CONTACT PRESSURES AND STRESSES.

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The application of a load over a very limited portion of the surface of a solid body generally causes intense stresses in the immediate neighbourhood of the place of application, while the manner in which the load is distributed over the area of contact, when two bodies are pressed together, may vary considerably with the physical characteristics of the materials brought into contact, also the form of each body and the mode of application of the load. The mode in which a load is distributed at a contact surface is all important as regards the stresses at neighbouring points, since St. Venant's principle of equi-pollent load systems does not then apply, nor can the distribution be determined except by experimental means. At a considerable distance away from the contact area the way in which the distribution occurs is relatively unimportant, since the stress distribution is hardly affected thereby. In practical applications great concentration of loading is avoided, as far as possible to prevent permanent injury to the material, but there are many cases in which intense loading pressures are inevitable, as, for example, knife-edge and roller bearings and the line contact of gear-wheels. The contact pressure of the drivingwheels of a locomotive on the rails affords another instance of intense local bearing and stress, the constant application of which has a deteriorating effect on both wheels and rails.

In the testing of materials there are also cases of great stress concentration, such as indentation tests for hardness, in which a spherical ball or a cone is pressed against the material by a load sufficiently intense to deform the material permanently. A variety of other examples will readily occur to engineers in which the study of contact pressures and the stresses produced by them is of practical importance, and offers a considerable field for theoretical and experimental investigation. A simple typical case is that of a single load applied normally to the edge of a plate, and such a case has been examined experimentally by Carus-Wilson* using a beam of glass resting with its narrow edge on a flat plate with a sheet of thin paper interposed between the surfaces.

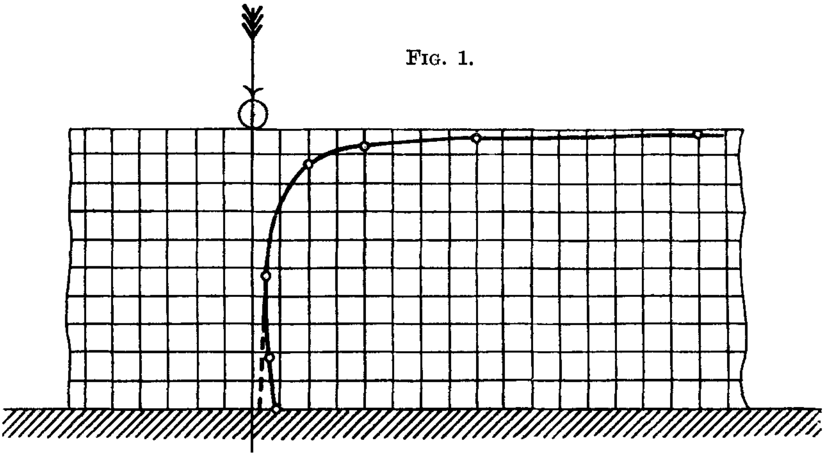
These experiments showed some interesting features, requiring further explanation, which is given later in this Paper, and they may be illustrated by an experiment made on a beam 6·1 centimetres long, 2·0 centimetres deep, and having a thickness of 0·65 centimetre. A steel roller of 0·2 centimetre diameter was pressed against the centre of this beam by aid of a screw and the temporary double refraction caused by loading was observed by aid of two Nicol's prisms crossed at 45° to the axis of the beam, with a quarter wave plate of mica placed in front of the analysing prism, so that the plane containing the optic axis was at right-angles to the length of the beam. With this arrangement the effect produced at any point in the normal line of load is that of no stress, wherever the difference of phase between the ordinary and extraordinary rays traversing the beam is equal and opposite to the difference of phase produced by the mica plate, and there will therefore be a black spot as the Nicols are crossed. The position A of this spot on the normal is determined and the value of p-q determined from the colour band at this point.

* "The Influence of Surface Loading on the Flexure of Beams," by Prof. C. A. Carus-Wilson. Phil. Mag., December 1891.

A second quarter-wave plate is now superposed on the first, and the black spot then moves up the normal to a point B, where the value of p-q is doubled.

The second mica plate is now removed and the load diminished until the black spot with one mica plate is brought to the point B. In this way a number of points on the normal are found, at any one of which the stress difference is twice that of the point below. As the stress at any point of the beam varies directly as the load on the roller, it is now possible to trace the variation of stress difference along the normal through the point of application of the load. One

FIG. 1.



set of observations is plotted in the accompanying Fig. 1 to a horizontal scale of phase differences, and it is found that on the upper half of the beam these points are found to lie very nearly upon a hyperbolic curve, but on the lower half there is a marked difference and the stress begins to increase again as the distance from the loading point becomes greater. As the roller indents the beam to a small extent, this asymptote to the hyperbola is actually not in quite the same position as is assumed in the simple theory, while the points of observation near the lower bearing surface lie somewhat off the hyperbolic curve.

In a further experiment to establish the hyperbolic law with greater certainty, the observations were limited to the upper half of the beam. The load was applied by a screw as before, until seven interference fringes appeared under the roller when viewed in light from a sodium flame. The distance along the normal from the point of contact to successive fringes was measured by a micrometer eye-piece divided to thousandths of an inch.

These distances require correction for the indentation caused by the loading pin, in order that they may be regarded as measured from the asymptote. To find this correction, it was assumed that it has a constant value δy, and by taking any pair of reliable observations the required correction can be found.

This correction is therefore to be added to the experimental numbers. The interference fringes are caused by the stress-differences N corresponding to the first seven ordinal numbers, and hence if the values of the corrected distance C and N are multiplied together, they should give a constant value, if the hyperbolic law holds. This is substantially found to be the case.

The experiments also showed that, for this kind of loading, all the points where the stress difference is constant lie on circles touching the upper surface of the beam at the point of contact.

These investigations, so far as they extend, are approximately consistent with the theory of a single concentrated load at a point in a semi-infinite plate, and before describing further experimental investigations in which stress distribution has been measured over a considerable area around the points of application, it will be convenient to give the results of analysis for this case.

The stress function $\chi = c \cdot r\theta \sin\theta$ is a solution of the general equation $\nabla^4_{r_1\theta}\chi = o$ (1) for plane stress, and from it we obtain :

$$\widehat{rr} = \frac{1}{r} \cdot \frac{\partial\chi}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2\chi}{\partial\theta^2} = 2c\frac{\cos\theta}{r}$$

$$\widehat{\theta\theta} = \frac{\partial^2\chi}{\partial r^2} = o$$

$$\widehat{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r} \cdot \frac{\partial\chi}{\partial\theta}\right) = o \quad . \quad . \quad . \quad . \quad (2)$$

The expression for $\widehat{rr}$ shows that this stress becomes very large as r approaches zero, and its form indicates that, at the origin, the state of stress is produced by normal concentrated stress at a point, or since we are considering plates, by loading along a line in the surface and normal to this bounding surface. The value of the constant c may be obtained from the consideration that the sum total of all the normal components of stress across a plane BC, Fig. 2, at any depth D parallel to the surface must equal the load at the origin. The radial stress rr at $\mathrm{A} = 2c\,\frac{\cos\theta}{r}$, and its

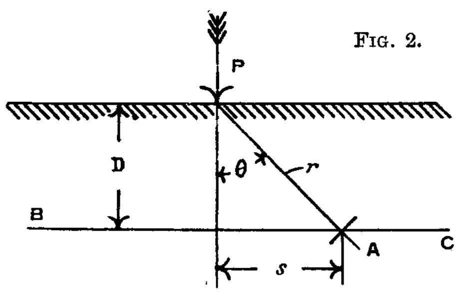


FIG. 2.

component normal to the plane BC is therefore $\widehat{rr} \cdot \cos^2\theta = 2c\,\frac{\cos^3\theta}{r}$. If, therefore, all these values are summed on the infinitely extended plane, the resultant value is—

$$\mathrm{P} = 4c\int_0^{\frac{\pi}{2}} \frac{\cos^3\theta}{r} \cdot ds \quad . \quad . \quad . \quad . \quad (3)$$

where $s = \mathrm{D}\tan\theta$, and therefore $ds = \mathrm{D}\sec^2\theta \cdot d\theta$. Also $r = \mathrm{D}\sec\theta$.

Putting these values into equation (3), we obtain—

$$\mathrm{P} = 4c\int_0^{\frac{\pi}{2}} \frac{\cos^3\theta}{\mathrm{D}\cdot\sec\theta} \cdot \mathrm{D}\cdot\sec^2\theta \cdot d\theta$$

$$\text{or}\quad \mathrm{P} = 4c\int_0^{\frac{\pi}{2}} \cos^2\theta \cdot d\theta = \pi c\,, \quad . \quad . \quad . \quad (4)$$

Hence a normal force P corresponds to a simple radial distribution

$$\widehat{rr} = \frac{2\,P}{\pi} \cdot \frac{\cos\theta}{r} \quad . \quad . \quad . \quad . \quad (5)$$

there being no other stress as appears from Equations 2 (page 368).

Along the normal at the point of application of the load

$$\widehat{rr} = \frac{2\,P}{\pi} \cdot \frac{1}{r} \quad . \quad . \quad . \quad . \quad (5a)$$

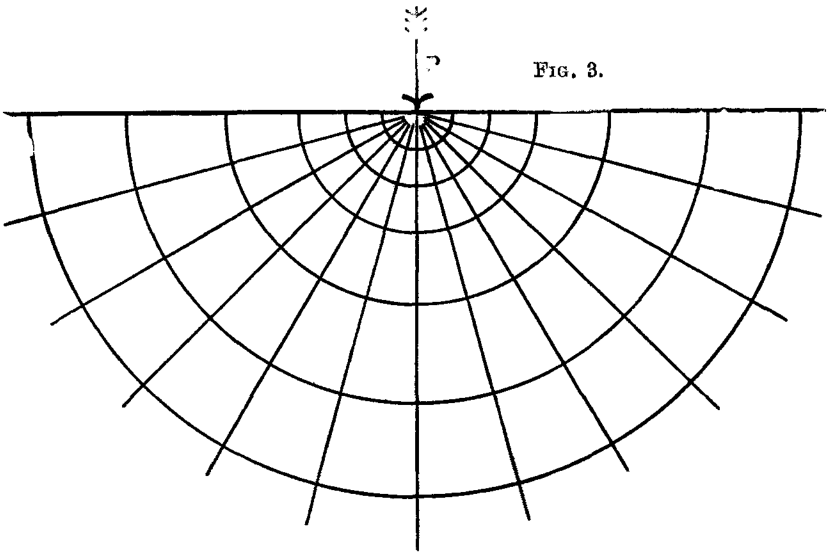


FIG. 3.

so that the stress falls away according to a hyperbolic law as Carus-Wilson's experiments show.

In order to obtain the form of the curves for which the stress difference is constant, we have, since $\widehat{rr}$ is the only stress, at any point R_o below the point of application of the load

$$\widehat{rr} = \frac{2\,P}{\pi} \cdot \frac{1}{R_o} = \text{a constant} = c_o.$$

Hence for any other point

$$\frac{2\,P}{\pi} \cdot \frac{\cos\theta}{r} = \frac{2\,P}{\pi} \cdot \frac{1}{R_o},$$

or

$$r = R_o \cos\theta \quad . \quad . \quad . \quad (6)$$

corresponding to circles touching the plane boundary at the point of application of the load.

The distribution of stress due to a line source P, normal to a semi-infinite plane, is therefore purely radial and the lines of principal stress consist of circles around a point of application intersected by straight lines passing through the origin as Fig. 3 shows. The magnitude of the radial stress reaches a maximum value for any radius on the vertical line through the origin as the

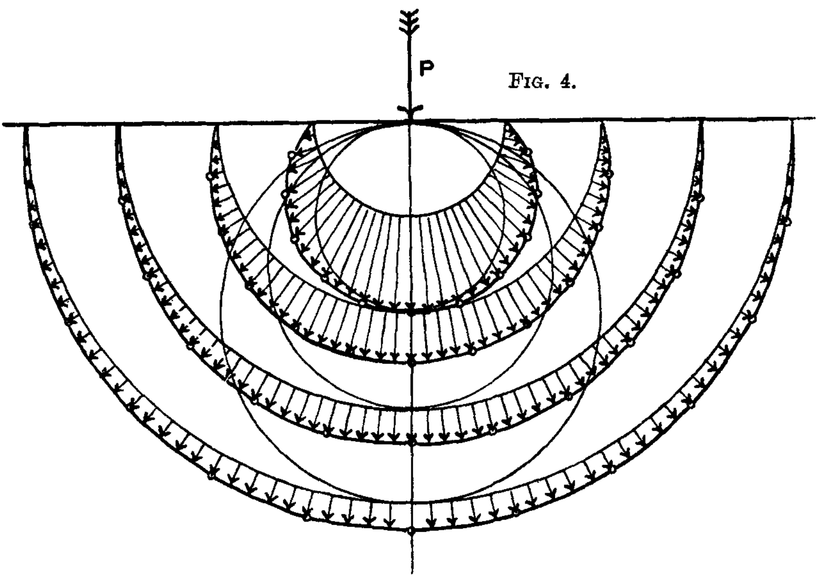


FIG. 4.

Fig. 4 indicates, while the colour bands marking the positions of equal stress difference should be circles touching the contour at the point of application.

Experiment, however, shows that such a simple stress condition is very difficult, if not impossible, to realize in practice, since line contact across the thickness of a plate cannot be maintained if any effect is to be measured or rendered visible optically. Moreover, the necessity of supporting the lower edge of the plate at a finite distance away introduces a system of applied forces which do not correspond with the conditions assumed in the simple theory above, and the disturbing influence of these latter is generally important. It seems probable that the discrepancy noticed by Carus-Wilson in the hyperbolic law of stress distribution along the vertical Fig. 1 (page 367) may be due to this latter cause, and that his curve of distribution is substantially correct for the experimental case taken.

Apparently the applied balancing forces along the lower side tend to concentrate around the central part of the bearing area, owing to the tendency of the edges of a block, of rectangular cross-section, to lift when one of its surfaces is pressed against another flat surface, of much greater area and approximately equal hardness, by a load applied at the centre of the face. This can be readily observed when a block of india-rubber, say 6 inches long, of square section, and 1 inch side, is pressed against a hard surface by a central load ; a visible uplift of the ends is then found, and contact is not obtained there. It seems probable, therefore, that even so hard a material as glass, when pressed against a flat metal plate with thin paper between, will behave in a similar manner, although the strains are now so minute that the uplift is not discernible.

In order to confirm this view, an experiment was made in a form of testing-machine* designed to give pure compression.

Calibration of this machine shows that the load applied by the lever arm is transmitted without perceptible loss to the specimen between the pressure plates. In order to test the correctness of the view expressed above as to the distribution of the load, a plate was prepared of as large a size as could be used in this testing machine. It was of rectangular form, 5·14 inches long and $2\frac{1}{2}$ inches deep, with one long side accurately bedded against a flat steel plate supported on the lower plate. Pressure was applied over an area extending 0·1 inch over the top edge and symmetrically disposed with reference to the central vertical line.

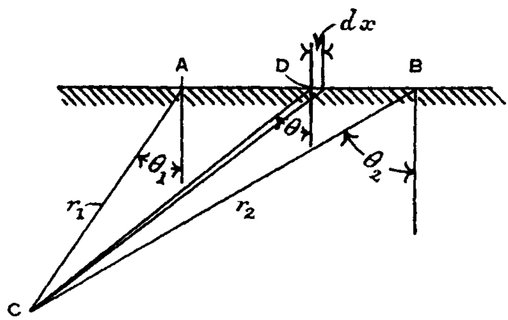
This form of distribution was chosen in order to prevent injury

* "A column testing-machine," by E. G. Coker, Proc. Phys. Soc. of London, vol. xxv, Part II, 1913.

to the plate and to avoid permanent deformation at the area of contact.

On the assumption of a uniformly distributed load over a limited portion of a semi-infinite plate, an expression for the stress distribution can be obtained by considering stress functions of the form $cr^2\theta$. If to a point A on the straight line boundary A B, Fig. 5, there corresponds a stress function $\chi_1 = cr_1^2 \cdot \theta_1$, the resulting

FIG. 5.



stress distribution at any point r_1, θ_1, with reference to the origin A is—

$$\widehat{r_1r_1} = 2c \cdot \theta_1,\ \widehat{\theta_1\theta_1} = 2c\,\theta_1 \text{ and } \widehat{r_1\theta_1} = -c.$$

If to any other point B on this boundary there is also a corresponding stress function $\chi_2 = -cr_2^2 \cdot \theta_2$ the stress at any point r_2, θ_2 is—

$$\widehat{r_2r_2} = 2c\,\theta_2,\ \widehat{\theta_2\theta_2} = -2c\,\theta_2 \text{ and } \widehat{r_2\theta_2} = +c.$$

For both systems on the line A B

$$\Sigma\,\widehat{rr} = o,\ \Sigma\,\widehat{r\theta} = o \text{ and } \Sigma\,\widehat{\theta\theta} = -2c\,\pi = \text{a constant of value say } p.$$

Also, as can be easily verified, there is no stress over the rest of the boundary, while at infinity all stress vanishes.

Hence the stress function

$$\chi = -\frac{p}{2\pi}\left(r_1^2 \cdot \theta_1 - r_2^2 \cdot \theta_2\right) \quad . \quad . \quad . \quad (7)$$

corresponds to the case of a normal pressure uniformly applied to a limited portion of the plane boundary of a semi-infinite plate.

The stress at any point C having co-ordinates $r_1, \theta_1, r_2, \theta_2$, with reference to the origins A, B, is obtainable by integrating the effect of all the elemental loads on the boundary A B. At any point D the load intensity p on an element dx of the boundary gives a radial stress at C of

$$\widehat{rr} = \frac{2p \cdot dx}{\pi} \cdot \frac{\cos\theta}{r} = \frac{2p \cdot r\, d\theta}{\pi \cdot \cos\theta} \cdot \frac{\cos\theta}{r} = \frac{2p}{\pi} \cdot d\theta$$

where $r_1\theta$ are the co-ordinates of C with reference to D as origin.

As will appear immediately, it is convenient to reckon the stress produced at C with reference to an origin D (r, ϕ) having a vector C D bisecting the angle A C B.

The normal, tangential, and shear stresses at C, due to the distributed load $p{\cdot}dx$, are then

$$\begin{aligned} \widehat{rr} &= \frac{2p}{\pi} \cdot \cos^2\phi \cdot d\phi \\ \widehat{\theta\theta} &= \frac{2p}{\pi} \cdot \sin^2\phi \cdot d\phi \\ \widehat{r\theta} &= \frac{2p}{\pi} \cdot \sin\phi \cdot \cos\phi \cdot d\phi \quad . \quad . \quad . \quad (8) \end{aligned}$$

Also $\theta - \phi = \frac{\theta_1 + \theta_2}{2}$ a constant and hence $d\theta = d\phi$.

Summing the stresses for uniformly applied load over the strip A B, we obtain at the point C

$$\widehat{RR} = \frac{2p}{\pi}\int_{\theta_1}^{\theta_2} \cos^2\phi \cdot d\phi = \frac{p}{\pi}\{(\theta_2 - \theta_1) + \sin(\theta_2 - \theta_1)\}$$

$$\widehat{\Theta\Theta} = \frac{2p}{\pi}\int_{\theta_1}^{\theta_2} \sin^2\phi \cdot d\phi = \frac{p}{\pi}\{(\theta_2 - \theta_1) - \sin(\theta_2 - \theta_1)\}$$

$$\widehat{R\Theta} = \frac{p}{\pi}\int_{\theta_1}^{\theta_2} \sin^2\phi \cdot d\phi = o$$

Or writing $\theta_2 - \theta_1 = \alpha$ we obtain—

$$\left.\begin{aligned} \widehat{RR} &= \frac{p}{\pi}(\alpha + \sin\alpha) \\ \widehat{\Theta\Theta} &= \frac{p}{\pi}(\alpha - \sin\alpha) \\ \widehat{R\Theta} &= 0 \end{aligned}\right\} \quad (9)$$

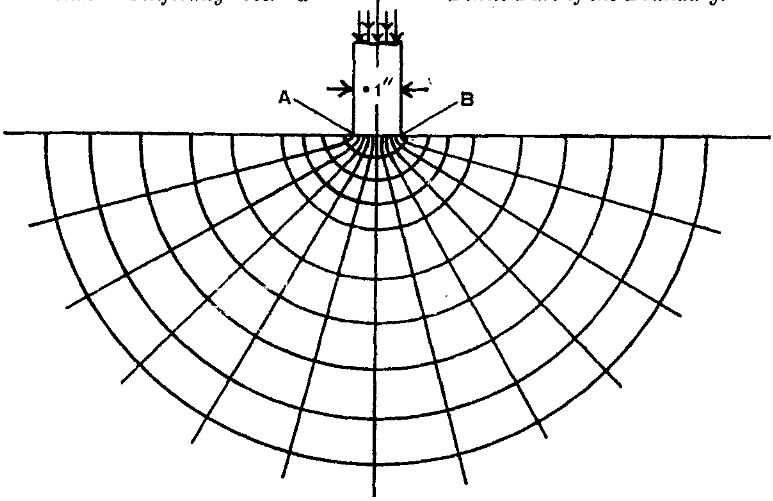
as the stress system at the point C.

TABLE 1.

1 inch Width		$\frac{p}{\pi} = 100.$		
$d = \frac{1}{2} \cot \frac{\alpha}{2}$	$\widehat{rr}$	$\widehat{\theta\theta}$	$\widehat{rr} + \widehat{\theta\theta}$	$\widehat{rr} - \theta\theta$
∞	0	0	0	0
11·4520	—	0·01	17·45	17·24
5·7150	34·81	0·09	34·90	34·72
2·8356	69·11	0·71	69·82	68·40
1·8660	102·4	2·36	104·72	100·00
1·3738	134·1	5·53	139·62	128·56
1·0723	163·9	10·66	174·52	153·20
0·8661	191·3	18·12	209·44	173·20
0·7141	216·2	28·22	244·36	187·94
0·5959	238·1	41·14	279·24	196·96
0·5000	257·1	57·08	314·16	200·00
0·4195	273·0	76·04	349·04	196·96
0·3501	286·0	98·01	383·96	187·94
0·2887	296·0	122·84	418·88	173·20
0·2332	303·5	150·30	453·80	153·20
0·1820	308·6	180·08	488·72	128·56
0·1340	311·8	211·80	523·60	100·00
0·0882	313·4	245·04	558·48	68·40
0·0438	314·1	279·34	593·40	34·72
0	314·16	314·16	628·32	0

These expressions have been obtained by J. H. Michell* in a somewhat different manner. The forms of the expressions show that the lines of the principal stress are ellipses, Fig. 6, intersected by orthogonal hyperbola, both systems having the same foci, A, B. At the area of contact, therefore, these lines are markedly different from the case of line contact, but at a moderate distance away the curves are hardly distinguishable from circles and orthogonal straight lines having an origin at the centre of the pressed area.

FIG. 6.—*Lines of Principal Stress in a Semi-infinite Plate Loaded Uniformly over a Finite Part of the Boundary.*



The most convenient method of examination is to compare the actual values of stress difference (p–q) along the central line with those obtained by calculation. These latter may be readily obtained from Equations 9, and are shown in Table 1 (page 375), where $\frac{p}{\pi} = 100$, with a bearing area of one unit in length for a plate of unit thickness. The values of $\widehat{rr}$ and $\widehat{\theta\theta}$ and their sum and difference are readily applicable to any case, and are plotted in Fig. 7

* "The Inversion of Plane Stress." Proc. London Mathematical Society, vol. xxxiv.

FIG. 7.—*Values of Stresses.*

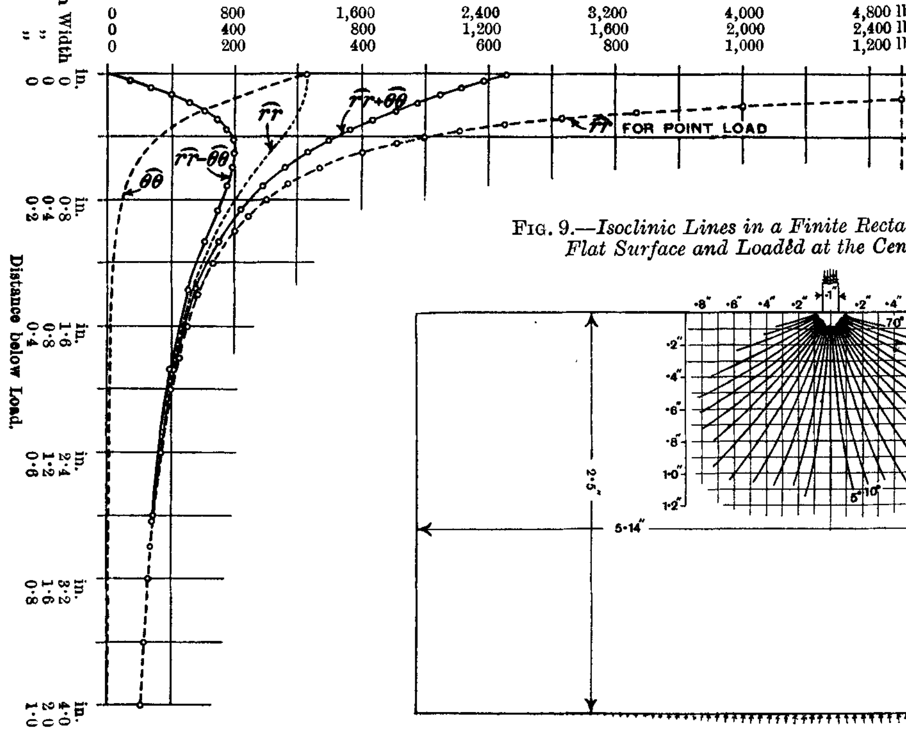
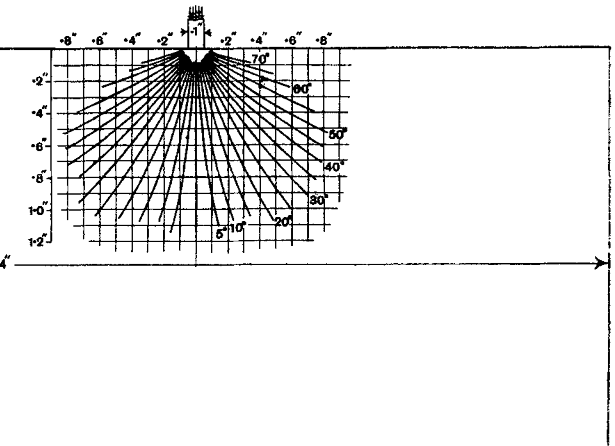


FIG. 9.—*Isoclinic Lines in a Finite Rectangular Plate Resting on a Flat Surface and Loaded at the Centre of the Upper Side.*



to three different scales to facilitate subsequent comparisons. It is of interest to note that $(\widehat{rr}-\widehat{\theta\theta})=p-q$ along the central line, has a zero value at the origin, and a maximum value when $\alpha=\frac{\pi}{2}$ corresponding to a distance of half the width of the contact area.

An examination of the colour bands, Fig. 8, Plate 16, for the plate described above shows, however, that the assumptions on which calculation is based are difficult to realize. The extreme edges of the contact area show brilliant colour banding, in addition to circles passing through points A and B, Fig. 6 (page 376), which calculation, based on uniform surface loading, lead one to expect. Circular bands of this type corresponding to the equation—

$$p-q=\sqrt{(\widehat{rr}-\widehat{\theta\theta})^2+4\,\widehat{r\theta}^2}=\widehat{RR}-\widehat{\Theta\Theta}=2\frac{p}{\pi}\sin\alpha$$

are very well marked, although usually not perfectly symmetrical and often slightly elliptical, but in addition there are lobed-shaped curves starting from the end points A and B, which penetrate into the plate and indicate a variation from uniform distribution.

The colour scheme, therefore, points to the conclusion that the pressure distribution is not uniform, but, as its nature is not completely determined from these optical measurements, the further elucidation of this matter is deferred until the method of separation of p and q has been described. There is, however, nothing to indicate from a superficial examination of the colour bands that the stress distribution at a considerable distance away differs from that obtained from the assumption of uniformly applied load, but measurements of the inclinations of lines of principal stress, by aid of plane polarized light, show however that the lines of equal inclination Fig. 9 (page 377) are curves bending outwards and indicating stress concentration at the centre of the support, and this is confirmed by the lines of principal stress, which are easily obtained from the former set. These lines bend over to meet the bottom edge, as Fig. 10 (page 380) shows, and probably meet the contour at an angle slightly different from 90°, owing to the tangential stress produced by friction between the two surfaces.

It seems probable, therefore, that this system of stress distribution will show greater values along the central line than are given by the expression $\widehat{RR} - \widehat{\Theta\Theta} = 2\frac{p}{\pi}\sin\alpha$, and this is found to be the case, as Table 2 and Fig. 10 show. In all cases the observed values are higher than those obtained by calculation, and particularly so when the distance below the contacting surfaces is such that the law of equi-pollent loads might possibly be

TABLE 2.

Stress Difference along Central Vertical Line.

Distance.	$p - q$ lb. per square inch.	
	Calculated.	Observed.
Inch.		
0	0	0
—	2,300	2,660
—	3,260	3,270
—	3,000	3,300
—	2,370	2,775
0·2	1,570	1,635
0·3	1,080	1,215
0·4	820	985
0·5	660	815
0·6	555	785

considered to be applicable. In this case, however, no part of the plate in the neighbourhood of the central line is really sufficiently removed from either stressed boundary to justify the application of this principle, and the contact pressures undoubtedly differ in a marked manner from the simplifying assumptions of uniform pressure distribution of ordinary theory.

At the upper surface the great stress at the extreme boundaries of the loaded area may possibly be due to intense local shear, and although the displacement of points near the centre appears to be greater than at the ends, the strain at the latter probably reaches

2 C 2

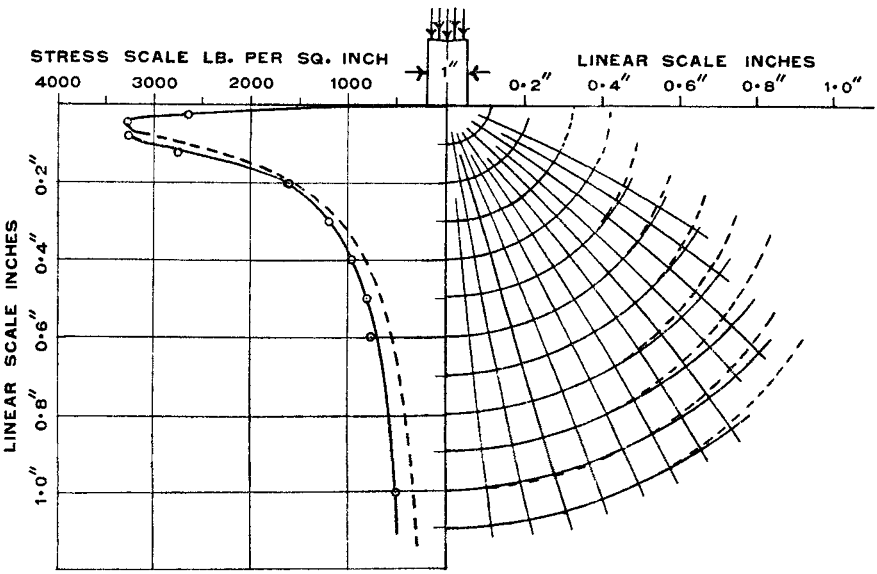
a maximum. This view appears to be confirmed by the phenomena sometimes observed at the bases of metal columns set on stone or concrete foundation blocks, where it is sometimes noticeable that the print of a very rigid base plate is bordered with flakes detached from the friable lower block.

FIG. 10.

Distribution of Principal Stress—Difference along the Central Vertical Line.

Lines of Principal Stress in a Finite Rectangular Plate Loaded at the Centre.

Load 100 lb.



At the lower surface the concentration is undoubtedly towards the centre, and on referring to the work * of Professor L. N. G. Filon, F.R.S., a confirmation was obtained on theoretical grounds. It is there shown that the pressure Q on the lower side of a block of depth b and of infinite length is expressed by an integral of the form—

$$-Q = \frac{2W}{\pi b}\int_0^\infty \frac{\sinh u + u \cosh u}{\sinh 2u + 2u} \cos \frac{ux}{b}\, du$$

* "On the approximate solution for the bending of a beam of rectangular cross-section under any system of load," by L. N. G. Filon, Trans. R. S. Series A., vol. 201.

where u is the variable, W is the concentrated load, and x the distance from the central line. This integral can be evaluated as a convergent series, and the distribution is found to have a high central value, which falls rapidly and becomes zero for a value of $x = 1 \cdot 35\, b$, and from that point the beam is no longer in contact with the lower surface. There can be no doubt, however, that if uniform pressure could be applied over a small finite area of a semi-infinite plate the measured stress distribution would agree with the calculated values.

As will be shown in a later section, the application of a uniform pressure over a limited area has only been approximately realized in one special case.

So far no attempt has been made to separate the principal stresses p and q from the experimental determinations of their difference and inclination, but this may be accomplished from this data by methods which have been described * but are often difficult to carry out with accuracy in the complicated cases which occur in engineering practice. It is, in general, simpler to measure an independent function of p and q, such as $(p + q)$—as suggested by Mesnager.† If a thin plate of any shape is stressed by forces in its own plane, the strains at any point r, θ in terms of the principal stresses $\widehat{RR}$ and $\widehat{\Theta\Theta}$ are given by—

$$m \mathrm{E}\, e_{rr} = m \cdot \widehat{\mathrm{RR}} - \widehat{\Theta\Theta}$$

$$m \mathrm{E}\, e_{\theta\theta} = m\, \widehat{\Theta\Theta} - \widehat{\mathrm{RR}}$$

$$m \mathrm{E}\, e_{zz} = -\left(\widehat{\mathrm{RR}} + \widehat{\Theta\Theta}\right)$$

where e_{zz} is the strain measured in the direction of the thickness, and therefore a measurement over the thickness of the plates of

$$-\frac{m \mathrm{E}}{d} \int_{-c}^{+c} e_{zz} \cdot dz$$

* "Experimental Determination of the Distribution of Stress and Strain in Solids." By Professors Filon and Coker. B. A. Report, 1914.

† "Mesure des efforts intèrieurs dans les solides et applications," by A Mesnager, Buda-Pesth Congress of The International Association for Testing Materials, 1901.

gives a mean value of the stresses $\widehat{RR}+\widehat{\Theta\Theta}=p+q$. These correspond *to the generalized stresses of Filon.** In practice, it is inconvenient to divide each measurement by the values of the constants m and E, and the sum of the stresses is more easily obtained by comparing the change of thickness observed with that obtained from a similar plate under uniform tension or compression stress. If P_t is a uniform tension load applied to a member of rectangular section $A=bd$ and P_c is a similar compression load. We have —

$$\epsilon^{1}_{zz} = -\frac{P_t}{bd}\cdot\frac{1}{m\,E\,d} = -\frac{p_t}{m\,E\,d}$$

$$\epsilon^{11}_{zz} = -\frac{P_t}{bd}\cdot\frac{1}{m\,E\,d} = -\frac{p_c}{m\,E\,d}$$

$$\text{also}\quad \epsilon_{zz} = -\frac{p+q}{m\,E\,d} = -\frac{\widehat{RR}+\widehat{\Theta\Theta}}{m\,E\,d}$$

$$\text{Hence}\quad \frac{\epsilon_{zz}}{\epsilon^{1}_{zz}} = +\frac{p+q}{p_t}$$

$$\text{or } p+q = +p_t\frac{(\epsilon_{zz})}{(\epsilon^{1}_{zz})}\ \text{ or } = p_c\,\frac{\epsilon_{zz}}{\epsilon^{11}_{zz}}$$

a relation which is independent of the constants.

The strains are very small, and, as the plate is usually thin, measurements of very small changes are required. With the transparent nitro-cellulose used in this investigation, a fair value of E is 300,000 in pound and inch units, and a corresponding value of m is 2·5.

The lateral alteration in a plate of 0·15 *inch thickness for a* stress of 1 lb. per square inch is 2×10^{-7} inches, and to determine stress distribution to an accuracy of 5 lb. per square inch a lateral extensometer is required reading to 10^{-6} inches. This offers no serious difficulty so far as the instrument itself is concerned, provided temperature conditions can be maintained very uniform, as the *coefficient of expansion of the material is of the same order as the* measurements.

An improved form of lateral extensometer was devised for these

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**Ibid.*, page 381.

measurements, reading to one-millionth of an inch. This was used throughout the investigations described below.

In general a large number of lateral measurements, and a corresponding number of optical observations, are required to obtain an accurate estimate of the stress distribution. A usual number for each is 20 per linear inch or 400 to cover a square specimen of one inch side. On this account an adjunct measuring instrument is required to bring the needles into their correct position. A convenient way of effecting this is to support the instrument by a three-wire suspension from a frame capable of adjustment horizontally and vertically by measured amounts.

For this purpose the frame may be carried on slides moving on horizontal bars and adjusted in position by a fixed horizontal micrometer bearing against a suitable projection of this frame. A second and vertical micrometer is arranged to lift the frame bodily through any required distance. The range of movement, in each direction, may be extended beyond the range of the micrometers, to any required amount, by using distance blocks of known dimensions. For both optical and mechanical measurements the most convenient method of working is to make observations with a small load, and then with a much greater one the difference between the measurements being taken to eliminate errors which may arise in working from zero loads and from the effects of any slight initial stress which may be present. This involves a double series of measurements, but the increased accuracy justifies the extra labour involved. A disturbing influence which requires elimination is due to the slightly varying thickness of a specimen cut from a plate. These differences are usually of the same order as the measurements, unless the faces are specially prepared.

A fair example of the condition of a piece cut from a sheet as received from the maker is shown by the measurements obtained on the lower half of a square of 1 inch side and approximately 0·251 inch in thickness. The variation with reference to a point for which $x = 0{\cdot}4$ and $y = 0{\cdot}2$ from the bottom left-hand corner is shown in the accompanying Table 3, and it is worthy of note that the principal differences occur in general along the edges, and

TABLE 3.

Variation of Thickness in a ¼-inch Plate.

One Unit = 10^{-5} inches.

—	0·01	0·05	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	0·95	0·99
0·5	−50	−18	− 6	− 6	− 8	− 5	+ 1	+ 8	+13	+16	+16	+ 8	−22
0·4	−42	− 9	+ 3	+ 3	− 1	+ 1	+ 3	+ 9	+13	+16	+20	+14	−15
0·3	−29	−11	+10	+ 7	0	− 3	− 4	− 3	− 2	+ 3	+ 8	+ 4	−21
0.2	—	+28	+26	+19	+ 9	0	− 6	−10	−12	− 9	−12	− 2	−16
0·1	+ 3	—	+39	+26	+12	+ 9	− 1	− 8	−11	− 9	0	0	−24
0·05	+23	+39	+57	+51	+38	+25	+14	+ 7	+ 1	+ 1	+11	+10	−13
0·01	−35	−11	+ 1	−11	−11	−21	−33	−39	−40	−48	−46	−51	−76

especially at the corners where the cut surfaces intersect. The exposure of a freshly-cut surface without subsequent polishing is, in fact, followed by a slight change at the edges due possibly to the escape of a small part of a volatile constituent. In a thick plate the effect is often noticeable from the appearance of a white line along the contour when the specimen is viewed in a polariscope. In thin specimens this effect is shown but rarely, even after the lapse of years.

In general, therefore, the stresses at the edges are somewhat more difficult to determine, but at a free contour the measurements are not complicated by the necessity of separating the principal stresses, since both $(p+q)$ and $(p-q)$ are equal, and the directions follow the contour.

The mechanical labour of recording all the measurements required proved far too great to allow of much progress, and in order to dispense with book-keeping, a cylindrical recorder was constructed having a barrel pivoted on a vertical axle and capable of vertical adjustment by aid of a lever operated by a screw, so that the zero can be adjusted as the extensometer is moved about. The positions of the reflected image are marked on a sheet of squared paper secured to the cylinder, and the differences obtained between the initial and final loads are measured between the corresponding curves drawn through the points of observation.

DISCUSSION OF THE POSSIBLE ERRORS AND CORRECTIONS OF THE MECHANICAL MEASUREMENTS.

Temperature Variations.—All the measurements were made in a room in the basement of the engineering wing of University College, where the temperature conditions vary somewhat, and this especially affects observations of change of thickness. This is especially noticeable during the winter months, when it is found that readings taken in a warm room on a Saturday differed materially from similar readings taken on the following Monday morning before the temperature conditions become normal. These differences disappear with the establishment of the working temperature of about 60° F. In some cases it has been found necessary to place a paper shield round the extensometer in order to avoid changes of temperature due to air currents.

Errors of Position.—The extensometer is supported independently of the specimen, and is registered thereto by reference to some convenient point or line when the initial load is applied. The readings are then taken for this load and also for a final load, which latter produces strains slightly altering the configuration of the specimen. In these experiments the diminution in the length of the central line and the changes in width of the specimen are both easily measurable, and by working with reference to this central line and the edge of the bottom pressure plate as axis, a correction can be applied to the position micrometer which renders this error practically negligible.

Corrections may also be required owing to the calibrating member for optical observations not being quite of the same thickness as the specimen under test, and also for the change in thickness caused by stress when both are compared, and the stress intensity measured by producing a dark field at the point required. This is most conveniently carried out with a tension member. If then the initial thickness of the specimen is x, and the calibrating member y, then at any point of the specimen the final thickness will be

$$x\left\{1 \pm \frac{p+q}{m\mathrm{E}}\right\}$$

where the $+$ sign is to be taken for all the cases examined here; while for the tension member the corresponding thickness is

$$y\left\{1 - \frac{f}{m\mathrm{E}}\right\}$$

where f is the corresponding stress.

When a black field is established, since the optical constants are the same for both specimens, we have

$$(p-q)\,x\left(1 + \frac{p+q}{m\mathrm{E}}\right) = fy\left\{1 - \frac{f}{m\mathrm{E}}\right\}$$

$$\text{or} \quad p - q = f\,\frac{y}{x}\left(1 - \frac{p+q+f}{m\mathrm{E}}\right) \quad \text{very nearly.}$$

In general $y = x$, although sometimes a slight difference occurs, while $(p+q)$ and f are rarely more than 1,500 lb. per square inch, and m and E have values of about 2·5 and 300,000 respectively. Hence $(p+q+f)\ /mE = 1/250$ or 0·4 per cent, so that occasionally corrections are required when high stress differences are found, but as a rule the error is small and vanishes entirely when $(p+q)$ and f are of opposite signs. The difficulty of providing a member in pure compression is, however, too great to make this a desirable feature of the measurements, as the results which follow show clearly.

Errors may also be caused by friction of the pivots, and of the measuring needles in their bearings, while minute scratches or irregularities at the surface show themselves by sudden jumps in the readings. If, however, the testing-machine used is gently tapped with a pencil, or, if massive, vibrated by the action of a small electric bell attachment, the instrument takes up the position due to the strain. The effect of a minute scratch sometimes renders it necessary to take readings on each side of the position, and to use these to obtain a value at the required point.

The usual tangential errors of the readings on the barrel are allowed for, and in some cases these are considerable. As an example an extreme case may be quoted, in which the reading h is 12 inches above the level of the reflecting mirror and the distance l is 32 inches. The movement of the mirror is approximately tan 0·175 or $\theta = 10° \ 37'$, and the error in assuming that $ld\theta = dh$ is $l\ (d \tan \theta) - ld\theta = l \tan {}^2\theta\ d\theta$, or very approximately $\tan {}^2\theta dh$. The error in reading is thus about 3 per cent. All readings are therefore corrected by reference to curves constructed for any required relation between h and l. In making this correction, the height of the reflecting mirror must be found relative to the readings on the drum, but any small error in the measurements of this height affects the correction by not more than one-tenth of the error made, and is therefore negligible. Moving the extensometer to different positions on the specimen also alters this height, but this can be corrected by the screw controlling the vertical position of the drum. A horizontal shift from the vertical plane passing through the spindle and the central position of the extensometer tends to increase the readings, since the image of the wire is received on a more distant part of the cylindrical drum. A movement of half an inch in either direction causes an error of one part in 2,000 for the drum used, which was $8\frac{3}{4}$ inches in diameter, and thus is insignificant unless the specimen is very wide. In this latter case the drum is arranged to move on a cross slide parallel with the extensometer. There are a number of possible minor errors, such as might occur owing to the recording cylinder not being vertical, or truly cylindrical, or due to imperfect ruling of the squared paper, but all these have been eliminated.

Early experiments showed some perplexing variations in the reading of the lateral extensometer under identical conditions, and these were found to be due to (1) errors in the dividing of the micrometer head, (2) errors in the screw, and (3) variations in the friction of the points and bearings of the measuring needles when moved over bounding surfaces of the specimen. The first was very much reduced by a newly graduated head, the second by using a very long screw and nut, and the last by increasing the rigidity of the supporting frame and careful rounding and polishing of the contacting areas.

Experiments were then made under good temperature conditions by two observers, who recorded their results quite independently, and a summary of these observations is given in the accompanying Table 4 (page 390), in which different positions of the screw and micrometer head are examined. It will be observed that the highest set of readings is obtained for the second position of the screw and the divisions 320°–360° of the head; and that the last three positions deal only with this range.

Taken as a whole, the deviations are small, and although the maximum difference from the mean is 2 per cent, the average difference from the mean is only 0·02 per cent for the whole series. The distance of the scale is taken much greater in some cases, to make the readings more sensitive, although this is usually accompanied by some loss of definiteness of the projected image of the cross wire.

DISCUSSION OF THE POSSIBLE ERRORS IN THE OPTICAL MEASUREMENTS.

An investigation * was also carried out to examine the mechanical properties of nitro-cellulose compounds used in this and other investigations and also the law of its optical behaviour, and it is material to the present experiments to remark that under tension stress the law of optical retardation is shown to be proportional to stress and not to strain, and that it is linear even when the material is stressed far beyond the elastic limit and very near the yield-point. Hence the optical measurements require no correction at high stresses, but the lateral measurements of the strain are affected thereby, and unless corrected from the stress-strain relation tend to give too high values. The polarizing and analysing Nicols, used throughout, were made by Ahrens, and are of a high degree of perfection. The arrangements for experimental work are of the usual kind, except that it is necessary to bring both the specimen under observation and the calibrating member into focus on the screen for comparison. We are indebted to Professor Filon, F.R.S., for suggesting an arrangement of lenses whereby this can be accomplished when the distance between the two pieces to be compared is considerable. This allows a much more simple arrangement of apparatus than would be required if both pieces were placed in juxtaposition, and it improves the definition of the projected images. Another matter to which attention may be called is that if the plane faces of the specimen are not placed perpendicularly to the plane polarized beam of light, but have the normals thereto inclined at a small angle θ to the direction of the ray, then the inclination of the path inside the material is $\frac{\theta}{\mu}$ and the change in the distance is $t \sec\frac{\theta}{\mu} - t = \frac{1}{2}\, t\, \frac{\theta^2}{\mu^2}$ if θ is small. If θ is taken as 5° and $\mu = 1\cdot5$ the change in optical path is $0\cdot2$ per

* "The Stress-strain Properties of Nitro-cellulose and the Law of its Optical Behaviour," by Prof. E. G. Coker, F.R.S., and K. C. Chakko, M.Sc. Trans. Roy. Soc., Phil. Series A, vol. ccxxi, 1920.

cent. It is believed that there is no appreciable experimental error from this cause.

There are probably small changes in the refractive indices due

TABLE 4.

Calibration of Lateral Extensometer.

Distance of Scale from reflecting Mirror, 51¾ ins.

Position of Screw.	Reading of Micrometer Head.	Readings.				Mean Values of Readings.	Percentage Difference from Mean Average Value.
		a.	b.	c.	d.		
	Degrees.						
I	50–90	144·9	144·3	144·0	--	144·4	+0
	140–180	142·0	142·2	142·6	—	142·3	−1·4
	230–270	144·9	144·2	144·2	—	144·4	+0
	320–360	144·4	141·8	143·3	142·8	143·1	−0·8
II	50–90	145·2	146·7	144·4	—	145·4	+1·0
	140–180	145·4	143·9	144·5	143·7	144·4	+0
	230–270	144·5	143·8	144·0	—	144·2	−0
	320–360	148·9	147·2	145·9	—	147·3	+2
III	320–360	144·0	143·4	142·8	144·1	143·6	−0·5
IV	do.	141·9	142·7	142·9	143·1	142·7	−1·1
V	do.	146·3	145·2	145·1	145·6	145·6	+1·0
		Average value of readings				144·3	
		Total numerical error . . .					7·8
		Mean numerical error . .					0·7=0·5%

to changes of temperature, but as this latter did not vary more than 6° F. while measurement was in progress, the effect is probably insignificant.

Errors due to defects in loading have been avoided as far as possible by adjustment of the stressing apparatus until the optical effects show a high degree of symmetry. Small initial stresses are sometimes found in the material, which latter was in all cases examined before cutting into shape, and only the most perfect pieces were used.

In plane stress, or generalized plane stress, such as is dealt with here, the distribution is completely determined when p, q, θ_p and θ_q are known. The directions are comparatively easy to measure, since a stressed plate in plane polarized light shows dark bands which mark these directions with respect to the principal planes of the polarizer and the analyser. The experimental solution is, therefore, complete and independent so long as the law of optical retardation holds and the lateral strains can be interpreted as linear functions of stress or corrected for the deviations from a linear law.

It may also be noted that the stress distributions obtained in all the various cases examined in this Paper may be directly applied to similar cases in other materials, since the fundamental equation $\nabla^4 \chi = o$ for plane stress in simply connected bodies contains no elastic constants, and the stress components are derived from this function χ in the form of partial differential coefficients, in which the co-ordinates are the independent variables.

Experimental evidence on this point has already been given,* and need not, therefore, be referred to further here.

STRESS DISTRIBUTION IN BLOCKS OF EQUAL BREADTH AND DEPTH WHEN SUBJECTED TO LOAD OVER A PART OF ONE EDGE AND SUPPORTED OVER THE OPPOSITE EDGE.

The distribution of stress in finite blocks of rectangular form when subjected to load is of fundamental importance, since the masonry and brick-work structures of the engineer and architect

* "The Optical Determination of Stress." E. G. Coker, Phil. Mag. 1910.
"Photo-Elastic and Strain Measurements of the Effects of Circular Holes on the Distribution of Stress in Tension Members." E. G. Coker, K. C. Chakko and Y. Satake. Trans. Inst. of Engineers and Shipbuilders in Scotland, 1919.

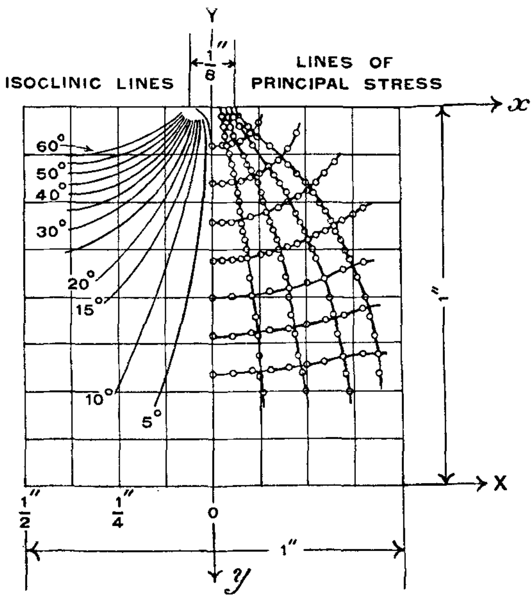
are essentially of this type, with pressures distributed over two opposed faces and generally over the whole area of each face, but in other cases where a great load is to be borne, as in columns, this is concentrated over a part of the upper face of the supporting base.

The earlier experiments described above showed that the stresses at the contact areas are not distributed uniformly over either face, and it is probable that a diminution in the size of the block will not change the essential character of the phenomena, although the distribution may be very different. This is borne out by the appearance presented by a block of square form 1 inch side and $\frac{1}{8}$ inch thick when loaded over the central part of one of its rectangular faces. Fig. 11, Plate 16, shows the appearance of such a block when a total load of 50 lb. is applied over a width of $\frac{1}{4}$ in. by means of a pressure plate of the same material. The colour bands there shown are not true circles, but are somewhat distorted curves passing through the extreme points of the upper block, and there are indications of extreme stress at these end-points, since subsidiary loops of colour appear with these end-points as origins.

In order to find the stress distributions for these cases, the principal stresses and their directions have been measured over so much of the areas of the blocks as is practicable.

As will be observed in the preceding Figure, there are areas neighbouring the upper angles for which there is practically no stress, as a dark field is shown for all angular positions of the plane polarized beam. The light band at the extreme edge appears to be due to a bending effect. In the other parts of the field the isoclinic bands are quite distinct except immediately under the load, where there is some difficulty in defining them. The positions of a sufficient number to determine the lines of principal stress are marked on the left hand side of the accompanying Fig. 12, and from any convenient point of this diagram tangent lines are drawn to the curves and the co-ordinates at the limits of a small square are found from the relation $\delta y = \delta x \times \tan \alpha$, where α is the inclination of the line of stress. Proceeding step by step in this manner, the co-ordinates of the curves of principal stress are found, Table 5, and are plotted on the right hand side of Fig. 12. It will be observed that they bear little resemblance to the case of uniform load applied to a limited portion of a semi-infinite plate, except immediately in the neighbourhood of the applied load. In the lower half of the plate, where observation is more difficult, owing to the smallness of the stress, the lines radiating from the upper contact area bend over and probably meet the lower edge very nearly

FIG. 12.



perpendicularly. The magnitudes of the stresses are also determined by observations of $(p + q)$ and $(p - q)$ and these are shown in Table 6 (page 398), from which it will be remarked that the stress distribution is found at a line one-hundredth of an inch away from the contact area, which is very nearly the limit of approach for the measuring needles. Along this line the directions of principal stress may not be quite perpendicular and parallel to the contour owing to unequal surface strains when two unlike rectangular blocks of the same material are present together, but the observations

TABLE 5.—*Co-ordinates of Lines of Principal Stress for 1 inch × 1 inch × ¼ inch Block with Pressure Face ⅛ inch wide.*

Curves System "A."

X (in.)	Y (inch).						
0	0·900	0·800	0·700	0·600	0·500	0·400	0·300
0·05	0·906	0·804	0·703	0·602	0·502	0·401	0·301
0·10	0·930	0·816	0·712	0·608	0·507	0·406	0·304
0·15	—	0·844	0·727	0·618	0·514	0·412	0·309
0·20	—	0·888	0·749	0·632	0·524	0·420	0·316
0·25	—	—	0·783	0·651	0·537	0·431	0·325
0·30	—	—	0·829	0·675	0·553	0·442	0·366
0·35	—	—	—	0·703	0·573	0·455	0·347
0·40	—	—	—	0·737	0·593	0·468	0·356

Curves System "B."

Y (in.)	X (inch).			
0·40	0·115	0·223	0·334	0·427
0·45	0·109	0·216	0·322	0·416
0·50	0·103	0·206	0·309	0·404
0·55	0·097	0·195	0·293	0·386
0·60	0·090	0·183	0·275	0·365
0·65	0·083	0·168	0·255	0·343
0·70	0·076	0·153	0·233	0·316
0·75	0·068	0·135	0·208	0·284
0·80	0·059	0·115	0·178	0·246
0·85	0·049	0·092	0·144	0·204
0·90	0·039	0·069	0·108	0·153
0·95	0·030	0·049	0·072	0·098

appear to show that the error in taking these lines perpendicular and parallel to the contact surface must be small, while along the remainder of the upper edge these conditions are fulfilled exactly. The distribution of normal stress along this section is therefore shown approximately by the value of p, and, as Table 6 shows, the stress over this contact area is very much concentrated at the ends, as the colour bands indicate. This kind of distribution is obtained when punching a plate, owing to the centre part under the punch bending and forming a cup-shaped depression. The cross stress rises to a maximum at the centre and dies away completely at the ends, in fact changes at 0·01 inch below the surface, so that it seems probable that this distribution of stress is accompanied by a corresponding distribution of shear along the contact surface.

As soon as the distance from the upper plane becomes of the same order as the breadth of the contact surface, the value of q ceases to be of much significance. This is shown by the curves of $(p+q)$ and $(p-q)$, Fig. 13 (page 396), which are nearly identical except within the range indicated above.

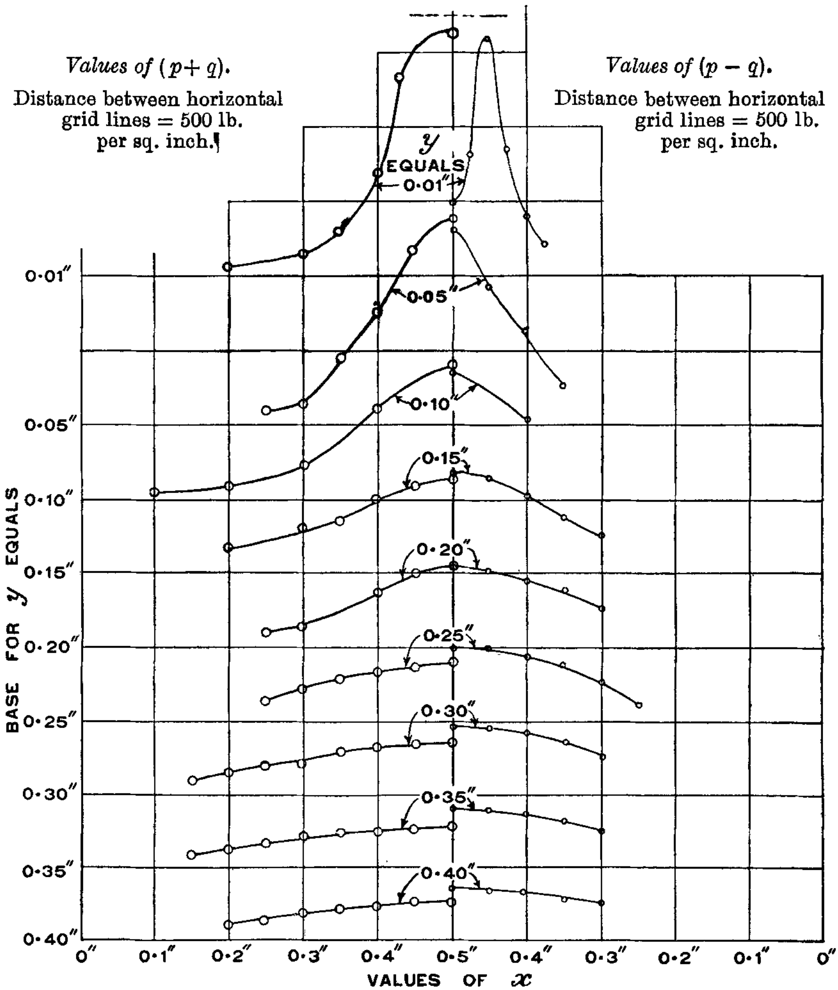
In order to obtain the normal stress distribution p_n over planes parallel to the surface, the values of p_n are calculated from the expression—

$$p_n = p \cos^2 \theta + q \sin^2 \theta$$

and are plotted in Fig. 14. These latter curves also permit the accuracy of the measurements to be checked by reference to the load, and with the exception of the last two planes the error is found to be less than $\pm$ 5 per cent, as the Table indicates.

On parallel planes more remote from the surface, the minor principal stress rapidly loses any importance and becomes insignificant at distances of more than 0·125 inch from the top surface, until the influence of the distribution at the lower surface is felt. This is indicated by the curves Fig. 13 (page 396), in which the maximum values at intermediate planes tend towards coincidence and then diverge again as the distance increases. Part of this divergence is possibly attributable to experimental error, as there is greater difficulty in determining small stresses optically as compared with large ones, and the presence of small initial stresses is

Fig. 13.


Values of (p+q).
Distance between horizontal grid lines = 500 lb. per sq. inch.
Values of (p − q).
Distance between horizontal grid lines = 500 lb. per sq. inch.
y EQUALS
0·01″
0·05″
0·10″
0·15″
0·20″
0·25″
0·30″
0·35″
0·40″
BASE FOR y EQUALS
0·01″
0·05″
0·10″
0·15″
0·20″
0·25″
0·30″
0·35″
0·40″
0″ 0·1″ 0·2″ 0·3″ 0·4″ 0·5″ 0·4″ 0·3″ 0·2″ 0·1″ 0″
VALUES OF x

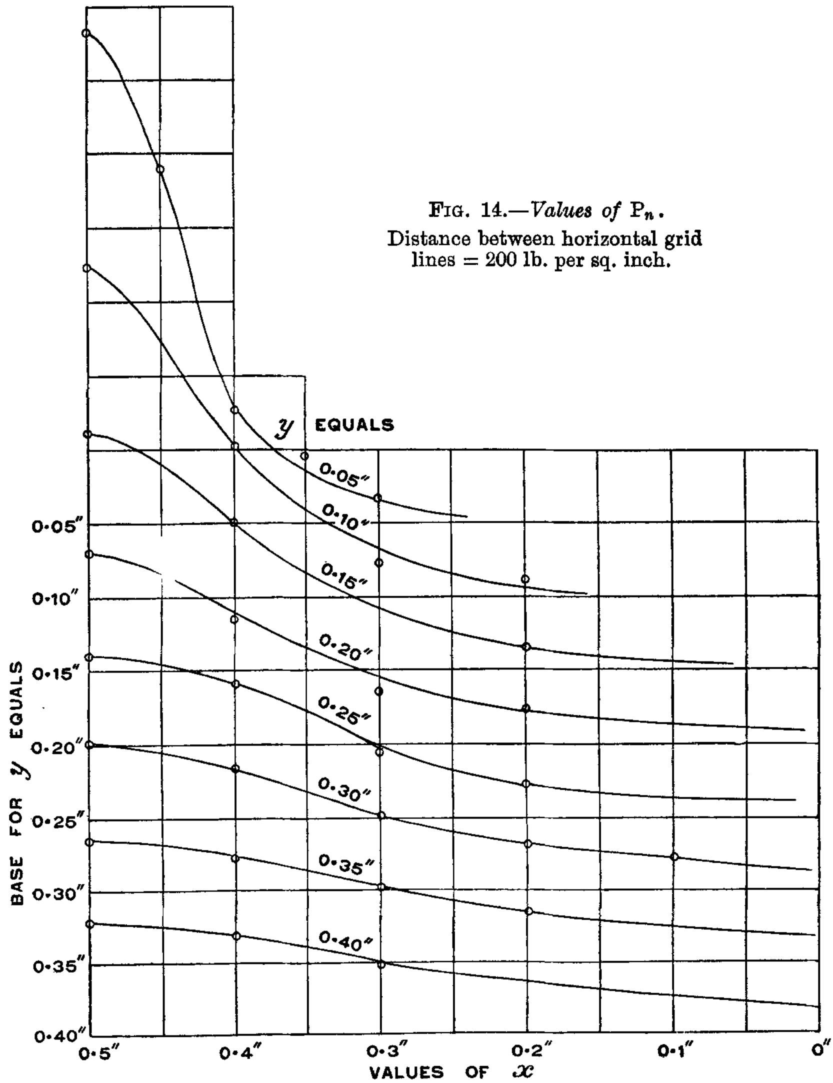


FIG. 14.—*Values of* P_n.
Distance between horizontal grid lines = 200 lb. per sq. inch.

TABLE 6.—*Table of Stresses.*

Actual load 50 lb.

y	x	θ	$p+q$	$p-q$	Cos 2θ	$(p-q)\times$ Cos 2θ	p_n	p_x	Measured Load.	Error.
Inch.	Inch.	Degrees.							lb.	Per cent.
0·01	0·2	—	50	—						
	0·3	—	120	—						
	0·35	—	290	—						
	0·4	—	680	412						
	0·45	—	1,540	1,648						
	0·5	—	1,640	474						
0·05	0·3	—	138	—	—	—	—	—	52·0	4
	0·35	62	425	249	−0·242	−60	183	242		
	0·40	50	721	619	−0·174	−108	307	414		
	0·45	15	1,103	935	0·866	809	956	147		
	0·50	0	1,376	1,289	1·0	1,289	1,332	44		
0·10	0·1	60	40	—	−0·50	—	—	—	50·6	1·2
	0·2	63	90	—	−0·588	—	—	—		
	0·3	57	210	—	−0·407	—	—	—		
	0·4	33	590	556	0·407	226	407	183		
	0·5	0	900	876	1·0	876	888	12		
0·15	0·1	50	—	—	—	—	—	—	49·1	−1·8
	0·2	53	112	—	−0·276	—	—	—		
	0·3	45	289	242	0	0	144	—		
	0·4	25	483	530	0·643	341	412	71		
	0·5	0	628	657	1·0	657	643	−15		

	0·1	40	—	—	—	—	—	—		
	0·2	43	—	—	—	—	—	—		
0·20	0·3	35	190	245	0·342	84	136	54	47·8	−4·3
	0·4	19	344	420	0·788	330	337	7		
	0·5	0	509	520	1·0	520	515	−6		
	0·1	35	—	—	—	—	—	—		
	0·2	36	80	—	0·309	—	—	—		
0·25	0·3	27	204	244	0·588	144	174	30	48·2	−3·6
	0·4	14	346	414	0·883	365	356	−10		
	0·5	0	390	485	1·0	485	438	−48		
	0·1	31	90	—	0·469	—	—	—		
	0·2	30	175	—	0·50	—	—	—		
0.30	0·3	22	213	249	0·719	179	196	17	48·4	−3·2
	0·4	11	299	398	0·922	368	333	−34		
	0·5	0	341	450	1·0	450	396	−55		
	0·1	28	75	—	0·559	—	—	—		
	0·2	25	150	—	0·643	—	—	—		
0·35	0·3	19	204	249	0·788	194	199	5	47·3	−5·4
	0·4	10	239	350	0·940	329	284	−45		
	0·5	0	272	387	1·0	387	330	−58		
	0·1	23	50	—	0·695	—	—	—		
	0·2	22	112	—	0·719	—	—	—		
0·40	0·3	16	176	248	0·848	210	193	−17	46·9	−6·2
	0·4	8	228	320	0·961	308	268	−40		
	0·5	0	255	350	1·0	350	302	−47		

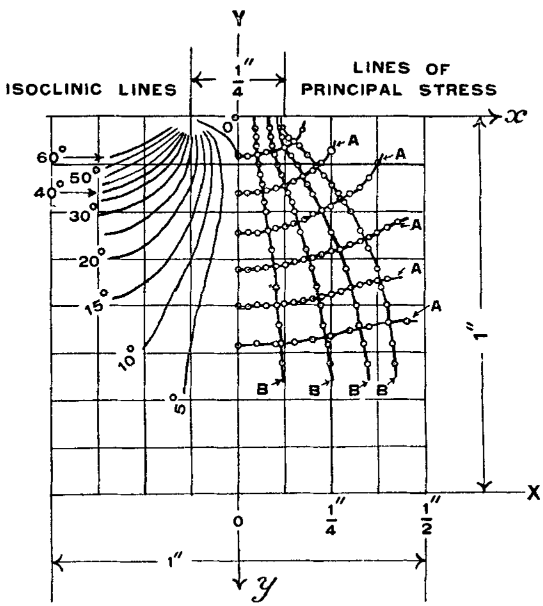
TABLE 7.—*Co-ordinates of Lines of Principal Stress for* 1 *inch* × 1 *inch* × ¼ *inch Block with Pressure Face* ¼ *inch wide.*

Curves System "A."							Curves System "B."				
X (in.)	Y (inch).						Y (in.)	X (inch).			
0	0·900	0·800	0·700	0·600	0·500	0·400	0·40	0·113	0·239	0·336	0·411
0·05	0·902	0·802	0·702	0·601	0·501	0·401	0·45	0·110	0·229	0·325	0·404
0·10	0·912	0·812	0·708	0·605	0·504	0·403	0·50	0·105	0·219	0·313	0·396
0·15	0·937	0·831	0·721	0·614	0·509	0·407	0·55	0·100	0·208	0·300	0·382
0·20	—	0·860	0·739	0·626	0·518	0·414	0·60	0·095	0·196	0·285	0·367
0·25	—	0·914	0·764	0·642	0·529	0·422	0·65	0·089	0·182	0·268	0·348
0·30	—	—	0·797	0·662	0·543	0·433	0·70	0·082	0·166	0·248	0·326
0·35	—	—	0·845	0·684	0·558	0·443	0·75	0·075	0·150	0·225	0·300
0·40	—	—	—	0·707	0·573	0·453	0·80	0·067	0·132	0·199	0·268
0·45	—	—	—	—	—	0·458	0·85	0·059	0·113	0·171	0·229
0·50	—	—	—	—	—	0·460	0·90	0·053	0·097	0·143	0·186
							0·95	0·050	0·085	0·118	0·140

more noticeable. The observations in fact agree very well with the load applied for planes near the upper surface, as the error column of Table 6 shows, but at the extreme planes of observations the error becomes comparatively large.

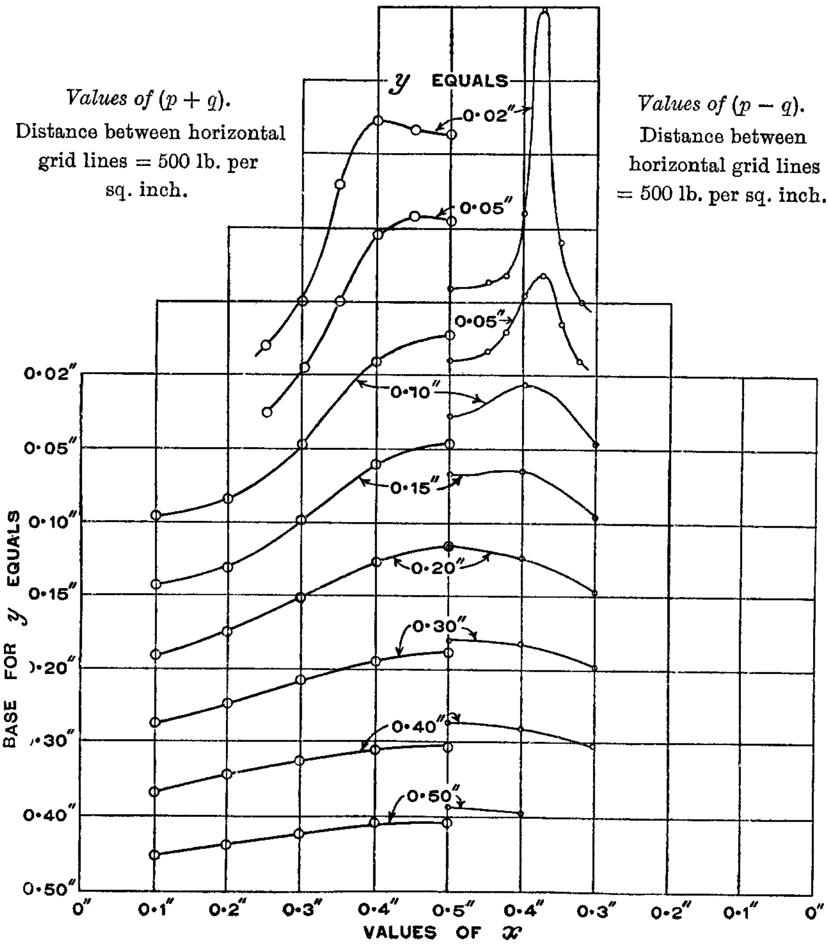
To obtain further evidence on these matters, the load was increased and distributed over a correspondingly greater width of $\frac{1}{4}$ inch as

FIG. 15.



shown in Fig. 15 to prevent the local failure of the material. In this latter case, the non-uniform character of the load distribution is very marked, as the observations at $y=0\cdot02$ inch show (Figs. 16 and 17 and Table 8). At the centre the normal stress is rather more than 1,100 lb. per square inch and rises to nearly 1,500 lb. per square inch immediately under the extreme edge of the upper pressure plate, while there is a cross stress of about 525 lb. per square inch at the centre, which diminishes to zero and changes sign very abruptly in the neighbourhood of this maximum normal stress. Effects of

FIG. 16.



Values of (p + q).
Distance between horizontal grid lines = 500 lb. per sq. inch.
Values of (p − q).
Distance between horizontal grid lines = 500 lb. per sq. inch.
y EQUALS
0·02″
0·05″
0·05″
0·10″
0·15″
0·20″
0·30″
0·40″
0·50″
BASE FOR y EQUALS
0·02″
0·05″
0·10″
0·15″
0·20″
0·30″
0·40″
0·50″
0″
0·1″
0·2″
0·3″
0·4″
0·5″
0·4″
0·3″
0·2″
0·1″
0″
VALUES OF x

Fig. 17.—*Values of* P_n.
Distance between horizontal grid lines = 200 lb. per sq. inch.

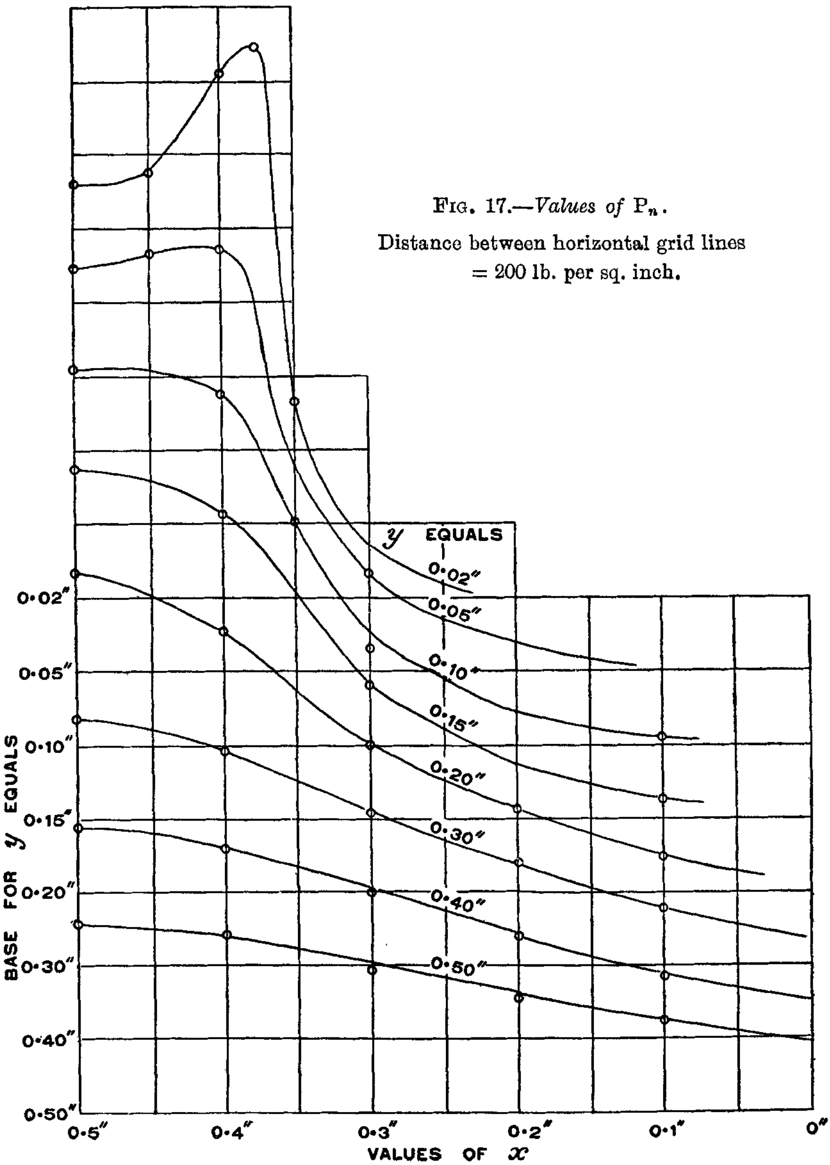


TABLE 8.—*Table of Stresses*

Actual Load 100 lb.

y	x	θ	$p+q$	$p-q$	Cos 2θ	$(p-q)\times$ Cos 2θ	p_n	p_x	Measured Load.	Error.
Inch.	Inch.	Degrees.							lb.	Per cent.
0·02	0·3	75	475	412 (?)	−0·866	−357	59	416	100·65	0·65
	0·35	55	1,400	927	−0·342	−317	541	859		
	0·375	30	1,675	2,575	+0·500	1,287	1,481	194		
	0·40	10	1,750	1,133	0·940	1,071	1,410	340		
	0·45	0	1,680	628	1·000	628	1,154	526		
	0·50	0	1,650	598	1·000	598	1,124	526		
0·05	0·25	70	240	—	−0·766	—	—	—	98·4	−1·6
	0·3	60	545	—	−0·5	—	—	—		
	0·35	50	1,000	955	−0·174	−167	416	584		
	0·4	20	1,450	1,056	0·766	809	1,127	323		
	0·45	2	1,560	679	0·998	679	1,120	440		
	0·5	0	1,550	618	1·000	618	1,084	466		
0·10	0·1	65	48	—	−0·643	—	—	—	94·5	−5·5
	0·2	60	180	—	−0·5	—	—	—		
	0·3	45	565	546	0	0	282	283		
	0·4	17	1,100	948	0·829	788	944	156		
	0·5	0	1,270	742	1·000	742	1,006	264		

	0·1	55	90	—	−0·342	—	—	—		
	0·2	48	220	—	−0·105	—	—	—		
0·15	0·3	35	530	530	0·342	186	358	172	95·7	−4·3
	0·4	16	905	850	0·848	744	825	80		
	0·5	0	1,040	845	1·0	845	942	98		
	0·1	30	95	—	0·5	—	—	—		
	0·2	30	250	—	0·5	—	—	—		
0·20	0·3	28	490	530	0·559	305	398	92	95·8	−4·2
	0·4	15	730	760	0·866	678	704	26		
	0·5	0	840	875	1·0	875	858	−18		
	0·1	25	150	—	0·643	—	—	—		
	0·2	25	285	—	0·643	—	—	—		
0·30	0·3	21	440	495	0·743	379	410	30	92·3	−7·7
	0·4	11	550	648	0·927	618	584	−34		
	0·5	0	610	721	1·0	721	666	−56		
	0·1	18	170	—	0·809	—	—	—		
	0·2	18	278	—	0·809	—	—	—		
0·40	0·3	15	365	475	0·866	423	394	−29	90·3	−9·7
	0·4	7	440	595	0·970	595	518	−78		
	0·5	0	475	670	1·0	670	572	−97		
	0·1	14	240	—	0·883	—	—	—		
	0·2	14	310	—	0·883	—	—	—		
0·50	0·3	11	390	—	0·927	—	—	—	93·9	−6·1
	0·4	5	435	525	0·985	534	485	−50		
	0·5	0	450	577	1·0	577	513	−63		

the same general character are observed up to a distance of nearly 0·1 inch from the edge, but from this distance onwards both $(p+q)$ and $(p-q)$ have maximum values at the central section. All the observations, in fact, go to show a distribution of load which is far from uniform both at the upper and lower surfaces of the block, while the stress conditions which obtain in the interior are largely conditioned by these and the proximity of the sides.

DISTRIBUTION OF STRESS IN BLOCKS LOADED OVER THE WHOLE EXTENT OF THEIR UPPER AND LOWER SURFACES.

The importance of this type of loading has been noted earlier, and the cases already examined naturally lead to the inquiry whether it is possible to so load a short rectangular block as to produce uniform compression throughout. It is sometimes assumed, where a central load is applied to the ends of such a block, by thick distributing plates such as are usual in most forms of testing machines, that it is uniformly stressed throughout, but this is not usually the case, for if the materials pressed together are of a different nature, the stress distribution is apparently always non-uniform, no matter how well the faces may be prepared. This is illustrated by the case of a square block pressed between brass plates, which shows zones of non-uniform stress at the ends which only disappear when the distance from the contact surface is appreciable. The pressure between these dissimilar materials apparently gives rise to considerable tangential stress at the common boundary and lateral expansion is prevented. If, however, a very extensible material, such as a thin sheet of rubber, is interposed, the tangential stress is reversed and the lateral strains become abnormally great, especially near the central vertical section; they are, however, fairly uniform throughout the depth. The tangential forces over the end boundaries produce considerable tensional stresses in the material in the direction of the thickness of the plates. This disruptive effect has been frequently observed in compression tests of stone blocks, when the faces under load are covered by thin sheets of lead.* The experimental evidence, therefore, goes to show that in order to obtain pure compression stress in a block the contact faces must be similar and equal, and of the same material. These conditions are approximately fulfilled by interposing similar end plates of sufficient length, as may be shown in the polariscope when a square is compressed between two plates of like material and of the same cross section. When loaded in this manner an approximately pure compression stress is obtained, but even with these precautions it may not be perfect, as the experimental analysis of Table 9 (page 413) shows. With a total load of 260 lb., corresponding to a mean stress of 1,030 lb. per square inch, there is nearly pure compression stress at the upper and lower bounding surfaces, although not quite uniform, as the plotted values of Fig. 18 indicate. The cross stress q is insignificant, but towards the centre this latter becomes an appreciable tension, and has an approximately parabolic distribution as the Figure shows, with a maximum value at the centre of the block of rather more than 5 per cent of the numerical value of the applied pressure.

A block of one-half this depth gives rather better results, and when loaded in this manner almost perfectly pure compression stress is obtained.

THE INFLUENCE OF CONTACT STRESSES IN THE TESTING OF MATERIALS.

It is a well-known and often disconcerting circumstance in experiments on the tensile strength of materials, that metal bars and rods often fracture at the gauge-points or even outside these limits, especially in hard materials. The reasons for the latter type of fracture have been examined recently,† but so far as the Authors are aware the exact effect of the minute indentations required for

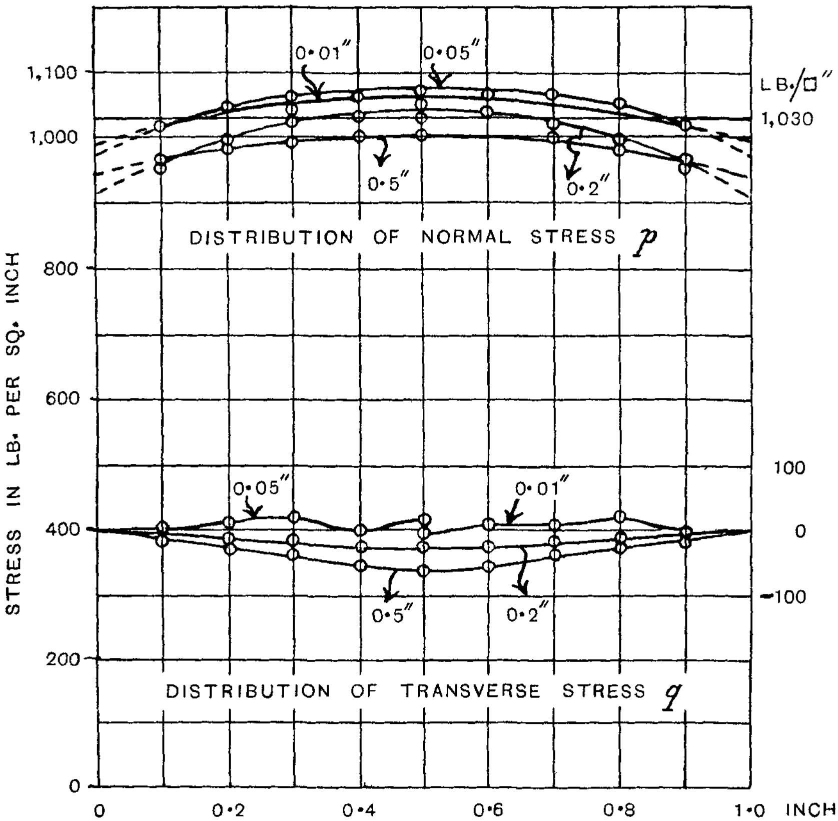
* "The Testing of Materials." W. C. Unwin, F.R.S. 1st edition, page 417.

† "Photo-Elastic Measurements of the Stress Distribution in Tension Members used in the Testing of Materials." E. G. Coker, Proc., Inst.C.E., vol ccviii, Part ii, 1918–19.

the attachment of extensometers has not been examined. Although fractures at the gauge-points are frequent in unyielding material,

FIG. 18.—*Pressure Distribution in* 1 *inch* × 1 *inch* × 0·254 *inch Block with Secondary Pressure Plates* 0·4 *inch deep.*

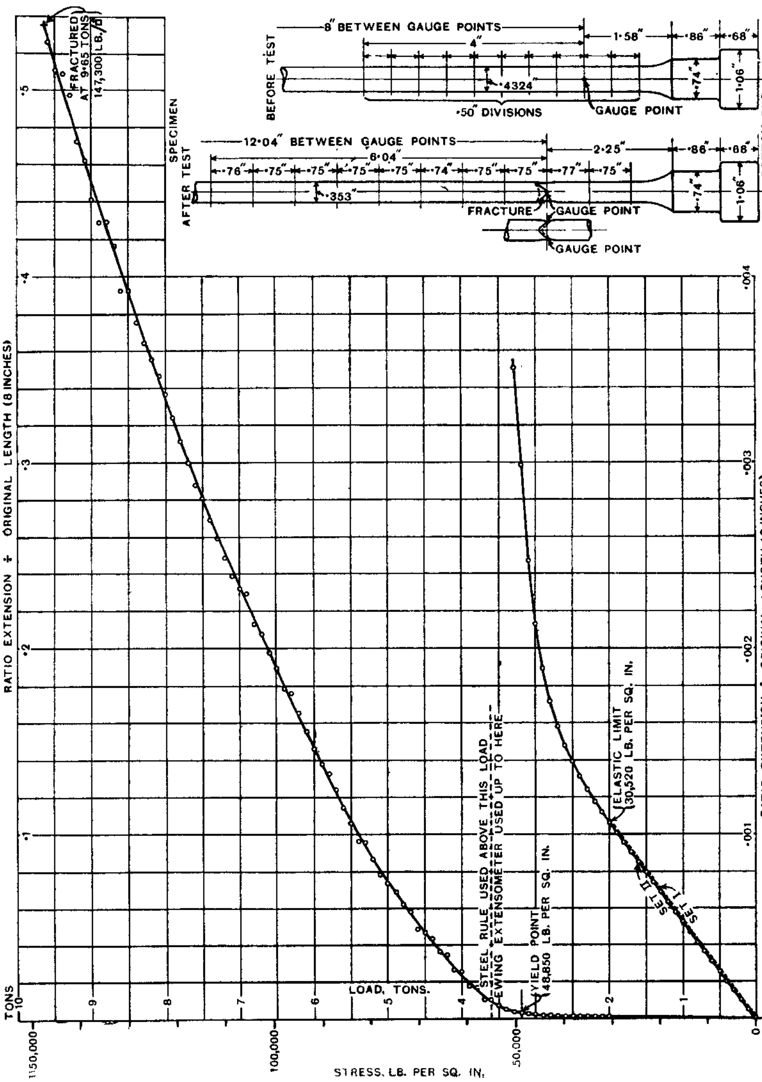
Load Applied 260 lb.



they also occur in cases where the substance becomes very plastic, and this is well shown by a specimen of forged manganese steel kindly sent by Sir Robert Hadfield for examination. This specimen was in the form of a round bar with enlarged ends, Fig. 19, machined to fit the shackles of a 10-ton Buckton testing-machine

FIG. 19.—*Hadfield's Test-Bar, A8725.*

Initial Condition of Specimen: Surface finished smooth by grinding.



at University College. Owing to the high tensile strength of the material, it was found necessary to make the specimen of rather small diameter, in order to carry the test to completion. The salient features of the test are recorded in Fig. 19 of the load

Data of Fig. 19 (page 409).

——	Initial.	Final.
	Inches.	Inches.
Distance between Gauge-Points . .	8·00	12·04
Mean Diameter	0·4324	0·3532
Modulus . . (between 0·2 and 1·0 ton)	$28 \cdot 5 \times 10^6$ = lb. per sq. inch	
Elastic Limit . . . . (2 tons)	30,520 lb. per sq. inch	
First Yield-Point . . . (3·2 tons)	48,850 lb. per sq. inch	
Breaking Stress . . . 9·65 tons)	147,300 lb. per sq. inch	
Equivalent Elongation in 8 ins. . (4·08 ins.)	51 per cent	
Minimum Diameter at Fracture . .	0·323 inch	
Reduction in Area	43 per cent	
Ultimate Stress at Minimum Section away from Point of Fracture. (Diam. 0·35 in.)	224,650 lb. per sq. inch	
Character of Fracture .	Fibrous with silky lustre.	
Place of Fracture . .	Lower Gauge-Points.	
Temperature of Laboratory	55° F.	
Date of Test . . .	1st April 1920	

extension diagram, from which it appears that this material has a high elastic limit and yield-point, and a great capacity in the plastic condition for stretching under load, coupled with a very high ultimate stress. The remarkable quality of this material is indicated by the data obtained, from which it will be observed that the total elongation at fracture, measured on an 8-inch length, amounted to 51 per cent, and was, moreover, distributed very uniformly throughout its length, as the Figure shows. The specimen broke at a load of 9·65 tons, corresponding to a stress of 147,300 lb. per square inch of the original section and 224,650 lb. per square inch of the actual section at fracture, which latter occurred at a gauge-point, although every care possible was taken to avoid this by using very minute centres for the support of the Ewing extensometer employed.

The test, therefore, does not show the true strength of the material at fracture, owing to the effect of the small indentations, causing a break at a definite place fixed by the conditions imposed by the test. It is also possible that the cross-sectional area at fracture is affected by the same cause.

This example is typical of the general difficulty which occurs in this kind of testing, and it has led to various devices for avoiding indentations or scratches in test-pieces. The problem, in fact, presents a case of contact stress of practical importance, which appears to warrant further investigation, in order to ascertain the changes in stress distribution caused by various methods used in determining the tensile strength of materials. A form of gripping device to imitate the action of point or line grips was therefore constructed, in which the pressure, applied laterally, could be measured. This consisted of a U-shaped frame, shown in side and end elevation, Fig. 20, to grip a tension member A. The frame B is made in two parts connected by a thin steel plate fulcrum C, and at a convenient distance from this latter there are a pair of adjustable screws D, the inner ends of which are bored out to receive various types of gripping devices. Those shown in the Figure are small cylindrical steel pins, but others were also used, such as knife-edges, cylinders and flat plates of various widths.

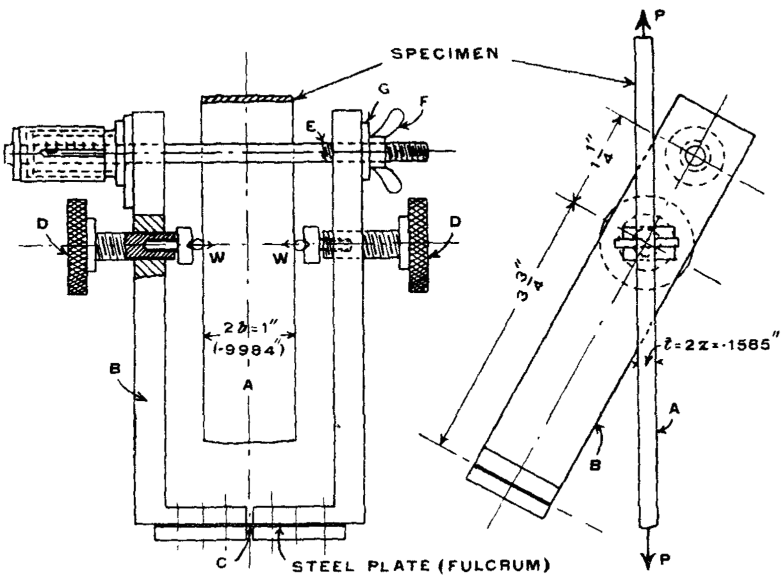
A measurable load is applied to the grips by a rod E screwed at one end and furnished with an adjustable nut F bearing against a pivoted washer G to ensure axial loading. At the opposite end, the rod E is attached to a calibrating spring, the compression of which measures the applied load, while a micrometer scale, not shown on the diagram, indicates when the long arms of the frame B are exactly parallel, and therefore producing oppositely directed loads. For convenience the holes in the frame B are arranged so that the screws D and the load measuring device E, F, G, can be interchanged to

FIG. 20.—*Loading Arrangement.*

Dimensions of Specimen . .	0·9984 inch × 0·1585 inch.
Comparison Test-piece . .	0·3750 inch × 0·1573 inch.

Thickness assumed for Calculation of Stresses (Standard) 0·1 inch.

Width of Specimen taken as 1·000 inch.



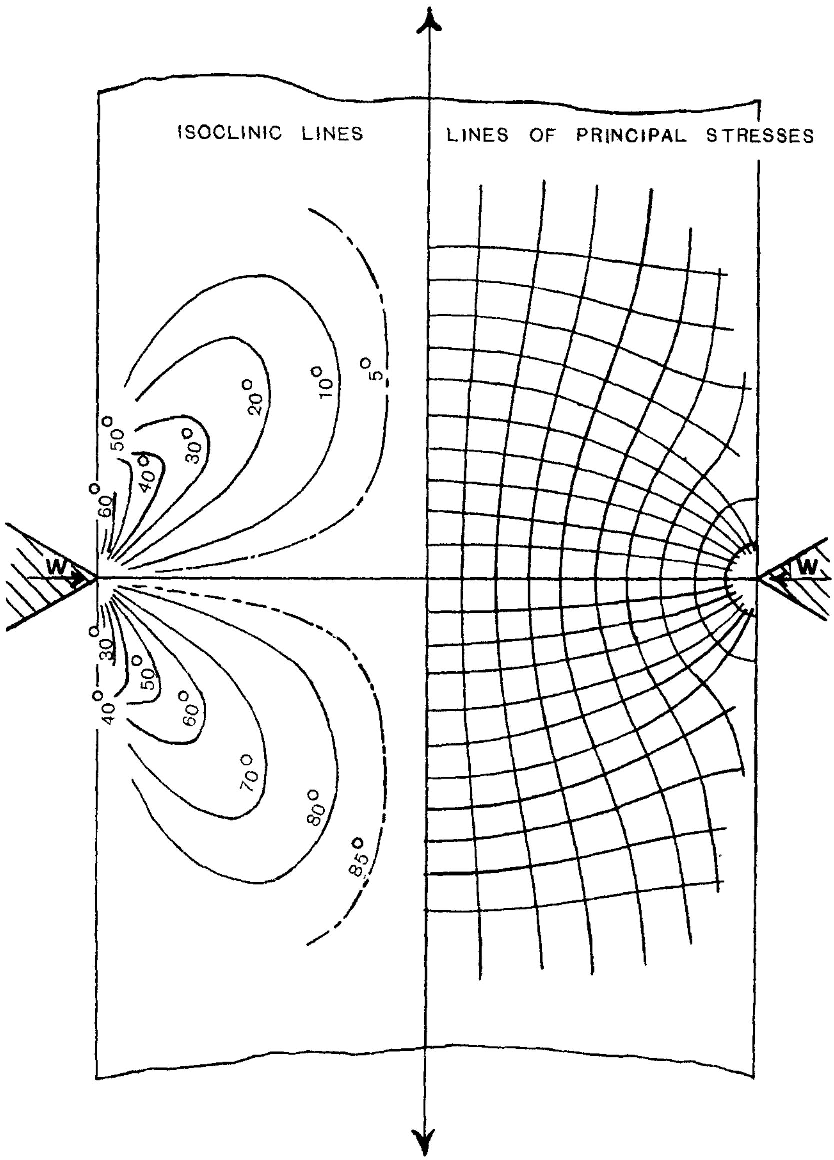
give a greater range for experiment. The weight of the gripping device is also balanced by counterpoise weights applied to the screws D.

The lateral loads due to screw points gripping the specimens not only injure the specimen and often determine the place of fracture, but they may also change the stress distribution, and this is well shown in Fig. 21 (page 414), in which lateral pressures of 15 lb., applied by knife-edges having an angle of 60°, change the

TABLE 9.—*Measurements on* 1 *inch* × 1 *inch* × 0·252 *inch Block with Secondary Pressure Plates* 0·4 *inch deep. Applied Load* 260 *lb.*

y	x	θ	$p+q$	$p-q$	p	q	Measured Load.	Error.
In.	In.	Degs.	lb.	lb.	lb.	lb.	lb.	Per cent.
	0·1	0	1,001	1,027	1,014	−13		
	0·2	,,	1,069	1,038	1,054	+16		
0·01	0·3	,,	1,042	1,038	1,040	+ 2	262·5	+1·0
	0·4	,,	1,069	1,060	1,065	+ 5		
	0·5	,,	1,050	1,060	1,055	− 5		
	0·1	0	990	1,027	1,009	−19		
	0·2	,,	1,075	1,038	1,057	+19		
0·02	0·3	,,	1,077	1,088	1,058	+20	262·9	+1·1
	0·4	,,	1,068	1,060	1,064	+ 4		
	0·5	,,	1,036	1,060	1,048	−12		
	0·1	0	1,020	1,016	1,018	+ 2		
	0·2	,,	1,042	1,027	1,035	+ 8		
0·05	0·3	,,	1,089	1,038	1,064	+26	263·0	+1·2
	0·4	,,	1,045	1,060	1,053	− 8		
	0·5	,,	1,084	1,060	1,072	+12		
	0·1	0	912	1,006	959	−47		
	0·2	,,	1,012	1,016	1,014	− 2		
0·10	0·3	,,	1,062	1,038	1,050	−38	253·4	−2·6
	0·4	,,	1,054	1,060	1,057	− 3		
	0·5	,,	1,035	1,060	1,048	−13		
	0·1	0	111	995	953	−42		
	0·2	,,	960	1,016	988	−28		
0·20	0·3	,,	1,001	1,038	1,020	−19	253·3	−2·6
	0·4	,,	1,004	1,060	1,032	−28		
	0·5	,,	1,046	1,060	1,053	−27		
	0·1	0	917	995	956	−39		
	0·2	,,	948	1,006	977	−29		
0·30	0·3	,,	969	1,038	1,004	−35	248·3	−4·5
	0·4	,,	1,007	1,060	1,034	−27		
	0·5	,,	1,012	1,060	1,036	−24		
	0·1	0	906	984	945	−39		
	0·2	,,	923	1,006	970	−42		
0·40	0·3	,,	957	1,038	998	−41	248·0	−4·6
	0·4	,,	964	1,060	1,012	−48		
	0·5	,,	960	1,060	1,010	−50		
	0·1	0	938	984	961	−23		
	0·2	,,	942	1,006	974	−32		
0·50	0·3	,,	942	1,038	990	−38	248·0	−4·6
	0·4	,,	942	1,060	1,001	−59		
	0·5	,,	929	1,060	995	−66		

Fig. 21.—*Isoclinic Lines and Lines of Principal stresses in a Tension Member.*

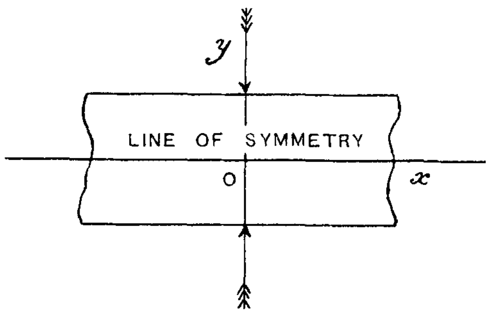


character of a simple tension stress due to a total load of 45 lb. into a distribution of considerable complexity. In this case the lines of equal inclination, determined experimentally, are shown on the left-hand side, and from these are drawn the orthogonal curves of principal stress. In a simple tension test, these lines should be parallel and perpendicular to the straight contours of the specimen, and the actual lines show how very great is the change in the type of stress due to the lateral pinch, and demonstrate that simple tensional stress is only regained at an axial distance from the loading of rather more than half the width of the bar. The experimental values in this case are important as indicating the character of the disturbance in the stress conditions, and why failure is likely to occur at the section under the combined effects of the lateral stress due to a permanent indentation combined with longitudinal pull.

The results obtained encouraged the Authors to proceed further with this part of the investigation, and it was resolved to measure the distributions obtained with various forms of contacts applied laterally to the bar, especially as a cylindrical roller contact had been found very useful in testing the mechanical properties of thin strip steel, used for aeroplane structures during the war. Such material is especially difficult to test, as a scratch or indentation generally determines the place of fracture. In order, therefore, to measure the extension of the material, an arrangement was devised which was found to work satisfactorily. Clips were provided, on the inner sides of which were slots containing cylindrical pins, housed so that they projected a very slight amount beyond the inner faces of each member of the clip, and these pins were pressed against the specimen when the clips were drawn together by small bolts. This arrangement gave a sufficient grip to support an extensometer from the clips without injuring the material, and in practice the strip broke within the gauge length. In the testing of nitro-cellulose, narrow blocks, cemented to the edges of a strip, have been employed instead of clips, and this latter arrangement also proved fairly successful.

An examination of the essential features of both these forms was carried out, therefore, and compared with some calculations, which are fortunately available. Prof. Filon has shown * that the stress produced by oppositely directed forces applied perpendicularly to the sides of a rectangular strip, of breadth $2b$, can be evaluated, and he obtains for the stresses the following values :—

FIG. 22.



$$\widehat{xx} = -\frac{2\,\mathrm{W}}{\pi b}\int_0^{\infty}\frac{\sinh u - u\cdot\cosh u}{\sinh 2u + 2u}\cdot\cos\frac{ux}{b}\cdot\cosh\frac{uy}{b}\cdot du$$

$$-\frac{2\,\mathrm{W}}{\pi b}\int_0^{\infty}\frac{uy}{b}\cdot\frac{\sinh u}{\sinh 2u + 2u}\cdot\cos\frac{ux}{b}\cdot\sinh\frac{uy}{b}\cdot du$$

$$\widehat{yy} = -\frac{2\,\mathrm{W}}{\pi b}\int_0^{\infty}\frac{\sinh u + u\cosh u}{\sinh 2u + 2u}\cdot\cos\frac{ux}{b}\cdot\cosh\frac{uy}{b}\cdot du$$

$$+\frac{2\,\mathrm{W}}{\pi b}\cdot\int_0^{\infty}\frac{u\sinh u}{\sinh 2u + 2u}\cdot\cos\frac{ux}{b}\cdot\sinh\frac{uy}{b}\cdot du$$

$$\widehat{xy} = \frac{2\,\mathrm{W}}{\pi b}\int_0^{\infty}\frac{u\cdot\cosh u}{\sinh 2u + 2u}\cdot\sin\frac{ux}{b}\cdot\sinh\frac{uy}{b}\,du$$

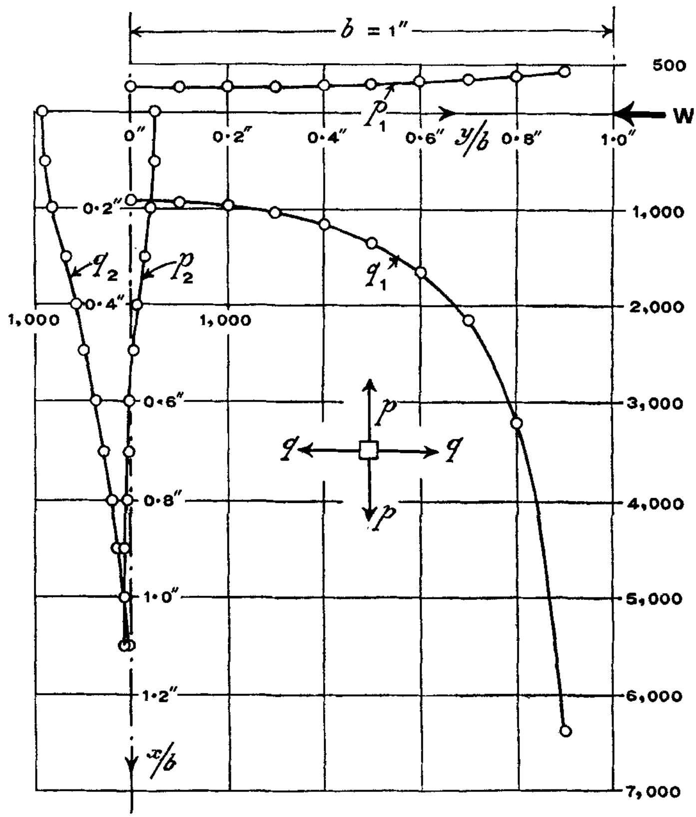
$$-\frac{\mathrm{W}y}{\pi b^2}\int_0^{\infty}\frac{u\sinh u}{\sinh 2u + 2u}\cdot\sin\frac{ux}{b}\cdot\cosh\frac{uy}{b}\cdot du.$$

The numerical values of the stresses along the axes of symmetry have been obtained, although with much labour, for the case where

* *Loc. cit.*, ante p. 18.

W = 100 lb., $2b = 1$ inch, and the thickness of the strip is $\frac{1}{16}$ inch, and their values are recorded in the accompanying Table 10 (page 418) for comparison purposes. It may be remarked in passing that these numerical values were found to agree with some given in the Paper

FIG. 23.—*Theoretical Results.*



already cited, but the Authors were obliged to extend the range to suit the particular purpose, while for convenience of calculation the values of p and q were obtained from the more convenient expressions of the sums and differences of the principal stresses.

Thus for $\frac{y}{b} = o$ the formulæ above give

$$p_2 + q_2 = -\frac{4\,W}{\pi b} \cdot \int_o^\infty \frac{\sinh u}{\sinh 2u + 2u} \cdot \cos \frac{ux}{b} \cdot du$$

$$p_2 - q_2 = +\frac{4\,W}{\pi b} \cdot \int_o^\infty \frac{u \cosh u}{\sinh 2u + 2u} \cdot \cos \frac{ux}{b} \cdot du$$

with expressions of a similar type for $(p_1 \pm q_1)$.

TABLE 10.

Data.

Breadth . . .	$2b = 2$ inches.
Thickness . . .	$t = 2z = 0{\cdot}1$ inch.
Pull	$P = 0$.
Pressure . . .	$W = 100$ lb.

Stresses along $\frac{x}{b} = 0$.

$\frac{y}{b}$	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
p_1	249	252	257	264	279	292	319	347	388	431
q_1	−920	−933	−972	−1046	−1166	−1341	−1647	−2150	−3200	−6370

Stresses alone Line $\frac{y}{b} = 0$.

$\frac{x}{b}$	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0	1·1
p_2	249	236	202	153	65	45	−4	−37	−63	−74	−81	−82
q_2	−920	−895	−830	−670	−586	−501	−388	−289	−205	−142	−93	−55

The values so obtained are plotted for convenient reference in Fig. 23 for comparison with the cases selected.

It will be observed that p is small along either axis and the labour of separating p and q is hardly justified, so that we are content to compare, in all cases, the optical values of stress distribution with

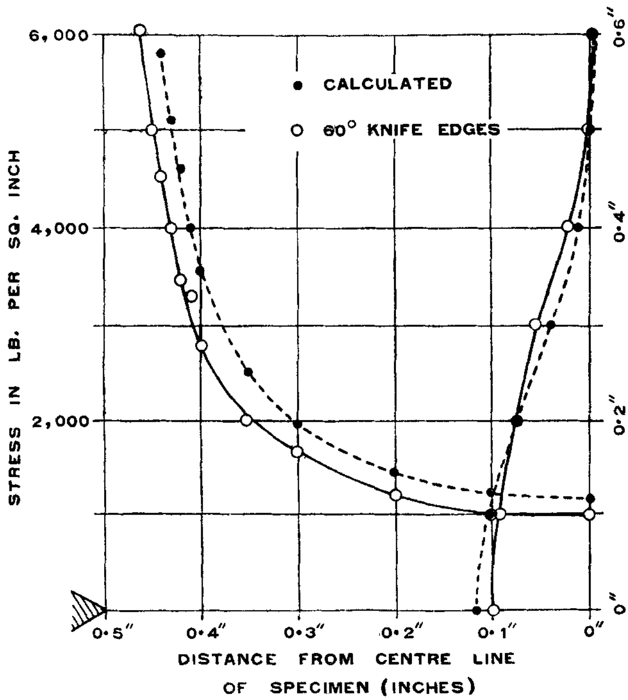
TABLE 11.

Table of (p − q) Values).

		Calculated.	60 deg. knife edge.	Flat.			Round Pin.			
				Width $\frac{1}{20}$ in.	Width $\frac{1}{8}$ in.	Width $\frac{1}{4}$ in.	Diam. $\frac{1}{16}$ in.	Diam. $\frac{1}{8}$ in.	Diam. $\frac{3}{16}$ in.	Diam. $\frac{1}{4}$ in.
$\frac{x}{b}=0$, $\frac{y}{b}=$	0·0	2,340	1,990	2,150	2,410	1,970	2,120	2,170	2,150	2,150
	0·2	2,460	1,960	2,200	2,490	2,075	2,260	2,160	2,150	1,320
	0·4	2,890	2,380	2,690	2,950	2,280	2,440	2,270	2,280	2,560
	0·6	3,930	3,290	3,520	3,815	2,560	3,460	3,370	2,580	3,400
	0·7	4,994	3,980	4,640	5,270	2,840	4,350	4,040	3,900	4,560
	0·8	7,116	5,510	7,140	6,065	2,800	6,880	6,720	5,920	7,240
	0·84	9,200	6,930	9,040	6,365	2,650	9,340	8,300	7,910	9,060
	0·88	11,600	8,530	14,000	6,165	1,880	12,840	11,100	13,080	10,[illegible]20
	0·90	13,600	9,960	15,700	5,220	1,650	15,300	13,240	14,600	12,600
	0·92	—	12,100	18,200	4,450	1,370	—	15,400	—	15,880
	0·94	—	18,300	—	4,050	930	—	—	—	—
	0·96	—	23,800	—	—	825	—	—	—	—
	0·98	—	—	—	—	—	—	—	—	—
$\frac{y}{b}=0$, $\frac{x}{b}=$	0·0	2,340	1,990	2,100	2,420	1,995	2,120	2,170	2,150	2,150
	0·2	2,064	1,814	1,820	2,350	1,720	1,620	1,680	1,880	1,880
	0·4	1,302	1,460	1,200	1,540	1,320	1,300	1,060	1,100	1,440
	0·6	768	1,100	440	830	610	800	740	530	1,040
	0·8	284	427	300	300	320	290	400	210	230
	1·0	24	0	20	20	40	60	60	60	40
	1·2	−120	−107	−20	−130	−150	−80	−120	−100	−110

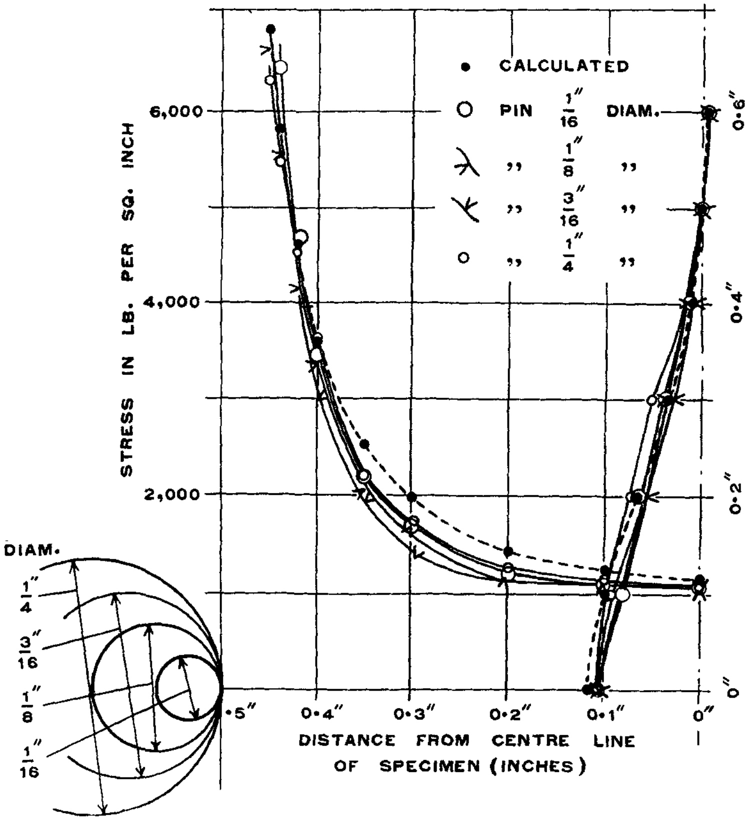
the calculations. All the results so obtained along both axes are recorded in the accompanying Table 11, and are plotted in diagram form to facilitate comparison. The distribution obtained with knife-edges is perhaps the most interesting from a theoretical aspect, since, as Fig. 24 shows, there is a good agreement, especially

FIG. 24.—*Stresses* $(p-q)$. *Set* 1.



along the centre line of the member where the stress is small and somewhat difficult to measure. Along the axis of the applied load the agreement is apparently less satisfactory, since all the measured values are less than those obtained from calculation, but the discrepancy is easily explained by a circumstance, already pointed out in the experiments of Carus-Wilson, that the application of pressure along a line by means of a knife-edge causes the latter to sink into the material, and for a true comparison the experimental curves must be depressed an appreciable amount. With this correction, amounting in this case to $\frac{1}{30}$ inch, the agreement is close and affords a verification, if such were needed, of the

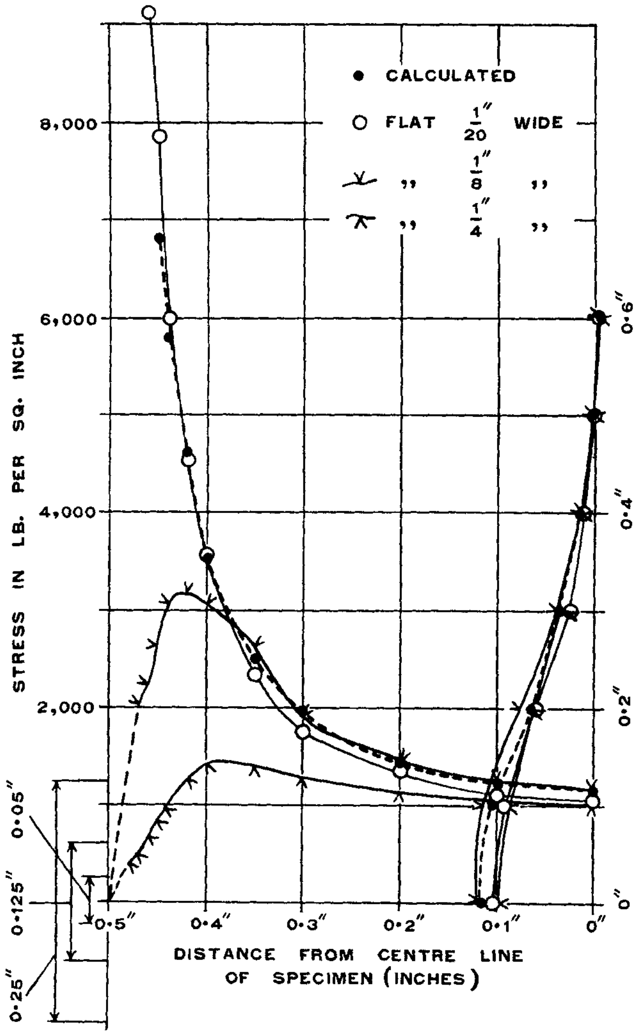
FIG. 26.—*Stresses* $(p-q)$.



accuracy of the analysis. The extent of the area affected by a lateral pinch is indicated by Fig. 25, Plate 17, and the disturbance produced by centres are easily realized from the colour bands.

If, instead of a knife-edge, a roller is applied, the agreement is still satisfactory, as Fig. 26 shows, although in each case the roller produces some depression, which tends to cause the experimental

FIG. 27.—*Stresses* $(p-q)$.



curve to lie above the calculated values. After a correction for this is made, both sets of values are in very good agreement. The same remark also applies to the smallest flat block, but if the area of the block is increased, there is a marked divergence, as Fig. 27 shows, and the curves have a maximum value with a turning point like those obtained when a finite length of a large plate is under compression and due to the same cause, namely, the approach to equality of p and q giving a zero value directly underneath the block, as indicated by dotted lines in the region where it was found very difficult to obtain an accurate measurement.

The inherent difficulty, which these experiments show, of determining the true strength of a material under a tension load, when it is necessary to make an indentation or scratch, however minute it may be, has resulted in attempts to avoid this source of error altogether, and it is of interest in this connexion to note that Prof. Dalby, who has devised an autographic testing-machine of great perfection, has found it necessary to abandon the ordinary methods of attaching an extensometer. In his arrangement the specimen is turned from a solid bar with collars spaced a convenient distance apart, and the extension between these latter is measured by suitable mechanism bearing lightly against the collars.

At Prof. Dalby's suggestion the Authors have examined the effects produced by the sudden enlargement of the cross section of such a test-piece.

The exact determination of the effect of a cylindrical collar is a difficult problem, although possibly not an insuperable one, but the case of a plane section of such a test-bar is relatively simple and possibly not very different from that of a specimen in the round. It has, in fact, been shown, * in another analogous case, that a change of section causes a somewhat more widely distributed local stress for round specimens than for a flat strip of the same contour. The type of stress distribution is obviously not quite the same, but the indications afforded by the problem in plane stress are valuable as a guide to what happens in the cylindrical specimen.

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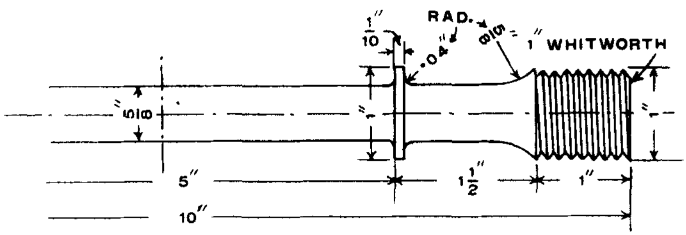
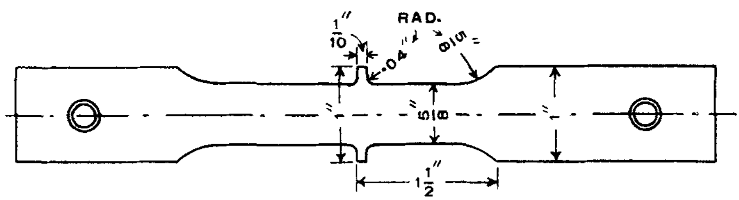
* *Loc. cit.*, ante p. 42.

The standard form used by Prof. Dalby is shown in the accompanying Fig. 28, and consists of a $\frac{5}{8}$-inch round bar provided with collars 1 inch in diameter, $\frac{1}{10}$ inch thick, and 5 inches apart, and having enlarged and screwed ends for applying load in the testing-machine at a sufficient distance away to ensure no uneven distribution of stress from these screwed ends.

FIG. 28.—*Standard Form of Dalby Test-piece.*



Dimensions of Plane Model of Nitro-Cellulose. Thickness 0·121 *inch.*



The merging of the collars into the bar is effected by fillets of 0·04-inch radius, and of the outer enlarged ends by curves of $\frac{5}{8}$-inch radius. The measurement of the distribution of stress in the screwed ends is a difficult matter and outside the scope of this investigation. It is not considered here. A slight increase of stress is known to occur near the join of the straight contour with the curves of large radius, but, as these latter effects have been measured and described elsewhere,* it is not necessary to refer to them further. The full sized model test-piece, which formed the subject of the experiments described below, was shaped in the manner shown in

* Proceedings, Inst. C.E. Vol. ccviii, Part II, 1918–19.

the lower part of Fig. 28, in which the exact contour is preserved up to the enlarged ends, and the load is applied through pins sufficiently distant from the discontinuities to ensure uniform stress in the enlarged ends before change of section takes place.

When this bar is subjected to a moderate load and viewed in the polariscope, the colour effects appear to indicate, Fig. 29, Plate 17, that in the neighbourhood of the flanges and at the rounded contour there is a local increase of stress, but the general effect of the enlargement appears to decrease the stress intensity along the lines of a truncated quadrilateral, while across the central section there are some indications of stress variation, as the photograph shows. The outermost parts of the flanges are under no stress, but their inner parts are subjected to a somewhat complex stress distribution, which gives rise to an increased tension near the join of the fillet with the parallel contours, and becomes equal to the mean value corresponding to the load at a small distance from the discontinuity. It is also fairly evident from the colour picture obtained in the polariscope that the stress is symmetrically disposed, and that the enlarged ends do not produce an unsymmetrical distribution at the flanges.

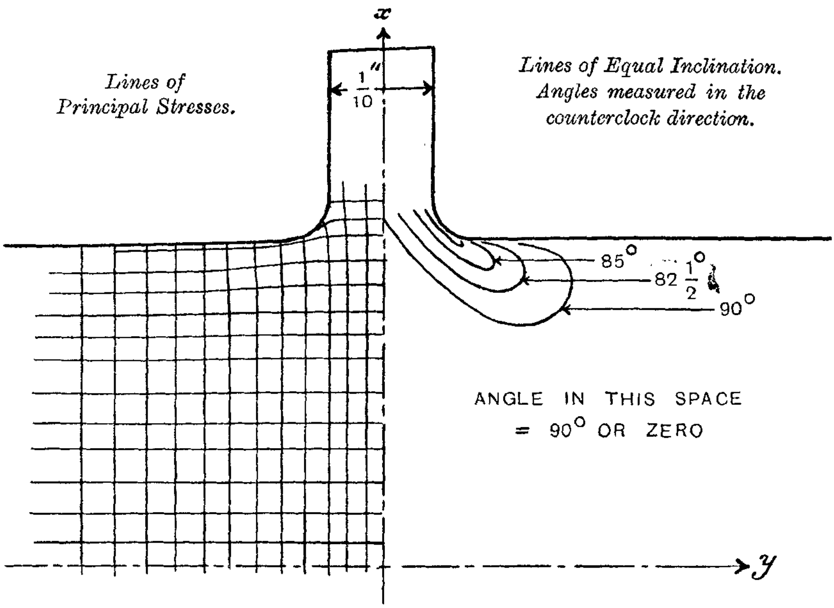
The experimental determination of the distribution in such a case can be completely solved by measuring the directions of principal stress and their magnitudes, on the assumption that the variation of stress throughout the thickness of the plate is negligible.

The former are determined from the isoclinic lines shown on the right hand of Fig. 30, from which it appears that at the enlargement caused by the flanges the lines of principal stress parallel to the axis bulge outwardly in a somewhat abrupt fashion accompanied by a divergence of the transverse orthogonal lines. The disturbing effect of this enlargement on the lines of principal stress is small, however, and, so far as this kind of evidence shows, it appears to indicate that the variation in the distribution of stress is not very great.

In order to determine the stress distribution accurately, a number of cross-sections were examined in the neighbourhood of discontinuity and the values of $(p \pm q)$ measured along them. These values are recorded in Table 12 (page 428), and a few of these are plotted in Fig. 31 to show their general character.

At the central section it is noticeable that the $(p-q)$ curve of distribution is above that of the corresponding $(p+q)$ curve for the greater part of the section, and that it rises slightly, as we proceed, away from the axis and reaches a maximum value at a

FIG. 30.—*Directions of Principal stresses.*

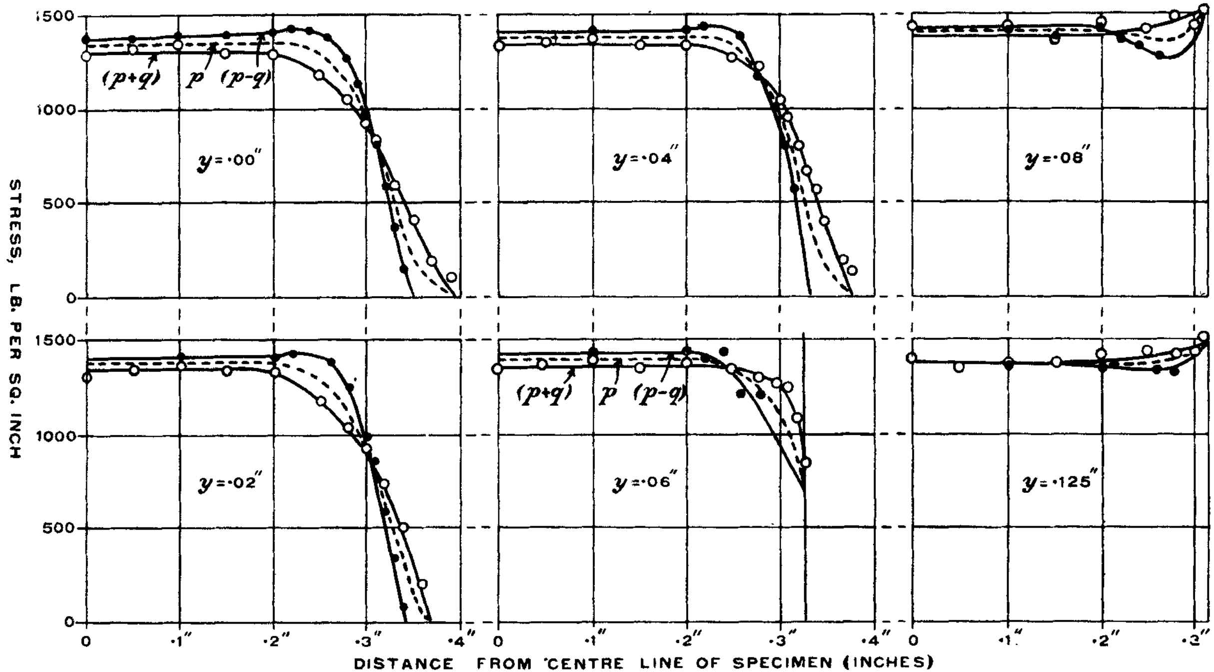


distance of about 0·23 inch from the central line, as the colour effects indicated. There is then a rapid decrease to a zero value.

The measurements show, in fact, that the tensional stress across this section is fairly uniform for two-tenths of an inch on each side of the central line, with a little rise as we go outwards, and then rapidly dies away. It is accompanied by a compressional cross stress of small magnitude, which reaches a maximum value at about $x = 0 \cdot 23$ inch and changes sign when $x = 0 \cdot 3$ inch approximately.

This distribution is characteristic of all the sections in varying

FIG. 31.



STRESS, LB. PER SQ. INCH
DISTANCE FROM CENTRE LINE OF SPECIMEN (INCHES)
(p+q)
p
(p-q)
y=·00"
y=·04"
y=·08"
y=·02"
y=·06"
y=·125"
1500
1000
500
0
·1"
·2"
·3"
·4"

TABLE 12.—*Sum of Principal Stresses* $(= p + q)$ *lb./in.*^{2}

$y =$	Inch. 0·00	Inch. 0·01	Inch. 0·02	Inch. 0·03	Inch. 0·04	Inch. 0·05	Inch. 0·06	Inch. 0·07	Inch. 0·08	Inch. 0·10	Inch. 0·125	Inch. 0·15	Inch. 0·20	Inch. 0·30
$x =$ inch. 0·00	1,302	—	1,308	—	1,320	—	1,342	—	1,438	1,302	1,374	1,302	1,302	1,373
0·05	1,328	—	1,353	—	1,354	—	1,353	—	1,380	1,302	1,335	—	—	—
0·10	1,406	—	1,368	—	1,368	—	1,380	—	1,432	1,276	1,368	1,250	1,275	1,315
0·15	1,302	—	1,342	—	1,322	—	1,335	1,400	1,353	1,302	1,368	1,302	1,263	1,368
0·20	1,302	1,250	1,328	1,432	1,335	1,380	1,360	1,471	1,452	1,263	1,413	1,343	1,380	1,413
0·25	1,185	1,172	1,178	1,308	1,262	1,302	1,348	1,367	1,419	1,308	1,432	1,302	1,387	1,302
0·28	1,055	1,035	1,042	1,250	1,224	1,275	1,290	1,287	1,483	1,353	1,419	1,380	1,368	—
0·30	925	866	911	1,042	1,042	1,210	1,250	1,380	1,438	1,335	1,438	1,445	1,510	1,530
0·31	840	800	—	917	950	1,087	1,237	1,367	1,530	—	1,510	1,484	—	—
0·32	—	—	729	768	781	925	1,068	1,191	—	—	—	—	—	—
0·33	598	592	—	664	657	814	845	—	—	—	—	—	—	—
0·34	—	—	508	527	573	638	—	—	—	—	—	—	—	—
0·35	404	293	—	—	390	—	—	—	—	—	—	—	—	—
0·36	—	—	202	338	—	280	—		Load on specimen = 100 lb.					
0·37	195	150	—	—	195	—	—	—	—	—	—	—	—	—
0·38	—	—	—	260	130	195	—	—	—	—	—	—	—	—
0·39	104	13	—	195	—	—	—	—	—	—	—	—	—	—

TABLE 12—*continued.*

Difference of Principal Stresses $(= p - q)$ *lb./in.*^{2}

$y =$	Inch. 0·00	Inch. 0·01	Inch. 0·02	Inch. 0·03	Inch. 0·04	Inch. 0·05	Inch. 0·06	Inch. 0·07	Inch. 0·08	Inch. 0·10	Inch. 0·125	Inch. 0·15
$x =$ inch.												
0·1	1,408	1,408	1,408	1,408	1,408	1,408	1,422	1,422	1,422	1,422	—	—
0·2	1,408	1,408	1,408	1,408	1,408	1,408	1,422	1,422	1,422	1,422	1,340	1,340
0·22	1,440	1,440	1,440	1,440	1,440	1,440	1,390	1,390	1,357	1,323	—	—
0·24	—	—	—	—	—	—	1,423	1,423	1,340	1,241	—	—
0·26	1,387	1,387	1,387	1,387	1,387	1,387	1,191	1,159	1,273	1,223	1,340	1,340
0·28	1,255	1,255	1,255	1,255	1,155	1,155	1,191	1,159	—	—	1,323	1,323
0·30	978	978	978	978	978	882	—	—	—	—	—	—
0·31	860	860	860	860	819	819	—	—	—	—	—	—
0·32	580	580	580	580	580	—	—	—	—	—	—	—
0·33	368	335	335	311	—	—	—	—	—	—	—	—
0·34	154	104	77	5	—	—	—	—	—	—	—	—
0·35	5	5	5	—	—	—	—	—	—	—	—	—

Load Calculated.

$y =$	Inch. 0·00	Inch. 0·01	Inch. 0·02	Inch. 0·03	Inch. 0·04	Inch. 0·05	Inch. 0·06	Inch. 0·07	Inch. 0·08	Inch. 0·10	Inch. 0·125	Inch. 0·15	mean.
w	104·7	102·7	104·2	104·2	103·8	105·9	105·0	106·0	103·6	100·2	102·8	99·3	103·53
Per cent difference.	4·7	2·7	4·2	4·2	3·8	5·9	5·0	6·0	3·6	0·2	2·8	−0·7	3·53

degree up to Y = 0·05 inch, where the main discontinuity ceases, and only the rounded corners remain to increase the breadth of the section. We find the $(p \pm q)$ lines still cross, but, instead of terminating at zero, they reach a maximum at the curved contour.

It is interesting to note in passing that these measurements show the same kind of variation which occurs when the enlarged portion is very much longer, and that some increase of stress appears to be inevitable within the elastic limits of the material whenever there is a symmetrical enlargement connected to the main tension member by any form of curve.

An idea of the general accuracy of these measurements may be obtained by integrating the normal stress p_n across each section and comparing the value so obtained with the actual load. It will, however, be noticed that q is small and that it changes sign, so that if it is neglected altogether the error is hardly appreciable. The other principal stress p is, moreover, nearly normal to the cross section everywhere, so that if t be the thickness, we have $W = t\int p_n \cdot dx = t \cdot \int p \cdot dx$ very nearly, and we may take this latter value without appreciable loss of accuracy. This approximate check on the general accuracy of the measurement is, under these circumstances, likely to account for slightly more than the load of 100 lb., as in fact actually turns out to be the case, Table 12, except one section where the error is slightly in defect.

Except in two instances the error does not amount to more than 5 per cent and the average error is +2·53 per cent, which would probably be reduced somwehat if q were taken into account, and the inclinations of both principal stress to the cross sections considered.

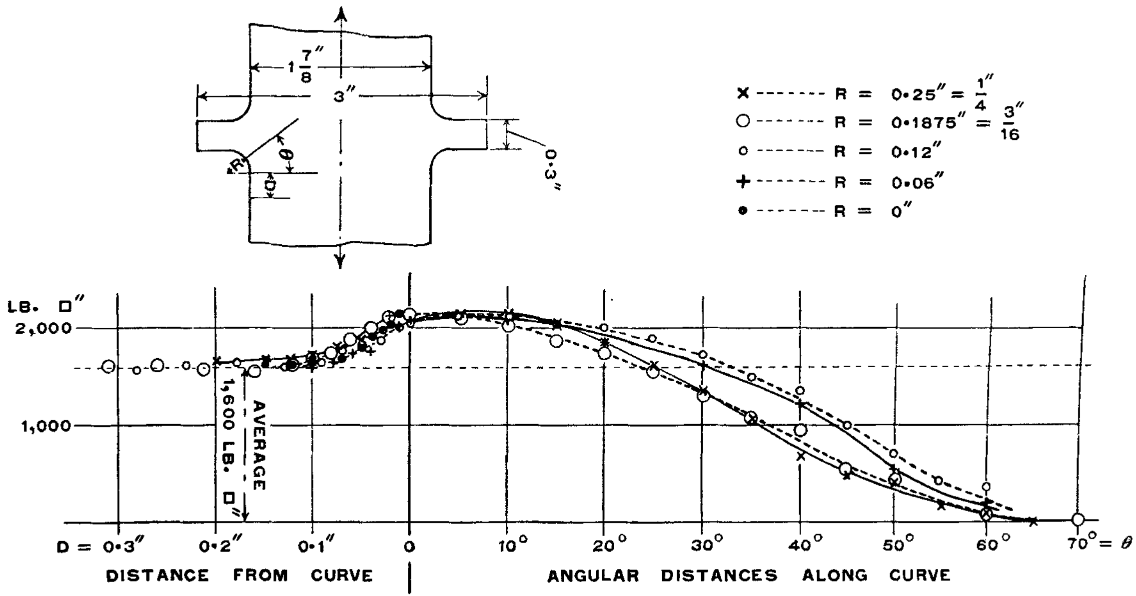
It is an interesting feature of this form of test-piece that a change in the radius of the fillet does not appear to exercise a very marked influence on the stress produced. It appears always to cause a local concentration of stress, but its effect does not change very much with curvature, whereas in re-entrant angles the maximum stress rises rapidly with the curvature.

In order to examine this with some accuracy, measurements

TABLE 13.

—	R=¼ in.	R=3/16 in.	R=0·12 in.	R=0·06 in.	R=0 in.
Degrees.					
70	—	0	0	9	—
60	137	65	370	201	—
50	405	438	720	560	—
40	679	952	1360	1258	—
θ=30	1398	1314	1764	1640	—
20	1870	1773	2004	1848	—
15	2015	1893	2056	—	—
10	2140	2015	2090	2100	—
5	2150	2100	2090	—	—
0=0	2060	2145	2035	2160	—
Inch.					
0·02	2020	2100	1960	2120	{2174 2060
0·04	1940	2020	1870	1780	1906
0·06	1860	1900	1770	1740	1780
0·08	1800	1760	1700	1658	1690
0·10 =D	1760	1730	1620	1620	1650
0·15	1710	1600	1600	—	1620
0·20	1700	1600	1630	—	—
0·25	—	1610	1600	—	—
0·30	1668	1620	1600	—	—

FIG. 32.—*Peripheral stresses at Flange of Dalby Test-bar. Load 300 lb. Width 3 inch total,* $1\frac{7}{8}$ *inch net. Assumed thickness 0·10 inch.*



were made along the boundary of an enlarged member, in which all the conditions were kept constant, except the radius of the fillet. At such a boundary the minor principal stress vanishes and either optical or mechanical measurements suffice, but the former are preferable owing to the ease with which they can be made at the boundary. Some of these are shown in the accompanying Table 13 and are plotted in Fig. 32, for a member three times full size under a load of 300 lb., which latter load corresponds to a mean stress of 1,600 lb. per square inch. In every case a maximum stress of more than 2,090 lb. per square inch is observed, but it is noteworthy that the extreme cases give very small differences, for with a radius of $\frac{1}{4}$ inch the stress rises to 2,150 lb. per square inch, and for the smallest radius that could be fashioned the corresponding stress is only very slightly increased. The experiments, in fact, appear to show that, although flanges cause a local increase of stress, the form of the curve, by which they merge into the main body of the member, is not very important provided it does not undercut the specimen.

On the whole, the evidence afforded by the above examination appears to us amply to justify the use of this form of test-piece for tension tests, although, as Fig. 32 shows, there is a local increase of stress at or near the join of the straight contour with the flanges. This local concentration is, however, met with in the usual standard forms under less advantageous circumstances.

In conclusion, the Authors desire to express their acknowledgments of the help they have received during the progress of this investigation from the Department of Scientific and Industrial Research, and the authorities of University College also for the great mechanical skill of Mr. F. H. Withycombe in the construction of the numerous pieces of apparatus and specimens required throughout this investigation which commenced early in 1914.

The Paper is illustrated by 28 Figs. in the letterpress, and by two Colour Plates Nos. 16–17.

Discussion in London on Friday, 18*th March* 1921.

The PRESIDENT pointed out the interest and value of the information contained in the Paper, which was the first of the kind read before the Institution. Papers of a similar nature had been published elsewhere, but clearly a great deal more still remained to be done, and he trusted that Professor Coker and his co-workers were continuing their experiments, because they could not fail to be of extreme interest to mechanical engineers. The colour prints which had been shown made one realize very vividly the distribution of the stresses, and one of them clearly showed how the strength of a structural member was weakened by *adding* material. He proposed a most cordial vote of thanks to the Authors for their valuable and interesting Paper.

Sir HENRY FOWLER, K.B.E. (Member of Council), formally seconded the motion, which was carried with acclamation.

Dr. H. S. HELE-SHAW, F.R.S. (Vice-President), said that any method of research, which enabled the engineer to investigate problems so that he could improve the design of both structures and machines, was most important, but when a method was optical, so that not only could the results be accurately measured but what was taking place during variation of stress could be actually seen, the value became enormously enhanced. In the application of the methods described in the Paper use was made of polarized light. The discovery and explanation of polarized light was probably rather more than a hundred years old. He had that day perused the wonderful and classic Paper by Sir David Brewster, who was the first to give a consistent theory of the subject; though Professor Coker had told him that Sir David had suggested the use of polarized light for investigating strains in an arch, he had not seen any reference to this in the above Paper. Professor Filon, Professor Carus Wilson—and others had also suggested applications of the method of polarized light.

There was a considerable difference between suggesting the use of a method and setting to work as Professor Coker had done, and by combining his mathematical and scientific knowledge with his engineering skill to follow up the work, and he had, after many years, given engineers a practical method of solving difficult problems. In the "Philosophical Transactions" last year a Paper had appeared without which the work under discussion could scarcely be regarded as complete. In the Paper to which he referred, the properties of transparent nitro-cellulose had been carefully examined and a large amount of detail had been given. The research showed that the results obtained by the use of varieties of this substance were not only true within the elastic limit but were also true outside that limit, which was very important because it did not require very high stresses to deform a block of celluloid or xylonite.

He believed the Paper before them was the first representing any practical attempt to deal with the obscure subject of contact pressures or the distribution stress at the point of contact. Members were familiar with the text-book treatment up to the present in use for dealing with transverse stress, as illustrated by beam and girder problems, and this was simply to ignore the contact effects at the supports and point of loading, which led often to most erroneous conclusions. For instance, a girder carrying a uniform load and supported at two points, if the distance from the ends of the latter supports was $(\sqrt{\frac{1}{2}}-\frac{1}{2})$ times the length, then the text-book result was that the maximum stress was equal at the centre of the girder and at the two supports. This might be absolutely wrong, as the stresses at the three points were in general totally different according to the actual form of the supporting surfaces. It was clear from the colour diagrams they had seen that the stresses at sharp corners might be enormous, and it might be asked why, if this was so, the girder or machine part did not *always* fracture at that point. The answer was that the stresses were local, the material was indented. Of course, if the material was very hard, it might break at the point of contact. It so happened that until the nature of the contacts had been investigated, the position of the really weakest part could not be ascertained. Engineers had been obliged up to the present to be contented with an assumption which was erroneous. They had now, however, a method by which accurate results could be arrived at in regard to contact pressures. He would like Professor Coker in his reply to say if he had combined the new contact results with calculations for transverse stress, as obviously this was necessary for the complete treatment of the subject.

There were three kinds of measurement taken by the Authors, namely, (1) the magnitude of the stress at certain points ; (2) the direction of the principle axes, and lastly (3) the isoclinic lines. These last, which gave the loci of equal inclination of stresses, were very valuable although not generally employed by engineers, and it was a matter of the greatest interest that mathematical investigations, which enabled these isoclinic lines to be predicted, had been found by measurements recorded in the Paper to be verified in a remarkable way.*

The speaker next alluded to the work in connexion with test-pieces, and expressed the view that it was a very satisfactory thing that the cause of the difficulties in the use of the extensometer could be understood by the optical method and that the evil effects of the centre punch marks on specimens in connexion with such instruments were so well explained in the Paper. The Paper gave a remarkable result with a specimen of Hadfield's manganese steel which had the breaking tensile strength of 100 tons, and, although showing great extension, had ultimately broken across the minute marks made for measuring elongation. Not the least valuable part of the Paper was the record of tensile stresses induced in the neighbourhood of the collar, which had been suggested by Professor Dalby with a view to obviating the almost universal custom of indenting the test-pieces, between which to measure such extensions, and of carrying an instrument for doing this.

He would in conclusion remark that beautiful as the coloured

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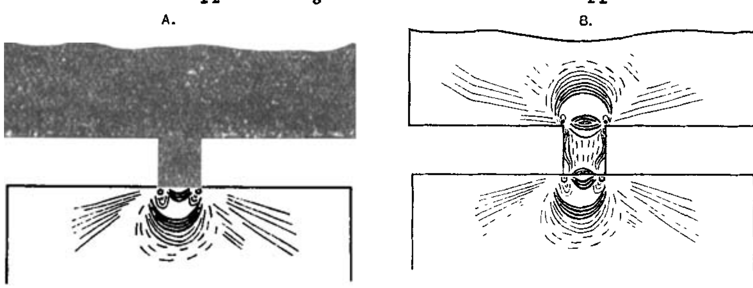
* The speaker wishes to say that he had not at that time had an opportunity of reading the Papers by Professor Filon (as he has since done), or he would have paid a tribute to his remarkable investigations on this subject

photographs had been, it was a great pity that the Authors of the Paper had not been able to produce on the screen the actual colour effects as the contact stresses were produced. The speaker had witnessed such experiments, only a day or so before, in the laboratories of University College; and it was only because the intense beam of light required for a large lecture hall was dangerous to the valuable Nicol prisms, that Professor Coker had not given them the same pleasure of seeing actual experiments that night. For instance, in the photographs shown, a metal pressure piece was used to make contact; whereas in the actual experiments the

FIG. 33.

A *is a diagrammatic Fig. of the coloured photograph, Fig.* 8, *Pl.* 16.
B „ „ „ „ *which shows what is really also happening to the Pressure Piece.*

Thickness of material ¼ inch. Specimen 1 inch square.
Pressure applied over ⅛ inch width. Load 55 lb. approx.



speaker had witnessed, a celluloid pressure piece was used, and Fig. 33 showed that, at the corners of the solid piece, curves started similar to those produced in the contact surface. This visibly brought home the evil effect of sharp corners, and that it was not only at the surface of contact but in the pressure piece itself that intense stresses were produced. He would have referred to other points of great interest which became evident by witnessing the actual formation of stress colour figures, for like his own method of stream-line colour bands, the changes taking place revealed many new facts to the observer.

As in various times his stream-line method had been suggested, and even used with the idea of giving stress diagrams in riveted joints in the structure of ships and other cases, he took this opportunity of saying that he, the speaker, had never suggested their use. Though in some cases bearing a certain similarity to the actual lines of stress as revealed by polarized light, the stream-line method could not really be applied to this purpose, any more than the colour bands of polarized light could be applied, as the stream-line method could be, for the flow of liquid, or for electric or magnetic problems. He believed that the method that the Authors had brought before them was the only method that had ever yet been suggested for investigating the hitherto obscure and unknown problems of contact stresses.

Professor L. N. G. FILON (University College, London) said it was not often that a mere mathematician like himself had the work of his callow youth disinterred out of the dust of oblivion by a real live engineer, and moreover had it tested, and he was very proud to say, verified, with the great conscientiousness that characterized all the Author's work, and which had been exemplified in the remarkable Paper that had been read. So far as the verification of the theoretical results was concerned, he thought the agreement was very striking. He desired to refer to some of the results of pages 420–1, to which Professor Coker had alluded very briefly that evening, namely, the comparison between the calculated and the observed values of the stresses in a beam immediately underneath a point of concentrated load. There were some discrepancies between theory and experiment in that instance, but what struck him most particularly was the curious divergence between the law of stress according to the particular form in which the load was applied If a 60 deg. knife-edge were used, for example, a series of stresses all smaller than the calculated value were obtained, and they got proportionately smaller as one moved away from the load. Again, if a flat punch were used discrepancies were obtained, but if the punch had a width of about $\frac{1}{8}$ inch, apparently the agreement for moderate depths was very much better. Again, if a round pin were used, the discrepancies varied considerably with the diameter of the pin. For a diameter of $\frac{1}{8}$ inch there was again a most remarkable agreement, which in that instance continued all the way down, with the results of theory. He had been somewhat exercised in his mind about those divergencies. Of course, it was possible to see how, even according to theory, the shape of the punch might affect the law of stress; but he would not have expected changes of the order shown, especially at a distance from the point of application of the load.

In that connexion he would like to call attention to certain possibilities that might have to be considered in regard to celluloid. It so happened that he had recently been carrying out some experiments on the physics of stress in celluloid, both in regard to its optical effect and to the extension of celluloid under tension. It appeared that that material, even under loads which were apparently well within the elastic limit, showed a considerable creep after several hours. It went on creeping for days and even for weeks. For a load corresponding to the order of extinction in some of the slides shown that evening as much as 17 per cent of optical creep could be obtained inside one hour, that is, the optical retardation, by means of which the stress difference $p-q$ was measured, might vary as much as 17 per cent of the whole. Further, the photo-elastic properties of the material appeared to be very profoundly influenced by previous treatment, especially if the loads had been considerable. If the specimen, after being unloaded, was allowed to rest for a couple of hours and then loaded again, it was found that the stress optical coefficient, as it was called, had varied to a certain extent, and that the optical effect had proportionately increased. For a moderately heavy load that change might amount to something like 4 or 5 per cent of the whole for an interval of measurement of about 90 minutes, that is, i one took 1½ hour over the measurement, the effect might come in between two successive sets ; but for smaller loads it was much less. He would like to ask the Authors whether they thought that might possibly account for some of the minor discrepancies.

He was very pleased that the Authors had taken up the method of the isoclinic lines. Much of Professor Coker's past work had been done by using the colour bands called isochromatic lines, which corresponded to relative retardations of 1, 2, 3 or 4 wave-lengths, and they required very considerable loads for their production, whereas the isoclinic lines, that is, the loci of the points where the axes of principal stress had all the same orientation—those loci which were shown by black brushes or bands in the field—could be produced with extremely small loads, and therefore the element of uncertainty to which he had just been referring was absent. He held in his hand a copy of the *Philosophical Magazine* for 1912, in which some of those isoclinic lines were shown on some plates ; and he remembered using the method about twenty years ago at Cambridge, in order to investigate that very problem of the stress in a rectangular bar of glass under a point of concentrated load.

With regard to the combination of the ordinary flexure stress with the local stress, which had been referred to by Dr. Hele-Shaw, he might say incidentally that glass was a very troublesome material with which to investigate the stresses in complicated engineering structures, and he thought one of the very great merits of Professor Coker's process, and one which really amounted to a great discovery, was the use of celluloid. He did not know of the photo-elastic properties of celluloid at the time, so he used glass. A rectangular bar of glass rested on knife-edges ; there was a central load and there were wooden projections which enabled him to apply a reverse bending moment in the part of the bar between the supporting knife-edges. By varying the reverse bending moment, he could vary the ordinary flexure stresses and adjust them so as to cancel local stresses at a given point, thus obtaining a null result. The method gave results which confirmed the theory to a certain extent (there were certain discrepancies), but there was no doubt about it that, in view of the methods that the Authors were now bringing to bear on such problems, his own early experimental attempts were very inferior, and probably the discrepancies between theory and observation that they seemed to show might have been due to that cause. He had been very much struck by the examples and slides shown by Professor Coker, the excellence of which he felt sure must have been noticed by everyone present. The reproduction of the actual colours was so accurate that, when the early slides were being shown, he really thought that the actual specimens under stress were being introduced into the lantern.

Mr. C. E. STROMEYER, C.B.E., said the Authors had done excellent spade work, which would throw a great amount of light on many problems with which engineers had to deal. The work was so complete that he could not offer any criticisms ; and he would, therefore, confine himself to a few suggestions. The first was with regard to Fig. 4 (page 371). The stress diagrams reminded him very much of the pressures which occurred in a bearing, and it seemed to him that this experiment could be brought nearer to mathematical conditions if, instead of a rectangular sheet resting on a flat surface, a celluloid half-disk were cemented into a nearly rigid bearing, and if instead of a point contact, the top of the disk were hollowed out as far, perhaps, as the first stress ring, and a cylindrical metal plug cemented into the hollow. It would then be possible to a certain extent to comply with the mathematical boundary conditions both for the centre and for the outside.

He thought that in spite of what had been said against stream-lines, they might be made very useful if, instead of resolving movements, as was done by Rankine, stresses were resolved. That, of course, implied that double the angles should be used instead of single ones. A far more important suggestion than either the above was that very considerable light would be thrown on many engineering problems if the Authors would deal with the problem of stresses in a curved beam, of which that of a thick walled cylinder was a unique case.

Professor L. LUIGGI, D.Sc., said he desired on behalf of Italian engineers to express gratitude to the Authors for the work they had done. In some countries, not England or Italy, engineers were being taught in these latter years almost entirely from a mathematical science point of view, it being forgotten that engineering was also, and principally, a physical science. He was glad that the Authors, by taking up the physical part of research, had brought engineering again into its proper

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channel. An engineer ought to do by experiment everything that he possibly could, in order to ascertain the structure of materials and the best way of using them, and he ought not to use the reasoning of his brain until all those experiments had been exhausted. If, on the other hand, an attempt were made to reason before all the experiments had been exhausted, assumptions might be made that were considerably away from the truth, and this had happened many times. For that reason the Authors' experiments, as well as those by Professor Mesnager, in Paris, had been followed with the greatest interest by Italian engineers. Not only engineers, but also men of scientific attainments, such as Professor Corbino and others, had followed the experiments with the greatest interest.

The Authors showed in a practical way how engineering ought to be studied and taught, namely, that a proper proportion of physical research work should be undertaken first, to be followed by not an excessive amount of mathematical reasoning. Mathematics had to be used when nothing else was available. High mathematics were like a microscope, which was to be used to see things which could not be seen with the naked eye ; but if a thing could be seen with the naked eye, this should always be used, and Professor Coker's experiments tended to that aim. That was also what Leonardo da Vinci taught. With reference to water, he said : " When you have to deal with water, first do all the experiments, then see what the facts can teach you, and leave the reasoning to the last." That remark could be applied by all students of engineering. Galileo, who was the first to experiment on the resistance of materials, also recommended students to experiment first and to reason about them afterwards. He was well aware of the fact that that was contrary to the teaching of the ancient school of the Greeks, who were great philosophers and not experimenters. The engineers of some countries, fortunately not England or Italy, were, he was afraid, going astray and following the Grecian school. He assured the members of the Institution that he felt that at the Meetings of the Institution of Mechanical Engineers he was really in an atmosphere of proper physical research.

Professor E. G. COKER, in reply, said it was extremely gratifying to his colleagues and himself that the Paper had been so favourably received, and he desired, on behalf of the Authors, including Mr. Ahmed, who was unable to be present, to thank the members for the interest they had shown. Dr. Hele-Shaw covered such a wide field in his remarks that he was afraid time would not allow him to make an adequate reply, but in the few minutes which had been placed at his disposal he would attempt to reply to one of the very interesting matters which had been raised. The point in connexion with the beam, where both bending moment and contact pressure were obtained, was a most interesting one. He believed that Professor Filon had attacked this problem mathematically, and the speaker had himself examined it experimentally (*Engineering*, 19th January 1921), although somewhat superficially. He could not say very much about it, therefore, as so little work had been accomplished on this complicated problem.

He had been extremely interested in Professor Filon's remarks. As many of the Members knew, Professor Filon had devoted many years to the study of the mathematical side of elasticity and to investigations of the physical laws of photo-elasticity. In the discussion he had drawn attention to the results which were obtained with regard to the contact pressures shown by Fig. 24 (page 420). The Authors also regarded these discrepancies between experiment and theory with some concern, especially as this seemed at first sight an ideal case for comparison purposes. There was, however, much better agreement for the cases of flat plates of small area and small cylindrical rollers applied at the sides than for the case of the knife-edge. The explanation of this was possibly due to the fact that a knife-edge in these experiments did not behave as an isolated load, as assumed in the calculations described with reference to Figs. 22 and 23. It pressed into the material like a wedge as the colour photograph, Fig. 25, Plate 17, showed, and under these circumstances the transverse stress q was not that of the formulæ anywhere, nor, for that matter, was the stress p. He was very glad to hear what so high an authority as Professor Filon had said about the physics of celluloid.

He would like to say as regards this Paper that the Authors were well aware that celluloid was not a perfect material (no material was perfect) ; but although not an ideal material to work with from the point of view of its elastic properties, it happened to be exceedingly convenient to use. Glass could not be freely used for engineering purposes, because it was impossible to make the models required. He desired to point out that nitro-cellulose, like most other materials, varied greatly in quality, and at the commencement of this work especial care was taken to select the most perfect pieces as regards optical quality from pre-war stocks at the factory. As the war progressed, makers found it more and more difficult to obtain their proper basic materials, and even now that difficulty had not been entirely overcome, and the optical quality of post-war material was only slowly approaching that of pre-war material. This made little difference to its commercial use, but as the material on which Professor Filon was working was all of post-war manufacture, he would, as time went on, probably find that the imperfections of this later material, to which he called attention, were less and less apparent. The speaker also desired to say that when he was making observations, all the colour effects which had been seen that evening, were compared with those of another standard member, which was stressed to the same extent. The model and the calibration member were stressed at the same time and continued to be so stressed from the nature of the experimental method adopted, so that if the discrepancy due to the effect of continued loading was present, it was going on at the same time in both bodies, and was therefore practically eliminated by his arrangements. He was perfectly familiar with the fact that creep could occur, and it must be recognized by those who desired to undertake such work. He thought the remarks he had made helped to show how the difficulty could be met.

With reference to Professor Filon's remarks on the subject of isoclinic lines, he had always used them to the fullest extent he could, as even his earliest Papers showed, but Professor Filon and he approached the subject from rather different angles, the one as a mathematician and physicist and the other as an engineer. The mathematician could choose his own problem, and he usually selected one which was mathematically solvable and for which therefore isoclinic lines could be calculated and tested by experiment, but the engineer could not so choose, because he was obliged to deal with existing problems which were generally beyond any form of mathematical reasoning available. The engineer investigator was, in fact, bound to take the machine or structure as it existed or had to be built, and try and find out what the stresses were. He was very glad to note that the maker of the colour photographs was present, to whom Professor Filon had paid such a high tribute, and he was sure that Mr. Norris appreciated it.

With regard to Mr. Stromeyer's remarks, he was afraid the suggested cure for preventing indentation was not very useful. As the Paper showed, there was some difficulty in measuring the distribution of pressure between two flat surfaces pressed together, and if two semi-circular surfaces of discontinuity were introduced, it would become necessary to make an elaborate investigation of the distributions across the surfaces of separation, while the problem became less interesting from a practical point of view owing to the cuts in the material.

He could not agree with the remarks Mr. Stromeyer had made with regard to the use of stream-lines, as that method was not applicable, except in very special cases. The fundamental equation of plane stress was $\nabla^4 \cdot \chi = o$ and if that equation could be solved generally, every kind of stress distribution in a plane was *ipso facto* solved. But this stage had not yet been reached. The fundamental equation in plane hydrodynamics was, however, $\nabla^2 \cdot \chi = o$, and there were apparently some cases in which these two equations became identical. These coincidences had proved very misleading, and many writers, both here and abroad, had assumed quite inaccurately that lines of stress and steam-lines were identical. It was impossible for stream-lines to give the information required for stress problems, because they were essentially different, as the fundamental differential equations showed, one being of a higher order than the other.

With regard to Mr. Stromeyer's further remarks, he desired to reiterate what he said in his opening remarks before reading the Paper, that an attempt was made here to deal with flat bodies pressed against flat bodies only. He might, however, say that the Authors had done a great deal of work on bodies of circular form, but he could not discuss this on the present occasion, beyond mentioning that several cases had been worked out, including some problems on rings and rollers.

The Authors were greatly obliged to Professor Luiggi for his interesting remarks and especially for the point of view he took as to the way engineering investigation ought to be conducted. They appreciated these remarks very much, especially from so high an authority on engineering as the President of The Italian Institution of Civil Engineers.

Discussion in Manchester, on Thursday, 21st April 1921.

The CHAIRMAN (Mr. CHARLES DAY, *Member of Council*) said he wished on behalf of the members to thank the Author for his most valuable Paper and for giving them a most interesting Lecture. To see the actual effects of stress illustrated by colour photography appealed to him, as opening up an entirely new line of investigation of stress problems, and he felt that Professor Coker had put before them ideas which, before many years had passed, would yield very important results.

Professor A. H. GIBSON, D.Sc., said he would like to express his deep appreciation not only of the Paper but also of the long series of research upon this subject which Professor Coker had carried out during the last thirteen or fourteen years. They all knew that, within certain well-defined bounds and under certain well-defined conditions, many problems of stress could be worked out mathematically, sufficiently closely, at all events, for engineering purposes. A point was reached, however, at which the stresses became so complex—especially if the structure or member was itself of complex form, or if the application of the stress was anything but simple—at which the distribution of stress passed the bounds of mathematical analysis. Under those conditions, the method outlined and developed by Professor Coker was practically the only one which gave any means of getting even approximate values. Errors of a few per cent might occur—it was difficult to get really accurate determinations of stress from this colour method, but when it was compared with the errors which might creep into a mathematical calculation, it was very evident that, even from that point of view, this method had very much to recommend it.

They had seen from the Paper and from the photographs shown that the application of a lateral load to a tension specimen increased the stress very considerably. Professor Coker had instanced the application of an extensometer by means of two contact points to a specimen. He had also instanced another case—rather a peculiar one—in which the addition of a small collar to a specimen increasing its diameter actually caused the stress in the specimen to increase by something like 30 per cent. They all knew that if the specimen were tested to destruction in a testing-machine, it would not break at that place. He had brought to the Meeting two specimens which bore upon that point rather well. One was a specimen of round iron and the other was of rectangular section. Both were tested in the rough in a testing-machine between ordinary tooth-grips. They would notice the depressions which the grips themselves had made in those specimens. That occurred in every case with such a specimen under test, but in spite of it, out of 100 specimens tested not more than two or three broke in the grips; they always broke between the grips where the stress was more or less uniform. It would also be observed that, in the round specimen for rough purposes of measurement, centre punch-marks were placed at intervals of an inch along its length. That day he had examined about 20 specimens, all of which had been marked in the same place and none of them had broken at a centre punch-mark. The point he wanted to make was that with a ductile material—such as most engineering materials were—tested under static load, these secondary stresses did not seem seriously to affect the ultimate strength of the structure. But it should be emphasized that it was not at all the case under different conditions. Take a material—say one of the specimens of Professor Coker—and subject it to a very rapidly applied load, especially a rapidly alternating load. Then the results became very different, more especially if the material was one of the modern high-tensile steels with a very high elasticity.

Some time ago a large number of tests were made at the Manchester University on specimens of the same general type as those shown by Professor Coker. They were short, about 1½ in. between the shoulders. The shoulders were well-rounded and the diameter of the specimens was about a quarter of an inch. Loads were rapidly applied, about 2,000 alternations a minute. In a large number of cases rupture took place, not in the parallel portion of the specimen, but at the shoulder where Professor Coker's diagrams had shown the maximum stress would occur. He had often wondered why they should break there, and he thought possibly there had been a small flaw in the material, but the true reason was made clear by Professor Coker's research. He thought it was in such cases—rapidly applied loads on materials not very ductile, and of very high tensile properties—that the results shown in the Paper could be more particularly applied.

Professor G. GERALD STONEY, F.R.S., said that Professor Coker's Paper was of the greatest importance to engineers, and explained many fractures that otherwise would be difficult to understand. Professor Coker had shown that in a riveted member, such as in a bridge, the stress at the edge of a rivet-hole was about three times the mean stress due to secondary stresses. Fortunately, by having a ductile material, which was absolutely necessary, such stresses would relieve themselves automatically by the material stretching. In the case of alternating stressses, however, as Professor Gibson had pointed out, such relief did not take place and they got fracture. Similar stresses occurred where there was a change of diameter in a shaft, and he believed engineers were not sufficiently alive to the importance of having a very large radius when there was a change in section. He was not sufficiently familiar with the work to know whether the investigation could be carried out as to what radius was advisable, but if it were, it would be of very great value to engineers.

Another thing which the Paper showed was the importance of having a smooth surface with no surface irregularities. It was well known that tensile test-pieces in many materials were liable to break at the punch-marks put on them to record the elongation. That was especially the case with some of the high tensile manganese bronzes, and other high tensile bronzes. In ordinary high-grade steel the state of the surface made a very considerable difference in the tensile test results. In a large number of test-pieces taken off the same forging of a high tensile steel, it was found that it made a very marked difference whether the test-piece was ground or only turned with the usual finish. Grinding was found to increase the tensile strength by about 1 per cent and the elongation by no less than 7 per cent, while at the same time the variation in the results obtained from individual test-pieces was largely reduced. In the case of the turned test-pieces, there was a variation in tensile strength between the highest and the lowest of the batch—there were twelve or fourteen in the batch—of 4·3 per cent, while in the case of those that were ground it was reduced to 3·7 per cent at the extreme limits. In the case of the elongation the effect was even more marked, as the maximum difference for the turned test-pieces was 14 per cent and for the ground ones only 8 per cent. It was not, perhaps, so much the case with mild steel, but certainly with high carbon steels, nickel chrome and so forth, it was most important to grind the test-pieces. A good finish was most important in all material which was exposed to high stress, especially alternating stress, and there was a real reason for polishing and finishing well such parts as the connecting-rods of aeroplane engines. By that they probably increased the strength very much. He had seen an engine made by Sulzer in which every part was most beautifully finished. It seemed a waste of money, but Professor Coker's results seemed to show it was a real benefit. Of course, in ordinary cases it was very doubtful whether high finish would pay.

He would like to draw attention to the habit of inspectors of making inspection marks all over an article. As a rule they chose for their inspection marks the place which was exposed to highest stress, and if it was analysed he thought it would be found that a perceptible number of breakdowns of aeroplane engines in the war were due to such stamping. In going over an aeroplane, he was astounded at the way beautifully finished pieces of work were marred by inspection marks stamped here and there all over. If Professor Coker's work induced inspectors to pay more attention to where they put their inspection marks, it would be of very great value.

It would be of great interest if endurance tests could be made on similar specimens taken from the same bar and carefully heat-treated in the same way—some turned with an ordinary finish, some rough turned, and others polished, in order to see whether under alternating stress there would not be considerably more endurance with the really well finished article. Mr. Stromeyer had done a great deal of that kind of work and perhaps he had investigated this point.

Mr. C. E. STROMEYER, O.B.E., said he had spoken on the Paper in London, and he did not wish to repeat his remarks, but he would reply to the inquiry which had been made about the subject of polishing. That subject had been very carefully investigated by Messrs. Eden, Cunningham, and Rose.* They made a very large number of tests and polished some specimens. As far as he could remember, the result was that the polished specimens gave better results than the others. Some years ago he analysed those experiments carefully and was struck with the great discrepancies which existed between specimens of the same bar. Fortunately when he made inquiry at The Institution of Mechanical Engineers, they were able to give him the numbers stamped on the specimens, and these numbers were said to be stamped in the order in which they were cut off the rolled bar. On plotting the results on the lengths of the bars, he found that there were "peaks" in the test results occurring about every four feet, and he suggested at the time that

* Proceedings, I.Mech.E., 1911, page 873.

the explanation of those irregularities was that the rolls in which the bars were rolled were not quite regular, and the rolled bars were therefore thicker at some points than at others. When they were finally drawn through the dies, the thicker parts would receive a heavier pressure than the other parts and would therefore be of different quality. This showed that a very slight difference in mechanical treatments might affect the strengths of the materials very much.

The question of the radius for a shaft had been raised. When at Lloyds' he condemned about 60 crank-shafts in marine engines, and, having kept records including indicator diagrams and other details, he was able, some years ago, to present an analysis to the Institution of Naval Architects, in which the conclusion was drawn that the strength of a crank-shaft under the working conditions in a marine engine was not proportional to the cube of the diameter, but to the cube of the diameter multiplied by the square root of the radius of the fillet. This result confirmed the general feeling amongst engineers, that the radius of the fillet ought always to be as large as possible.

The present subject of stress distribution was exceedingly interesting, and all engineers must be grateful to Professor Coker for having started and continued his experiments. If, at some future time, local stresses in structures could be estimated, it would be possible to reduce the factor of safety. He was the first to have coined the expression that the factor of safety was a measure of our ignorance, ignorance as regards the quality of the material and especially ignorance as regarded local stresses. To a large extent, engineers still designed parts of machinery by rather rough though very simple rules. Not knowing the exact behaviour of materials nor the exact local stresses, they had to allow a factor of safety which was at present perhaps not large enough for complicated parts and too large for simple parts.

In the early days of manganese steel, he was at one of the German steel works where they had adopted Hadfield's process of making manganese steel. They showed him large piles of ingots which they had made in a small Siemens furnace. It was, he thought, during the Boer War, and the question of broken bayonets had been very much discussed. It occurred to him at the time that it would be a very good thing to make bayonets of manganese steel. They forged a sword of that steel and it was struck several times on an anvil. It stood the test very well ; the edge of course, was blunted, but otherwise there was no harm done. But one of the managers began to fence with it, and accidentally struck something, when the steel broke off in his hand. It showed that some materials were very unreliable. He believed some of the discrepancies mentioned in the Discussion were due to local strains which existed in the specimens themselves. Of course, mild steel was less likely than harder steels to have these local strains.

Mr. E. G. HERBERT said that one of the photographs showing stresses in a tensile specimen with enlarged ends reminded him of a fact that had often struck him in making autographic tests, namely, that fracture usually occurred in the region between the gauge-points and only rarely at the ends outside the gauge-points. According to what was generally believed as to the effect of a change of section, one would expect fracture to take place usually near the enlarged ends, but this was quite contrary to his experience. As to plain specimens without enlarged ends, he was in complete agreement with a previous speaker (Professor Stoney). Although such specimens were mutilated and sensibly reduced in section by the sharp teeth of the grips, fracture rarely took place in the mutilated portion which was subjected to compound stresses, but usually in the uninjured central portion where the stresses were uniform.

One of the photographs, showed a tongue of lightly stressed material extending from the enlarged end a long way down the centre of the central or reduced portion of the specimen, and one might assume that a similar tongue of lightly stressed material would appear in a plain specimen, extending from the portion held in the grips. Unstressed material would not ordinarily be regarded as a source of strength. If parts of a specimen were not taking a due share of the load, other parts must be taking an undue share.

Nevertheless, the phenomenon exhibited in the photograph, in conjunction with the comparative immunity from fracture which was observed in the regions where it occurred, seemed to call for further investigation, for which Professor Coker's method would be admirably adapted. He would merely throw out the suggestion that the rather sharp surface of demarcation between the stressed and unstressed regions (a roughly conical surface in cylindrical specimens) might be the natural surface of cleavage when rupture took place, thereby giving rise to the familiar conical type of fracture, and that this conical area being greater than the cross-sectional area of the specimen, the liability to fracture would be less. He invited Professor Coker to confer a benefit on the engineering profession by proving that a change of section was a source of strength.

Dr. GILBERT COOK said he was not clear about one point in regard to the test of the manganese steel bar which broke at the point of attachment of the extensometer. Professor Coker had described the stress distribution produced by the pressure of the extensometer pivots, and he (Dr. Cook) understood Professor Coker to attribute the fracture at that point to the disturbance thus caused. Presumably the extensometer was removed before fracture occurred, and it would therefore be only the residual stresses due to local overstrain at the points of attachment that could modify the stress distribution. He would be glad to know whether the Author had measured such residual stresses.

He was very much interested to hear that Professor Coker had applied this method of stress determination to cases in which the surfaces were not plane. He referred to this because he was at present interested in a problem which, although it had no direct connexion with contact stresses, the description of Professor Dalby's test-piece afforded some excuse for doing so. If this bar, instead of being solid, were hollow, that is, if it represented a pipe with comparatively thin walls, and the collar represented a flange, or one of a series of rings which were now occassionally used to strengthen high-pressure hydraulic pipe-lines—what was the effect of that ring or flange when internal pressure was applied? Mathematical analysis appeared to show that, so far from strengthening the pipe, the rings might in certain cases actually weaken it; that is to say, the stresses in the pipe might be increased instead of reduced. He would, however, hesitate to accept such a conclusion in the absence of experimental confirmation. This was a case in which a flat plate, however adequate for the determination of the stresses in Professor Dalby's test-piece, could not in any sense represent the conditions. He would be glad to know whether Professor Coker had investigated the stresses in tubes and pipes by the optical method. If it could be developed to do so, considerable light would be thrown not only on that problem, but on the analogous and much more important problem of the stresses in a boiler-shell near the end-plates.

Mr. HARRY RICHARDSON said he could understand the various colours of the spectroscope at the point of contact of the teeth due to the pressure exerted, but would like an explanation of the colours near the root of one of the teeth of the gear-wheels.

Mr. HERBERT CARRINGTON remarked that he would like to add a few words further to what had been said about the tensile test illustrated on page 409. The load extension curve continued to rise right up to the point of fracture, whereas for ordinary mild steel and wrought iron the curves fell near the end, the fall corresponding with local drawing out of the bar in the vicinity of the fracture. It would appear that the material tested by Professor Coker was somewhat exceptional, and that it fractured like a brittle material although the elongation was considerable.

From a consideration of the dimensions of the fractured bar, there was a slight drawing out, but an examination of the curve strongly suggested that this was not purely local, but extended some distance on either side of the fracture. He would like to ask Professor Coker whether he thought that centre dots and scratches tended to determine the place of fracture only in the case of brittle material or material which broke like a brittle material, that is, with little or no local drawing out, and that the tendency was negligible in cases where the local drawing out was considerable, as, for instance, for ordinary mild steel.

Mr. LEONARD F. MASSEY said that he had been much struck by the distribution of stress shown when the material was subjected to an evenly distributed load, the greatest intensity of stress in such a case being at the extreme ends of the loaded surface. His interest lay in the fact that a very similar phenomenon was already familiar to him in hot forging of steel. When striking a steel bar with a narrow pallet in a steam-hammer, lines exactly similar to those shown in the figure could, under favourable circumstances, be distinguished by an intenser redness in the material. If a cold bar were struck hard and continuously by such a pallet, the phenomenon could sometimes be seen by the bar heating to redness down the lines in question. These lines marked an inverted isosceles triangle whose base was the line of the pallet, the other two sides being the red lines in question.

It would seem therefore that the phenomena revealed by Professor Coker's experiments on solid materials had a very close parallel when dealing with a plastic material, and the speaker had been very much pleased to see something of a theoretical demonstration of that with which he had already been familiar by closely watching the process of hot forging.

Professor COKER, in reply, said that many of the matters raised in the discussion were not only interesting but also difficult to answer. He was afraid he had not made it clear during the evening that all the natural colour photographs of the material which were shown on the screen were of stress distributions within the yield-point, except where an obviously black patch occurred due to the crushing of the material locally. It was important to emphasize that in a ductile material like mild steel or nitro-cellulose the distributions changed in the plastic region and the stresses tended to equalize. In a brittle material, on the other hand, such as cast-iron, the distributions shown probably persisted up to fracture, and results were obtained which corresponded closely to those which one would expect from photo-elastic observations taken within the elastic limit.

He was very interested to hear what had been said about alternating stresses. Apparently alternating stress gave results in a ductile material corresponding to what might be expected in a brittle material under steady load. Therefore, as Professors Gibson and Stoney pointed out, they obtained fractures such as were indicated as probable by the colour photographs ; although they might not be so obtained in a static test.

Professor Stoney had drawn attention to the fractures which sometimes occurred at changes of section in turbine-shafts and at the rectangular grooves turned in them. This was a very interesting matter and one of his assistants had been working on it recently, but the measurements had not advanced far enough to enable an adequate reply to be given ; speaking generally, the distributions appeared to resemble corresponding cases described by the speaker in a recent Paper on stresses in tension members referred to above. Another question which had been raised by one of the speakers, Dr. Cook, with respect to the stresses in members of circular cross-section, and here again the speaker would not take up time by describing the flat cell method referred to in the above Paper in order to obtain an experimental solution.

As regards the effect of surface finish upon fracture and extension, Professor Stoney had made some interesting observations of great value, and possibly not very well known. In regard to these he would like to mention some experiments made a short time ago in the laboratory at University College upon the effects of slight roughnesses and the like. These could be easily produced and their effects observed at the edges of a nitro-cellulose specimen. If, for example, one took an ordinary flat bar and loaded it in a testing-machine, a uniform colour was obtained corresponding to the load and there would be no unusual effect at the edges if these latter were quite smooth. If, however, a slight mark was made, say, by scoring with a penknife, and the region was examined by aid of a microscopic attachment, bands of colour could be observed of a characteristic form and generally consisting of two bright rays at about 45° to the edge, and tending to persist as the load increased. With alternating stress well below the yield-point of the main body of the material, these colour effects became permanent and penetrated more and more into the material as the number of alternations increased, and finally they seemed to cause fracture of the specimen. That, he believed, was what roughness of surface tended to produce in stressed material. It initiated high local stress which enlarged the scope of its effect with increasing load and also with repetition of load. It appeared to the speaker that this subject could be investigated still further by optical means, and he would like to point out that an advantage of such a method of attack was that microscopic methods seemed unnecessary. Scratches and roughnesses of surface when magnified could be represented in the plain bar by some form of notch, and provided there were no other disturbing boundary condition or applied force near, one could take any size of notch which happened to be convenient for examination. This appeared to be a sound line of attack, and mathematical reasoning seemed to support it. So far as the speaker was aware, there were no accurate mathematical solutions of the stress distributions caused by notches, but, in the related subject of holes of circular and elliptical orms in tension members, analysis showed that the size of the hole made no difference to the distribution in its neighbourhood, provided the bounding edges of the plate were sufficiently far away, and the inference that this also held for notches under similar conditions seemed a legitimate one. Some experiments on this basis were now n progress for investigating scratches in materials, and the speaker hoped to be able to describe them at a later date.

He was very interested in Mr. Stromeyer's remarks, and especially in his formula for the strength of a shaft which brought in the radius of the fillet. The speaker did not quite understand how it was obtained, but it appeared to him to be something new and important because all the experimental work done so far seemed to show that the radius of the fillet had a very marked effect.

The point which Mr. Herbert raised could be answered by reference to the detailed measurements of stresses in tension members used in testing materials and referred to above, and as he was bound

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to be brief, he would content himself with indicating this source of further information.

With regard to Dr. Cook's inquiries, his earlier remarks concerning the method of enclosing solid cylindrical specimens in flat cells were intended to explain what had been done in this direction. So far as the speaker was aware nothing had been done with regard to pipes, but he saw no reason why a similar arrangement should not be devised on these lines for examining pipes in which strengthening rings were used to reinforce them in hydraulic mains, particularly those used for distributing water to turbines in hydraulic power-houses. The same remark also applied to boiler shells, and although it might be extremely difficult to construct such experimental apparatus, he did not think it was inherently impossible.

The gear-wheels which Mr. Richardson had drawn attention to were only intended to illustrate some general introductory remarks in the Paper, but it might be of interest to mention that the stresses at the boundaries of the teeth forms had been measured, except at the points of contact, while the directions of the stresses throughout were also known, but so far they had not worked out the distribution of stress in the interior of a wheel. Mr. Richardson had remarked on the high value of the stress at the roots of the teeth in contact, and this high stress was generally found, but what was still more worthy of note in large-toothed wheels with radial arms, was the large stresses at places in the rim and in the arms far away from the teeth in gear.

In the test on the "Hadfield" steel bar, referred to by Mr. Carrington, there was a slight reduction in section at the fracture. It was important to remember in this connexion that the fracture occurred at one end of the gauge length where two small indentations occurred owing to the attachment of the extensometer in the early part of the test. As the data showed, the extension was remarkably uniform throughout.

The remarks of Mr. Massey on forging seemed important and worth following up. They seemed to be very interesting cases of the behaviour of stresses in plastic material, although possibly there might be other features of interest in them. One of the advances which would have to be made sooner or later—he hoped it would be soon—was to find out what were the stresses which occurred when a body had passed its yield-point and got into a more or less plastic state. It was really the most interesting work that was outstanding in the subject of strength of materials. The forging of hot materials was one of those cases. Possibly it was more complicated than the case of a cold material because of the temperature conditions, but in any case, the distribution of plastic stress was a line of investigation which would have to be tackled almost immediately, if they were to obtain useful results with regard to cutting tools in which The Institution of Mechanical Engineers was so much interested.

Communications.

Sir HENRY FOWLER, K.B.E., Member of Council, wrote that he would be glad if Professor Coker would say whether he had made any experiments to note the effect of vibration on material or specimens while under stress, as, practically speaking, in all cases where structures were in such a condition, they were accompanied by some vibration, and if the effect of this, which was often great, could be shown, it would be of very considerable interest.

With regard to the floating surfaces in contact, could Professor Coker say to what degree of accuracy these surfaces were true. It seemed to be almost impossible to get a satisfactory flexible or semi-flexible substance to interpose between two surfaces in contact, as in nearly all cases the intermediate substance tended to "flow."

With regard to the illustrations which showed the excess of pressure on the edge of the block bearing on the larger surface, a considerable time ago in an effort to find a suitable substance to interpose between two surfaces, the writer used several sheets of hard paper; these were put on the top of an ordinary compression test-piece, and upon pressure being applied, it was found that the surfaces of the test-piece became concave, the paper being finally cut through all along the circumference of the test-piece, although its thickness in the centre was very little increased. This, he thought, bore out the illustrations which the Authors had given.

In connexion with the pressure that was applied through a V surface, what was the radius of the edge of this? He thought this had a very material effect on the stresses set up at the point of contact. He would also like to have further information with regard to the relative behaviour of xylonite and steel, and what was the composition of the Hadfield's steel mentioned in connexion with Fig. 19 (page 409). Steel was of course a very complex substance, and its behaviour depended very largely upon its composition.

Dr. W. H. HATFIELD wrote that the Authors were to be congratulated on the useful application of the optical method of attack to important practical problems. They showed that so far as the mathematical results could be reduced to numerical values, the latter fully substantiated the results of the optical analysis, thus giving further confidence in the results for those cases which could be solved optically but not mathematically.

With regard to the value of results of tensile tests, the inaccuracies caused by putting centre dots or scratches on test-pieces, either for simple measurement of elongation, or for the purpose of carrying an extensometer, had long been felt to be serious, but just how far they interfered with the results obtained was hitherto a matter mainly for speculation. Professor Coker had now put the matter on a firmer footing. A simple way of translating the results from the small models to actual test-pieces, was to compare the stresses set up in the models by the transverse pressures with the longitudinal stresses that would have resulted if forces of equal value had been applied along the length of the specimen, remembering that generally, similar relative values would hold good for the ordinary test-pieces loaded similarly. Taking the figures given in Table 10 (page 418), the longitudinal stress due to a load W of 100 lb. would have been 500 lb. per square inch. The stresses set up when this load was applied transversely by knife-edges, varied from 249 to 431 lb. per square inch longitudinally, at the critical cross-section and from 920 to over 6,000 transversely, presumably reaching indefinitely high values in the neighbourhood of the knife-edges. On a tensile test-piece carrying extensometer of appreciable weight, (such as the Ewing type), it appeared then, that the material in the immediate neighbourhood of the knife-edges was seriously broken down, and the damage might extend an appreciable distance on either side. How far would this invalidate the extensometer readings? The distinct rounding-off of most load-extension curves just beyond the limit of proportionality was possibly exaggerated due to this cause, giving an apparent value for the limit, appreciably lower than the real one. The stress longitudinally due to the transverse pressure fortunately was relatively small compared with the stresses applied longitudinally in carrying out the test—the former being probably limited to say ½ ton per square inch locally. This meant, however, that at best one could not depend upon proportional limit determinations to a much finer limit than ½ ton per square inch. Would the Authors say whether, in their opinion, the inaccuracy was likely to be more than this? Would the fact that the transverse stresses, due to the clamping of the instrument, were much higher, cause still earlier yielding of the test-piece longitudinally, and so increase the error of the determinations of the limit of proportionality?

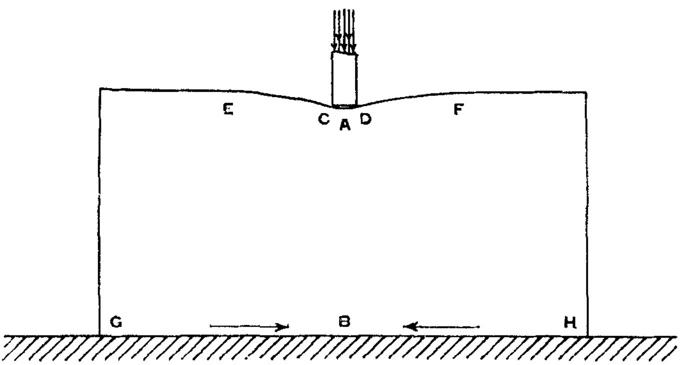
With regard to the form of test-piece employed by Professor Dalby, having collars turned with the piece, this seemed to be an improvement in every way on the ordinary method of centre-dotting. It was unfortunately outside the range of everyday practice.

The Authors referred to the stresses due to line contact in roller bearings, gear teeth, etc., and the writer hoped that in a further Paper on this subject, the Authors would deal with the local intensities of the pressures set up, and their distribution over the areas of contact, in such cases.

Dr. WALTER A. SCOBLE wrote that his association with Professor Coker when he commenced his researches by the use of polarized light, allowed him to appreciate the large amount of careful work, requiring delicate manipulation, which had been done in connexion with this Paper. It was probable that many members would not have time to study this long Paper in detail, and he would suggest that Professor Coker should add a conclusion to give the general results and to indicate their bearing on engineering practice If engineers would study these general investigations and picture the behaviour of the material which led to the results obtained, they would be rewarded by having a clearer understanding of the stress distribution in the complex cases which so often arose.

The Authors emphasized the fact that when a bar of material was supported on a slab and was subjected to a pressure at the centre of the top surface, the pressure between the bar and the slab was confined to the region near the line of the pressure and the ends of

FIG. 34.



the bar bent upwards. Further, the pressure on the top of the bar was most intense at the edges of the surface on which the load was applied. These phenomena could be explained in a general fashion as follows: *The compression in the direction AB, Fig. 34,* was accompanied by elongations normal to AB. In the lower portion of the bar the elongations might actually occur, or, if this would involve sliding on the surface GBH, and the frictional resistance was great, the elongation was opposed by frictional surface shears in the directions of the arrows. No doubt there was lateral elongation in the region of the application of the pressure, but this was less than the increase of length required by the new shape of the indented surface, ECDF. Consequently there was a virtual shortening here under the load. The similarity to the case of a beam was evident, compression above and elongation below, and so the ends of the bar canted up. The natural, continuous shape of the depressed surface was like ECADF, curved at all points except where the curvature changed sign. Excessive pressures at C and D were required to depress these points to the depth of A. A rigid base-plate of a column would exert more than the average pressure near its boundaries, but if the base-plate were not rigid the ends were relieved, and, in the case of insufficient stiffening, the pressure was greatest on the central region.

He was gratified to find that the lateral strain-meter which he introduced had been developed by the addition of slides and micrometers to control the position of the needles, and that with this addition the instrument gave records which permitted a more complete solution of the problems investigated than was furnished by the optical method alone. It was stated that, under tension stress, the law of optical retardation of nitro-cellulose compounds was proportional to stress and not to strain, and that it was linear, even when the material was stressed far beyond the elastic limit and very near the yield-point. Hence the optical measurements required no correction at high stresses. But surely, if the strain were no longer proportional to the stress, that fact caused the stress distribution to be altered, it was no longer similar to that in an elastic material under the same system of loading, such as steel. It appeared that the Authors measured the stress in the xylonite model, but at the places of high stress the distribution was only very approximately similar to what it would be in steel. But many materials had no better elastic properties than nitro-cellulose, so for them the agreement was sufficiently good.

It appeared to the writer that Professor Coker did not truly present the effect of centre dots on a specimen under a tension test. Contact pressure had little if anything to do with the fracture taking place through the dots. The case was more closely comparable with the notched bars previously investigated. Even so, in his experience, although the centre dots sometimes localized the fracture, particularly with the harder materials, the effect on the breaking stress was within the limit of experimental error. The effects investigated by Professor Coker were smoothed out after yield. The example given in the Paper corresponded to a pressure at the extensometer screws one-third the tension on the specimens. This corresponded to a very low tension on a metal specimen. The effect of the revised stresses on the elastic constants would be too small to be detected by an extensometer. At the elastic limit of a low tensile steel the initial screw-pressure was probably about one-hundredth of the tension on a standard specimen of 0·564 inch diameter. But the initial screw-pressure was reduced or disappeared through the lateral contraction of the specimen, because most extensometers had fixed and not spring-loaded screws. Immediately after yield, rigid clamps were often found to be loose in the centre dots. There seemed to be little doubt that the discontinuity at the centre dot, and not the pressure applied there, localized the fracture, and the fact that specimens of similar material marked for elongation measurements often failed in the same way supplied a confirmation of this view.

Mr. C. HUMPHREY WINGFIELD thought that the value of this highly mathematical Paper would be greatly increased if the Authors would summarize their conclusions in a form which could be easily made use of in general practice. He gathered that they would be somewhat as follows, but he hoped that the Authors could state them in a more definite manner and perhaps add to their number :—

1. It was of no use attempting to calculate the distribution of stress, as this could not " be determined except by experimental means " (page 365).

2. Lateral pressures of only 15 lb. transmitted by knife-edges to the sides of a 1-inch by $\frac{1}{10}$-inch test-piece materially altered the distribution of stress due to a longitudinal pull of three times that amount, the stress being greater in their immediate neighbourhood (page 412).

3. A mere scratch or indentation, although without continuance of lateral pressure, often altered the distribution sufficiently to determine the position of the fracture in high-tension steel

(page 407), or even when the material had become very plastic, Fig. 19 (page 409).

4. An increase of stress occurred near the enlarged end of a standard test-piece, which was met with under more advantageous circumstances (page 433), in the case of a thin flange such as used by Professor Dalby (page 420).

5. The radius of the fillet round such flanges did not matter much (page 430), nor was the form of curve, if other than a radius, very important (page 433).

6. In compression-tests uniform stress could not be approached : (*a*) unless the specimen was bedded on parallel blocks (page 406) of the same material ; (*b*) if the thickness was more than about one-quarter of the width (page 407).

Professor COKER wrote that he quite appreciated the suggestions made by Dr. Scoble and Mr. Humphrey Wingfield that in a somewhat lengthy Paper, a summary of the main conclusions might be useful, but, as the discussions in London and Manchester had shown that no difficulty had been experienced in understanding what these were, he had not attempted to add further descriptive matter. The surface effects in a loaded block no doubt contributed much to the changes in the form of the block, and Dr. Scoble emphasized their importance, possibly unduly so, having regard to the colour effects observed in the body of the block shown in Fig. 11, Plate 16.

The writer also desired to acknowledge the help of Dr. Scoble in devising a simpler form of extensometer to that which he used originally for lateral measurements of metals and rocks * under stress. There seemed some ground for supposing that, for very

* "A Laboratory Apparatus for Measuring the Lateral Strains in Tension and Compression Members, with some Applications to the Measurement of the Elastic Constants of Metals," by E. G. Coker. Proc. Roy. Soc. Edin., vol. xxv, 1904–5.

"An Investigation into the Elastic Constants of Rocks, more especially with reference to Cubic Compressibility," by Frank D. Adams and Ernest G. Coker. Carnegie Institution of Washington, 1906.

high stresses, there was some redistribution of stress even when the law of optical effect held. If this was so, it afforded an explanation of some small discrepancies between calculation and experiment which had been observed in one or two cases in which along a given line there was an extremely great stress variation, which latter, when measured by optical means, was found to agree within one or two per cent along the whole line except at the very highest points, when differences in defect of from 4 to 5 per cent were measured. Allowing for possible experimental errors these results seemed to indicate a stage at which equalization was commencing. Dr. Scoble appeared to be under some misapprehension as to the points raised in the Paper regarding the harmful effects which might be caused by extensometer screws. It was clear that these effects arose, in general, from the permanent indentations, and this was especially emphasized by the description of the test on "Hadfield" steel, in which the cross-sectional area was finally about 43 per cent smaller than the original cross-section, except at the fracture, where it was even less. Further, it would be noted (Fig. 19, page 409) that the load at which the extensometer was removed was about one-third of the breaking load.

Dr. Hatfield raised the question of the influence of the indentations upon the form of the load-extension curve, and especially the effect on the limit of proportionality, and on the form of the curve just beyond this point. There was no doubt that indentations tended to change the limit of proportionality in their neighbourhood, and their influence increased with decreasing gauge length. It was evident from Fig. 21 (page 414), that the conditions in that case were of such a character that simple stress distribution was not obtained for a length of the specimen at least equal to its width in the neighbourhood of the applied transverse load, and as extensometer readings were a summation of the alterations of form over the whole gauge length measured in the axial direction, it seemed evident that the lowering of the limit of proportionality would depend on this length and become more marked as it decreased. The cross-sectional shape and area also introduced other variable factors. It seemed probable, therefore, that given conditions in which these variables could all exert their maximum influence together, it would be quite possible to obtain a limit of proportionality much more than half a ton per square inch below its true value, accompanied by an abnormal rounding off of the load extension curve beyond this limit.

As regards vibration effects mentioned by Sir Henry Fowler, no measurements had been made as yet on their influence. The flat contacting surfaces were true planes within one or two ten thousandths of an inch, and were prepared by the usual scraping processes. The effect of hard paper on the end surfaces of compression blocks was very interesting, and it was worth noting that the writer had obtained the same type of effect with nitro-cullulose blocks under similar conditions. The V knife-edges were made by grinding the inclined surfaces, and the edges produced were similar in character to the steel knife-edges sometimes used in precision balances, but the radii of curvature of the edges were not measured, and as these pieces had been used for other work since, it is now impossible to say what their exact condition was when the measurements were made. The relative behaviour of xylonite and steel was difficult to describe briefly, but fortunately these matters had been examined recently * in such a manner that an effective comparison could be made.

Lastly, the writer was informed by Sir Robert Hadfield that the composition of the test bar A8725 was as follows:—

Iron	86·09	per cent
Manganese	12·64	,,
Carbon , . . .	1·27	,,
	100·00	

* "Researches on the Elastic Properties and the Plastic Extension of Metals," by Prof. W. E. Dalby, F.R.S. Phil. Trans., 1920.

"The Stress-Strain Properties of Nitro-Cellulose and the Law of its Optical Behaviour," by Prof. E. G. Coker, F.R.S., and K. C. Chakko, M.Sc., Phil. Trans., 1920.

Fig. 8. *Stress in a Large Rectangular Block loaded over a small part of its upper surface.*

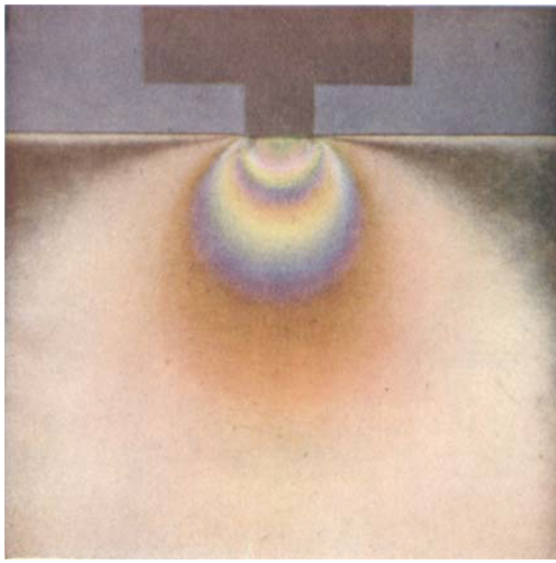


Fig. 11. *Stress in a Square Block loaded centrally over a part of its upper surface.*

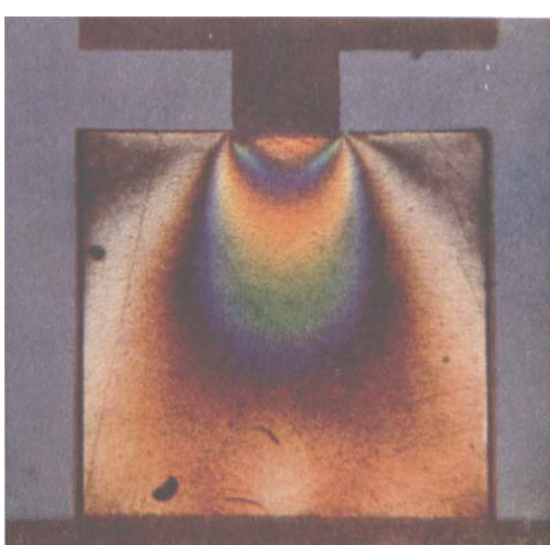


Fig. 25. *A Tension Test-piece gripped laterally by knife edges.*

Fig. 29. *The Stress in a Flat Tension Test Specimen of the Contour used by Prof. Dalby.*