but it is easy to see that this only differs from the length R by a quantity of the second order of small quantities.

the usual way small quantities of the order $\frac{X_1 e}{mc^2}$, we readily obtain

$$\cos \psi = \left(1 + \frac{X_1 e t}{2mc \cos \theta}\right) \left(1 + \frac{X_1 e t}{mc} \cos \theta\right)^{-1/2} \cos \theta.$$

Putting $et \cos \theta = x$ in terms multiplied by $X_1 e/m$ we find

$$\cos \psi = \left(1 + \frac{X_1 e x}{2mc^2} \tan^2 \theta\right) \cos \theta. \quad \dots \quad (16)$$

Differentiating, we readily obtain

$$\sin \psi \cdot d\psi = \left\{1 - \frac{X_1 e x}{mc^2} \left(1 + \frac{1}{2} \tan^2 \theta\right)\right\} \sin \theta d\theta. \quad (17)$$

Hence, since θ may be replaced by ψ in terms multiplied by $\frac{X_1 e}{mc^2}$ (12) becomes

$$\Delta n = \frac{c \delta n}{4\pi \lambda R^2} \left\{1 + \frac{X_1 e x}{mc^2} \left(1 + \frac{1}{2} \tan^2 \psi\right)\right\} \cos \psi \epsilon^{-R/\lambda} \delta \omega. \quad (18)$$

It will now be convenient to take E as origin. ψ becomes the angle made by the radius vector to the point O with the normal to PM, and writing $\delta n = A \epsilon^{-hmc^2} c^2 dc$, the total number of electrons reaching E per square centimetre per second from the ring of volume $2\pi R^2 \sin \psi \cdot d\psi \cdot dR$ passing through O and parallel to the plane PM is

$$(\Delta n)_1 = \frac{A}{2\lambda} \epsilon^{-hmc^2} c^3 dc \left\{1 + \frac{X_1 e x}{mc^2} \left(1 + \frac{1}{2} \tan^2 \psi\right)\right\} \sin \psi \cos \psi \epsilon^{-R/\lambda} d\psi \cdot dR. \quad \dots \quad (19)$$

To obtain the current density i_1 corresponding to the flow from left to right we must multiply this by e and integrate for all the electrons which succeed in getting across the plane.

If \dot{x} is the x component velocity with which an electron reaches the gap, then in order that the electron shall be able to cross, it is necessary that $\dot{x}^2 > u^2 + \frac{2\eta e}{m} - \frac{V e}{m}$ as was pointed out on p. 477. Now from (13)

$$\dot{x} = c \cos \theta + \frac{X_1 e}{m} t = c \left(1 + \frac{X_1 e x}{mc^2 \cos^2 \theta}\right) \cos \theta.$$

Replacing $\cos \theta$ by its value in terms of $\cos \psi$, as found above, we have, on substituting ψ for θ in the terms multiplied by $X_1 e/mc^2$, and neglecting second-order quantities

$$\dot{x} = c \cos \psi \left(1 + \frac{X_1 e x}{mc^2} + \frac{X_1 e x}{2mc^2} \tan^2 \psi\right).$$

Thus, writing $u^2 + \frac{2\eta e}{m} - \frac{Ve}{m} = v^2$, we have to integrate (19) between the limits given by

$$c = \frac{v}{\cos \psi} \left\{ 1 - \frac{X_1 e x}{mv^2} \cos^2 \psi - \frac{X_1 e x}{2mv^2} \sin^2 \psi \right\} \text{ and } c = \infty,$$

then replace x by $R \cos \psi$ and integrate from $R=0$ to $R=\infty$, and finally integrate from $\psi=0$ to $\psi=\pi/2$. If $\delta\Omega$ is the result of the first integration, we can write $\delta\Omega =$

$$\begin{aligned} \frac{Ae}{2\lambda} dR \cdot d\psi \int_{\frac{v}{\cos \psi}}^{\infty} \left\{ 1 + \frac{X_1 e x}{mc^2} + \frac{X_1 e x}{2mc^2} \tan^2 \psi \right\} \sin \psi \cos \psi \epsilon^{-R/\lambda} \epsilon^{-hmc^2} c^3 dc \\ + \frac{Ae}{2\lambda} dR \cdot d\psi \int_{R\Phi/\cos \psi}^{v/\cos \psi} \epsilon^{-R/\lambda} \sin \psi \cdot \cos \psi \cdot \epsilon^{-hmc^2} c^3 dc, \end{aligned}$$

where Φ is written for $1 - \frac{X_1 e x}{mv^2} \cos^2 \psi - \frac{X_1 e x}{2mv^2} \sin^2 \psi$, and

where in the second integral we have omitted the terms multiplied by $X_1 e x / mc^2$, since such terms would only contribute quantities of the second order, owing to the fact that the limits of integration only differ by first-order quantities.

Again, for the same reason we can write the second integral as

$$\frac{Ae}{2\lambda} \epsilon^{-R/\lambda} \sin \psi \cos \psi \cdot \epsilon^{-hmv^2/\cos^2 \psi} \frac{v^3}{\cos^4 \psi} \left\{ \frac{X_1 e x}{mv} \right\} \left\{ \cos^2 \psi + \frac{\sin^2 \psi}{2} \right\}$$

so that writing $x = R \cos \psi$ and $\cos \psi = z^{-1}$ we have

$$\begin{aligned} i_1 = \frac{Ae}{2\lambda} \int_1^{\infty} dz \int_0^{\infty} \epsilon^{-R/\lambda} dR \left[\frac{1}{z^3} \int_{vz}^{\infty} \epsilon^{-hmc^2} c^3 dc + \frac{X_1 e R}{2m} \int_{vz}^{\infty} \left(\frac{z^2+1}{z^4} \right) \epsilon^{-hmc^2} c dc \right. \\ \left. + \frac{X_1 e R v^2}{2m} \left(\frac{z^2+1}{z^2} \right) \epsilon^{-hmv^2 z^2} \right]. \end{aligned}$$

It is hardly necessary here to give the full analysis; the result of performing these integrations is

$$i_1 = \frac{Ae}{2(hm)^z} \left[\frac{\epsilon^{-hmv^2}}{4} + \frac{X_1 e h \lambda}{3} \left\{ (1 + hmv^2) \epsilon^{-hmv^2} - 2(hmv^2)^{\frac{3}{2}} \int_{(hmv^2)^{\frac{1}{2}}}^{\infty} \epsilon^{-\xi^2} d\xi \right\} \right].$$

The current density i_2 due to the flow from right to left is obtained by writing $-X_1$ for X_1 in the above, and μ for v where $\mu^2 = u^2 + \frac{2\eta e}{m} + \frac{Ve}{m}$. The resultant current density is then equal to $i_1 - i_2$. If we write $hmu^2 + 2\eta eh = a^2$ we have

$$hmv^2 = a^2 - heV, \text{ and } h\mu^2 = a^2 + heV.$$

It will be unnecessary to give the full steps in the evaluation of i . The result, after a certain amount of reduction, neglecting terms of smaller order than $(heV)^3$ compared with unity, and writing I for $\int_a^\infty \epsilon^{-\zeta^2} d\zeta$ amounts to

$$i = \frac{Ae}{2(hm)^2} \left[\frac{\epsilon^{-a^2}}{2} \sinh(heV) + \frac{2}{3} heX_1\lambda \left\{ (1+a^2) \epsilon^{-a^2} - 2a^3 I \right. \right. \\ \left. \left. + \frac{3}{4} (heV)^2 \left[\epsilon^{-a^2} - \frac{I}{a} \right] \right\} \right]$$

Discussion of the case where the complicated cell structure cited on page 476 is assumed.

Let us, as a preliminary problem, consider the case of a gap which is not perpendicular to the resultant field in the metal. Let X_1 as before be the actual field in the metal, and let us resolve X_1 into a component X_n perpendicular to the gap, and a component X_s parallel thereto. Suppose for the moment that X_s were absent. Then obviously, just as in the above problem, we obtain for the number of electrons crossing the gap per square centimetre per second the result

$$N = \frac{A}{2(hm)^2} \left\{ \frac{heV}{2} \epsilon^{-a^2} + \frac{2}{3} X_n he\lambda \left[(1+a^2) \epsilon^{-a^2} - 2a^3 I \right] \right\}, \quad (22)$$

where we have neglected the small quantities of the order $(heV)^2$.

But the number of electrons flowing per square centimetre per second past a plane in the metal parallel to the gap is $\frac{Ahe\lambda}{3(hm)^2} X_n$. The continuity of the flow requires that these shall be equal, so that we find on equating them

$$V = \frac{2}{3} \lambda X_n F(a), \quad (23)$$

where $F(a)$ is the quantity defined on p. 479.

Now this result will not be altered by the presence of the component X_s , for the effect of this component is merely to shift the position in which each electron strikes the gap by a small amount parallel thereto. It does not shift all of the electrons equally. If we divide the electrons up into groups, such that the members of each group take the same time to reach the gap after collision, then each of the individual members of any one group will be shifted by the same amount, but by an amount which is different from that by which the members of any other group are shifted. The

fact that the different sets of electrons are shifted by different amounts does not, however, affect the total number which pass through the gap, which number will consequently be the same as if the force X_s were absent, except for a small error at one end of the plane, which only exists, however, for a length of the plane comparable with the mean free path*. Thus, in the general case (23) represents the condition which must hold between the value of V at a gap, and the normal component of the force resolved perpendicular thereto.

Now returning to the complicated cell-structure cited on p. 476, we see that the fields inside the groups and the drops of potential at the gaps must satisfy three conditions.

(1) If we draw parallel to the applied field X a line of length L great compared with the dimensions of a group, and if ds is an element of this line, we must have

$$XL = \int_0^L X' ds + \Sigma V$$

for all such lines parallel to the field X . Here ΣV represents the sum of all the drops of potential occurring at gaps along the line of length L , and X' is the field at any point.

(2) If ABC (fig. 8) represent three regions separated by bounding gaps it is easily seen by considering the state of affairs at the point O , for instance, that we must have

$${}_A V_B = {}_A V_C + c V_B.$$

(3) The force and the potential drop at each point in the neighbourhood of a bounding gap must satisfy the relation

$$V = \frac{4\lambda F(a)}{3} \cdot X_n.$$

Now in the general case the condition (1) is readily satisfied owing to the averaging effect due to the integration

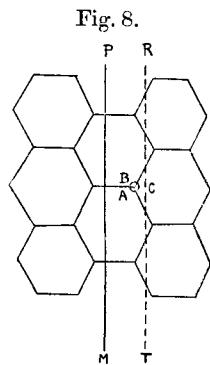


Fig. 8.

* It may be noted that if we consider the electrons coming from any one element, and draw a plane through that element perpendicular to the gap and to the force X_n , then since the electrons which strike the gap on the side of the plane to which the force X_s points have had their journeys increased by the action of the force, fewer of them will reach the plane than would otherwise have done so. It is to be noticed, however, that this effect is cancelled by the diminution in the journey travelled by the corresponding set of electrons striking the gap on the other side of the plane; at least this is true if we neglect the second order of small quantities.

over a sufficiently large number of groups, but in the general case the distribution of the field inside the groups, necessary to satisfy (2) and (3), is extremely complicated. There is one particular case, however, in which we can obtain a solution, and we can, I think, see by arguing from this case that the solution for the general case cannot be far different from this.

Let us consider the case where the bounding gaps form the faces of hexagonal solids filling up the space, and suppose the field to act parallel to the line MP (fig. 8) which is perpendicular to one set of faces. Then a uniform field throughout the system will satisfy all the conditions. For suppose that we assume such a constant field X_1 . In order to satisfy condition (3) the value of V at any plane will be

$$V = \frac{4}{3} \lambda F(a) X_1 \cos \gamma,$$

where γ is the angle between the normal to this plane and MP.

This adjustment of V will satisfy (2) since we shall have

$${}_A V_B = \frac{4}{3} \lambda F(a) X_1 \cos(O),$$

$${}_A V_C + {}_C V_B = \frac{4}{3} \lambda F(a) X_1 \left\{ \cos \frac{\pi}{3} + \cos \frac{\pi}{3} \right\}.$$

If we now adjust the absolute value of X_1 so that $X = X_1 - \frac{\bar{V}}{l}$

where \bar{V} is the potential drop across one of the gaps perpendicular to MP, and l is the distance between an adjacent pair of such gaps, (1) will be satisfied along the line MP, and it will also be satisfied along such a line as TR, since, although this line intersects twice the number of gaps per centimetre as the line MP, the potential drop across each of

them is only $\bar{V} \cos \frac{\pi}{3} = \frac{\bar{V}}{2}$.

Thus we see that the solution for this particular case is exactly the same as if the gaps were all perpendicular to the field X and separated by distances l , *i.e.* the solution is exactly the same as for the problem we have already solved in this paper, and the specific resistance is given by the expression in equation (10).

If the field is not perpendicular to a set of bounding gaps

the above is, however, not absolutely true. Suppose, for example, that we imagine the direction of the field to remain fixed, and the whole of the material to be rotated through an angle α about one of the lines where the bounding gaps meet, say the lines whose end is represented by the point O. A constant field, with V adjusted in the above manner, would satisfy (3) and it would satisfy (2), as far as those planes are concerned which intersect on lines parallel to the lines of rotation, for the angles which these planes now make with MP, counting in a counter-clockwise direction from the plane which was originally perpendicular thereto, are

$$\alpha, \quad \frac{\pi}{3} - \alpha, \quad \frac{\pi}{3} + \alpha,$$

so that

$${}_A V_B - {}_A V_C - {}_C V_B = \frac{4}{3} \lambda F(\alpha) X_1 \left\{ \cos \alpha - \cos \left(\frac{\pi}{3} - \alpha \right) - \cos \left(\frac{\pi}{3} + \alpha \right) \right\} = 0.$$

If we consider any of the planes which are not included in the above class, however, we shall see that this condition will not strictly hold. This tells us that a uniform field throughout the hexagonal figures would not satisfy the conditions of the problem. A little consideration will, however, show that the conditions are not violated to any large extent, and when we consider that as the direction of the field is varied throughout a solid angle of 2π there are seven different positions in which the ideal condition represented by fig. 8 become satisfied, we see that there is no direction in which the field can act in which the condition departs very far from the ideal condition represented by that figure, and we are, I think, justified in assuming that the solution given for that case, and represented by equation (10), is a very good approximation to that for the most general case.

Department of Terrestrial Magnetism,
Carnegie Institution of Washington,
February 25, 1914.