ON THE SINGULAR SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS OF THE FIRST ORDER WITH TRANSCENDENTAL COEFFICIENTS

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I. *The theory as at present accepted.*

1. If we start from the complete primitive

$$f(x, y, c) = 0, \tag{I}$$

and form the equation
$$\frac{Df}{Dx} + \frac{Df}{Dy}\frac{dy}{dx} = 0, \tag{II*}$$

and then eliminate c between (I) and (II), we obtain a differential equation of the form

$$\phi\left(x,\ y,\ \frac{dy}{dx}\right) = 0,$$

or as we shall write it
$$\phi\,(x, y, p) = 0. \tag{III}$$

Let us now replace c in (I) by a certain function of x, y, viz., C, so that the relation between x, y is

$$f(x, y, C) = 0, \tag{IV}$$

where C satisfies
$$\frac{Df(x, y, C)}{DC} = 0: \tag{V}$$

then on differentiating (IV), and using (V), it follows that

$$\frac{Df(x, y, C)}{Dx} + \frac{Df(x, y, C)}{Dy}\frac{dy}{dx} = 0. \tag{VI}$$

* D denotes partial differentiation when the independent variables are x, y, c or x, y, C or x, y, p; ∂ denotes partial differentiation when the independent variables are x, y; δ denotes partial differentiation when the independent variables are x, c; Δ denotes partial differentiation when the independent variables are x, p; d denotes total differentiation with regard to x; and p denotes $\frac{dy}{dx}$.

Now if C be *not* such a function of x, y as to make

$$\frac{Df(x, y, C)}{Dx} = 0, \qquad \text{(VII)*}$$

and

$$\frac{Df(x, y, C)}{Dy} = 0, \qquad \text{(VIII)*}$$

then it is possible to eliminate C from (IV) and (VI) and the result is again (III).

Hence (IV), on the hypothesis made as to C, is an integral of (III).

It is to be observed that in discussing any particular differential equation, whilst it may be impossible to deduce the singular solution from one form of the primitive (IV) by the use of (V), it may be possible to deduce it from some other form of the complete primitive.

Lagrange supposed that y was expressed in terms of x and c, and instead of using the condition (V) he used the condition here represented by

$$\frac{\delta y}{\delta c} = 0,$$

which is not necessarily equivalent to (V). (See Examples 1, 3, 4.)

The theory of singular solutions as at present accepted assumes that every singular solution of (III) can be obtained in this way, but this is an assumption which is the converse of what has been proved. I have never met with any proof of the converse theorem, nor on the other hand with any particular case in which the converse theorem is not true.

2. If we now turn to the problem of determining the singular solution from the differential equation (III), then the theory as at present accepted states that, if a singular solution of (III) exist, it must satisfy simultaneously (III) and the two following equations

$$\frac{D\phi(x, y, p)}{Dp} = 0, \qquad \text{(IX)}$$

$$\frac{D\phi(x, y, p)}{Dx} + p\frac{D\phi(x, y, p)}{Dy} = 0, \qquad \text{(X)}$$

unless p be infinite, in which case $\dfrac{D\phi(x, y, p)}{Dy}$ must vanish.

Now it is true that, if a singular solution exists which satisfies (IX),

* If C satisfy (IV) and (V), then (VII) and (VIII) are equivalent to a single condition only.

then it must also satisfy (X), except in the exceptional case noted above. But the theory as at present accepted is an assumption of the converse of this proposition.

Cases are given below (see Examples 1, 3, 4) in which it is impossible to satisfy equation (IX) at all. A case is given (see Example 5) in which the singular solution satisfies (IX), but reduces the left-hand side of (X) to an indeterminate expression.

[2*a*.* Commencing with the primitive in the form taken by Lagrange

$$y = f(x, c), \tag{XI}$$

we obtain the differential equation

$$\frac{dy}{dx} = \frac{\delta f(x, c)}{\delta x}, \tag{XII}$$

or, writing $p = \dfrac{dy}{dx}$,

$$p = \frac{\delta f(x, c)}{\delta x}. \tag{XIII}$$

Solving (XIII) for c, and substituting in (XI), we obtain an equation of the form

$$y = \chi(x, p). \tag{XIV}$$

I proceed to calculate the partial differential coefficient of y with regard to p, treating x as constant.

Let Δ denote partial differentiation when x and p are treated as independent variables.

Then differentiating (XI) and (XIII) partially with regard to p, we obtain

$$\frac{\Delta y}{\Delta p} = \frac{\delta f(x, c)}{\delta c} \frac{\Delta c}{\Delta p},$$

and

$$1 = \frac{\delta^2 f(x, c)}{\delta c\, \delta x} \frac{\Delta c}{\Delta p};$$

therefore

$$\frac{\Delta y}{\Delta p} = \frac{\delta f(x, c)}{\delta c} \bigg/ \frac{\delta^2 f(x, c)}{\delta c\, \delta x}. \tag{XV}$$

If now the differential equation be taken in the form

$$\phi(x, y, p) = 0, \tag{III}$$

* This article was added to the paper in February, 1917, subsequently to the date at which the paper was presented to the Society. It is based on Mansion's work, which is referred to in the Historical Section below. The parts so added to the paper are enclosed in square brackets. Mansion's reasoning is differently expressed, but it leads to the same conclusions.

then $$\frac{D\phi}{Dy}\frac{\Delta y}{\Delta p}+\frac{D\phi}{Dp}=0. \tag{XVI}$$

Equating the values of $\frac{\Delta y}{\Delta p}$ in (XV) and (XVI), we obtain

$$\frac{\delta f(x,\,c)}{\delta c}\Big/\frac{\delta^2 f(x,\,c)}{\delta c\,\delta x}=-\frac{D\phi}{Dp}\Big/\frac{D\phi}{Dy}. \tag{XVII}$$*

Mansion shows that if $\frac{\delta^2 f(x,\,c)}{\delta c\,\delta x}$ vanishes then each curve represented by the complete primitive has contact of the second order with the envelope. He takes as the condition for finding a singular solution

$$\frac{\Delta y}{\Delta p}=0.$$

It is clear from the equation (XVII) that when

$$\frac{\delta f(x,\,c)}{\delta c}=0,$$

we must have $$\frac{D\phi}{Dp}\Big/\frac{D\phi}{Dy}=0;$$

and therefore it is not sufficient to take

$$\frac{D\phi}{Dp}=0,$$

an equation which it may not be possible to satisfy. It is necessary to consider also the possibility that $\frac{D\phi}{Dy}$ may be infinite.

Mansion did not in his paper apply his condition to any equation having transcendental coefficients. This has been done in Examples 1, 3, 4, 6, 7.]

II. *History of the Subject*, §§ 3–7.

3. The preceding theory, excepting (*a*) the restriction upon C involved in the condition that the equations (VII) and (VIII) are not to be satisfied, and (*b*) the work of Mansion referred to in § 2*a*, dates back to the memoir by Lagrange, "Sur les intégrales particulières des équations différen-

* Mansion does not give this equation.

tielles" (*Nouveaux Mémoires de l'Académie royale des Sciences et Belles-Lettres de Berlin*, année 1774), printed in the 4th volume of his works.

Lagrange assumes the converse theorems referred to in §§ 1 and 2 above. Thus in § 5, p. 12, of his memoir, after giving the substance of § 1 of this paper, he says:

"Il est facile de démontrer qu'il n'y a pas d'autres combinaisons possibles qui puissent fournir des intégrales de cette espèce non comprises dans l'intégrale complète."

And again, after obtaining the conditions (IX) and (X) (of § 2 of this paper), in § 14, p. 26, on the hypothesis that the differential equation contains no radical signs and no transcendental functions, he says, in § 15, p. 28:

"Et quoique la démonstration précédente soit fondée sur l'hypothèse que l'équation proposée ne renferme aucune fonction transcendante il n'est cependant pas difficile de se convaincre que la même conclusion aura lieu quelles que soient la nature et la forme de cette équation."

It is strange that with this last mentioned idea in his mind Lagrange should have limited his examination of particular cases to equations containing no transcendental functions; and the strangeness is increased by the fact that he refers, *l.c.*, p. 6, to a memoir by Laplace, "Sur les solutions particulières des équations différentielles" (*Mémoires de l'Académie royale des Sciences de Paris*, année 1772), printed in the 8th volume of Laplace's works, in which Laplace compares the differential equations

$$\frac{dy}{dx} = qy^n, \quad \frac{dy}{dx} = q(\log y)^{-r}, \quad \text{and} \quad \frac{dy}{dx} = qe^{-1/y};$$

where q is some function of x, y, which is neither zero nor infinite when $y = 0$, and n and r are both positive (*l.c.*, pp. 330, 331).

4. Darboux in a paper, "Sur les solutions singulières des équations aux dérivées ordinaires du premier ordre", published in the *Bulletin des Sciences Mathématiques*, 1873 (pp. 163, 164), bases his work on the conditions (IX) and (X).

He refers to a paper by Mansion, entitled "Note sur les solutions singulières des équations différentielles du premier ordre" (*Bulletins de l'Académie royale des Sciences, des Lettres et des Beaux-Arts de Belgique*, 1872, pp. 149–169); but he dissents from the view, taken by Mansion, that the condition

$$\frac{D\phi}{Dp}\Big/\frac{D\phi}{Dy} = 0,*$$

should be substituted for the condition

$$\frac{D\phi}{Dp} = 0.$$

Darboux says, "Mais ces dernières règles ne s'appliquent qu'aux cas tout à fait singuliers où la fonction ϕ contient des expressions mal déterminées, radicaux, etc.; tout au plus si elles peuvent fournir la solution $p = \infty$, que ne donne pas l'équation

$$\frac{D\phi}{Dp} = 0,$$

et qu'on peut toujours écarter par un changement d'axes coordonnés" (*l.c.*, p. 158, footnote 2).

It will be seen, however, in Examples 1, 3, 4, that their singular solutions are obtainable by Mansion's condition, though not obtainable by Lagrange's condition; and further that they are not merely cases in which $p = \infty$.

In a later memoir, "Sur les solutions singulières des équations aux dérivées partièlles du premier ordre", published in 1883 in T. 27 (sér. 2) of the *Mémoires presentés par divers savants à l'Académie des Sciences de l'Institut National de France*, Darboux constructs his theory on equations corresponding to (IX) and (X). These equations he describes as a convention adopted to give a definition of the singular integral applying to all cases (*l.c.*, p. 114). On p. 2, he says that the hypothesis of the existence of a complete integral made by Lagrange was not justified. He says (pp. 2 and 3) that Lagrange supposed the existence of a complete integral which was finite and continuous throughout the whole extent necessary to determine the envelope of the curves represented by the complete integral. He says that this hypothesis excludes, without mentioning it, the case in which the curves represented by the complete primitive have singular points or cease to be continuous or even to exist beyond a limited region of the plane.

Darboux, like Lagrange, does not apply his theory to any particular differential equation with transcendental coefficients. It appears to me to be difficult to see how Darboux's theory, which is based (*l.c.*, pp. 4, 5) on

* Or the equation obtainable from this by the interchange of the dependent and independent variables.

the properties of equations obtained as the result of the elimination of constants, and corresponding closely to the theory of envelopes of algebraic curves and surfaces, could apply without exception to the envelopes of transcendental curves; but if I rightly understand Darboux's reference to "all cases" his theory was intended to include these.

5. Cayley says (*Messenger of Mathematics*, Vol. 6, 1877, p. 24) that the theorem he had proved regarding the existence of an envelope of a family of algebraic curves does not extend to transcendental curves.

6. Hamburger's paper "Ueber die singulären Lösungen der algebraischen Differentialgleichungen erster Ordnung" (*Crelle*, Bd. 112, 1893) is based on the equations (IX) and (X).

But he differs (p. 206) from the opinion expressed by Darboux that the hypothesis made by Lagrange as to the existence of a complete primitive was unjustified, and he shows (pp. 221–229) how the complete primitive is to be found. In this connection he refers not only to the work of Madame Kowalewsky but also to that of Darboux himself.

He also differs from the opinion of Cayley quoted above. His opinion is (pp. 206, 207) that the existence of envelopes or singular solutions has nothing to do with the transcendental or algebraic nature of the general integral.

He points out (p. 238) that inasmuch as the conditions I have marked (VII) and (VIII) [which if taken in conjunction with (IV) and (V) are equivalent to one independent condition] have to be satisfied in order that there may be no envelope, the general case is that in which an envelope exists; whilst, on the other hand, when we start from the differential equation (III), it is necessary that a condition amongst the coefficients [arising from the fact that it must be possible to satisfy simultaneously (III), (IX), and (X)] should be satisfied in order that an envelope may exist, so that the case in which the envelope does not exist is the general one.

7. Laplace, on pp. 327–331 of his memoir, deals with the question of determining whether a given solution of a differential equation is or is not comprised in the general integral, it being supposed that the general integral is unknown. His conclusion is that if the values of the differential coefficients from the second* onwards, as determined from the differential equation, are respectively equal to their values as determined

* The first differential coefficients are necessarily the same.

from the given solution, then that given solution is a particular integral, comprised in the general integral.

If this is not the case, then the solution is a singular integral. This last statement is certainly true, but the one preceding it cannot in all cases be relied on, as will be seen from the discussion of Example 1 below.

Laplace's argument rests on the possibility of expanding the ordinates of points on the complete primitive and on the given solution in ascending powers of the abscissa.

The demonstration therefore may not apply when the points on the given solution are points of discontinuity on the curves represented by the complete primitive.

Laplace's test gives a correct result in Example 2, but fails in Example 1.

It follows from Hamburger's investigations (*l.c.*, p. 219) that the equation considered by Laplace, mentioned above, viz.

$$\frac{dy}{dx} = qy^n,$$

has $y = 0$ for a singular integral when $0 < n < 1$, but a particular integral when $n \geqslant 1$; and thus Laplace's test gives the correct result in this case.

Hamburger's investigations, which are limited to differential equations in which the coefficients are algebraic, do not apply to the other two equations, mentioned by Laplace.

In one of these $$\frac{dy}{dx} = q(\log y)^{-r} \quad (r > 0);$$

therefore $$\frac{d^2y}{dx^2} = \frac{dq}{dx}(\log y)^{-r} - rq^2y^{-1}(\log y)^{-2r-1},$$

so that $$\frac{d^2y}{dx^2} = \infty, \quad \text{when} \quad y = 0.$$

But the solution $y = 0$ makes $\dfrac{d^2y}{dx^2} = 0$.

Hence $y = 0$ is a singular solution, and Laplace's test gives the correct result.

In the other case $$\frac{dy}{dx} = qe^{-1/y}.$$

Consequently the first and every following differential coefficient tend to zero as y tends to $+0$.

If Laplace's test could be relied on, this would make $y = +0$ a particular integral.

But the equation is not integrable, and I do not know of any method of deciding the question. It should also be observed that, since $e^{-1/y}$ is undefined when $y = 0$, it is probably useless to call $y = 0$, or even $y = +0$, an integral.

I discuss in detail a somewhat similar case: see Example 9 below.

III. *Construction of Differential Equations with Singular Solutions not obtainable from the Differential Equations by the Condition*

$$\frac{D\phi}{Dp} = 0.$$

8. Differential equations with transcendental coefficients have not, I believe, been studied with the object of determining their singular solutions.

As has been said Hamburger's investigations are definitely limited to differential equations with algebraic coefficients.

With regard to Darboux's investigations the passages referred to above would seem to indicate that he held the view that the theory he constructed completely covered all cases of whatever kind.

If, then, the existing theory is incomplete it will be best tested by considering cases in which the differential equations have transcendental coefficients.

Some preliminary theorems which are required are proved in §§ 9–12.

Preliminary Theorems, §§ 9–12.

9. If v be an analytic function of x, y, such that all the values of x, y which make $v = 0$, also make $\frac{\partial v}{\partial x} = 0$, then they also make $\frac{\partial v}{\partial y} = 0$, and

$$\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y^2} - \left(\frac{\partial^2 v}{\partial x\, \partial y}\right)^2 = 0.$$

For suppose x, y and $x+\delta x$, $y+\delta y$ to be neighbouring points on $v = 0$. Then we must have $v = 0$ at $x+\delta x$, $y+\delta y$; therefore

$$v + \delta x\,\frac{\partial v}{\partial x} + \delta y\,\frac{\partial v}{\partial y} + \text{higher powers of } \delta x,\ \delta y = 0.$$

But $$v = 0, \quad \frac{\partial v}{\partial x} = 0$$

Hence $$\delta y \frac{\partial v}{\partial y} + \text{higher powers of } \delta x,\ \delta y = 0.$$

This cannot be satisfied unless $\frac{\partial v}{\partial y} = 0.$

Again, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 0$, at $x+\delta x$, $y+\delta y$; therefore

$$\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\delta x + \frac{\partial^2 v}{\partial x \partial y}\delta y + \text{higher powers of } \delta x,\ \delta y = 0$$

and $$\frac{\partial v}{\partial y} + \frac{\partial^2 v}{\partial x \partial y}\delta x + \frac{\partial^2 v}{\partial y^2}\delta y + \text{higher powers of } \delta x,\ \delta y = 0.$$

These reduce to $$\frac{\partial^2 v}{\partial x^2}\delta x + \frac{\partial^2 v}{\partial x \partial y}\delta y + \ldots = 0,$$

$$\frac{\partial^2 v}{\partial x \partial y}\delta x + \frac{\partial^2 v}{\partial y^2}\delta y + \ldots = 0,$$

and cannot be simultaneously true unless

$$\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y^2} - \left(\frac{\partial^2 v}{\partial x \partial y}\right)^2 = 0,$$

at all points for which $v = 0$.

If v be a rational integral function of x and y, then the geometrical interpretation of the conditions

$$v = 0, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0, \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y^2} - \left(\frac{\partial^2 v}{\partial x \partial y}\right)^2 = 0,$$

is that every point on the curve represented by $v = 0$ is a cusp. But this is impossible.

The equation $v = 0$ cannot represent a proper curve. Consequently v must break up into factors; and as every point on $v = 0$ satisfies the above conditions, it is necessary that every factor in v should be repeated twice at least.

As examples of functions of v satisfying the above conditions which are not rational, we may take

$$v = (1+w^2)^{\frac{1}{2}} - 1,$$

or $$v = (x^2+y^2+w^2)^{\frac{1}{2}} - (x^2+y^2)^{\frac{1}{2}},$$

where w is a rational function of x and y, and the radicals are all taken with the positive sign.

10. *If v be any analytic function of x, y which does not satisfy the conditions in the preceding article, and u be any function of x, y which is not infinite when $v = 0$, and* $\dfrac{\partial(u, v)}{\partial(x, y)} = 0$ *when $v = 0$, then some constant k exists such that $u - k = 0$ when $v = 0$.*

It is supposed, as in § 9, that all the values of x, y which make $v = 0$ do not make $\dfrac{\partial v}{\partial y} = 0$. Such values define y as a function of x, say

$$y = \psi(x).$$

Then the equation $v(x, y) = 0$ gives $v[x, \psi(x)] \equiv 0$, and therefore

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\psi'(x) = 0,$$

when $y = \psi(x)$. Now we wish to examine the value of u when $v = 0$, *i.e.* $y = \psi(x)$. Then

$$u(x, y) = u[x, \psi(x)] = U(x),$$

say. And now

$$\frac{dU(x)}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\psi'(x),$$

where $y = \psi(x)$. But $\psi'(x) = -\dfrac{\partial v}{\partial x}\Big/\dfrac{\partial v}{\partial y}$, where $y = \psi(x)$, and therefore

$$\frac{dU(x)}{dx} = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)\Big/\frac{\partial v}{\partial y},$$

where $y = \psi(x)$. But $\dfrac{\partial v}{\partial y} \neq 0$, when $v = 0$, *i.e.* when $y = \psi(x)$; and

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 0,$$

when $v = 0$, by hypothesis. Thus $\dfrac{dU(x)}{dx} = 0$, *i.e.* $u(x, y)$ is a constant, say k, when $v(x, y) = 0$: *i.e.* the integral $v = 0$ is included in the integral $u(x, y) = k$.

The application which will be made of this theorem is the following.

If $u =$ const. is a complete primitive of (III), and $v = 0$* any solution

* The case where v satisfies the conditions of § 9 is excepted. For if all the values of x, y which make $v = 0$ also make $\dfrac{\partial v}{\partial x} = 0$, then, as has been shown, they also make $\dfrac{\partial v}{\partial y} = 0$, and in this case the Jacobian $\dfrac{\partial(u, v)}{\partial(x, y)}$ vanishes, but the vanishing of the Jacobian when $v = 0$ implies no relation between u and v. I give two examples (Nos. 6 and 7), in which v is taken to have the form referred to.

of (III) such that $\frac{\partial(u, v)}{\partial(x, y)}$ vanishes when $v = 0$, then a particular integral

$$u - k = 0$$

exists which includes the integral $v = 0$.

11. The converse of the preceding theorem is required.

If the integral of $v = 0$ be included in the integral $u = k$, then

$$\frac{\partial(u, v)}{\partial(x, y)}$$

will vanish when $v = 0$.

We proceed as in the preceding article as far as the point

$$u(x, y) = u[x, \psi(x)].$$

Then, if $v = 0$ is included in $u = k$,

$$u[x, \psi(x)] = k,$$

and the left-hand side of this equation does not include x at all.

Consequently $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \psi'(x) = 0,$$

where $y = \psi(x)$. But, as in the preceding article,

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \psi'(x) = 0,$$

where $y = \psi(x)$. Hence, since $\frac{\partial v}{\partial y} \neq 0$ when $v = 0$, *i.e.* $y = \psi(x)$, we get, by the elimination of $\psi'(x)$,

$$\frac{\partial(u, v)}{\partial(x, y)} = 0,$$

when $v = 0$.*

12. The two preceding theorems make it possible to construct a test for distinguishing a singular solution from a particular solution of a differential equation of the first order, whenever it is possible to express the complete primitive in the form

$$u = k.$$

For, let v be any solution of the differential equation, and then take u in such a form that u is not infinite when $v = 0$, and then form

$$\frac{\partial(u, v)}{\partial(x, y)}.$$

* I have substituted the argument of §§ 10 and 11, which was suggested to me by Mr. G. H. Hardy, for that contained in the original version of the paper.

It follows from the preceding work that if $\dfrac{\partial(u, v)}{\partial(x, y)}$ vanishes when v vanishes, then v is included in the complete primitive (§ 10); and if v is included in the complete primitive, then $\dfrac{\partial(u, v)}{\partial(x, y)}$ vanishes when v vanishes (§ 11). This gives the following test.

If $\dfrac{\partial(u, v)}{\partial(x, y)}$ *do not vanish when* v *vanishes, then the solution* $v = 0$ *is not included in the complete primitive, and therefore it is a singular solution.*

13. It sometimes happens that a solution of a differential equation is both a particular and a singular solution of a differential equation.

E.g., $y = 0$ is both a particular integral and a singular solution of the differential equation

$$p^4 = 4y(xp-2y)^2,$$

of which the complete primitive is

$$y = c^2(x-c)^2.$$

Hamburger, *l.c.*, p. 218, gives tests by which a solution could be recognised as both singular and particular.

In the example just referred to, the solution $y = 0$ arises in two ways :—

(1) by putting $c = 0$, which shows it is a particular integral;

(2) by putting $c = x$, which shows it is a singular integral, and possesses the envelope property.

In this paper I am concerned solely with those solutions which can only be obtained from the complete primitive by replacing the arbitrary constant by a function of the variables. In special cases, as in the one referred to above, a solution may have a double origin, but these are not considered here.

14. The following primitive has a considerable degree of generality, but it is not quite so general as that discussed in § 19 below, of which it is a particular case.

Consider the equation

$$f(x, y) + \phi(w)\,\psi(x, y) = c, \qquad \text{(XVIII)}$$

where w is some function of x and y which is such that all the values of x, y, which make $w=0$, do not make $\frac{\partial w}{\partial x}=0$ or $\frac{\partial w}{\partial y}=0$; where $\phi(w)$ vanishes when $w=0$; where $\phi'(w)$ is infinite when $w=0$; where $f(x, y)$ neither vanishes, nor reduces to a constant, nor becomes infinite when $w=0$; and where $\psi(x, y)$ neither vanishes nor becomes infinite when $w=0$.

Differentiating (XVIII), we get

$$\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}\right)+\phi(w)\left(\frac{\partial \psi}{\partial x}+\frac{\partial \psi}{\partial y}\frac{dy}{dx}\right)+\phi'(w)\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\frac{dy}{dx}\right)\psi(x, y)=0. \quad \text{(XIX)}$$

The last term is indeterminate when $w=0$, because $\phi'(w)$ is infinite, and

$$\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\frac{dy}{dx}=0.$$

Dividing (XIX) by $\phi'(w)$, the equation can be put into the form

$$\frac{1}{\phi'(w)}\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}\right)+\frac{\phi(w)}{\phi'(w)}\left(\frac{\partial \psi}{\partial x}+\frac{\partial \psi}{\partial y}\frac{dy}{dx}\right)+\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\frac{dy}{dx}\right)\psi(x, y)=0. \quad \text{(XX)}$$

Now $w=0$ is an integral of (XX).

But $w=0$ is not a particular case of the complete primitive, for if we put $w=0$ in the complete primitive, then in order that the equation may be satisfied c must be equal to the value of $f(x, y)$ when $w=0$. But, by hypothesis, $f(x, y)$ is not constant when $w=0$. Hence $w=0$ is not included in the complete primitive, and must therefore be a singular solution.

To see whether $w=0$ satisfies the condition (IX), we observe that the equation (XX), after putting $\frac{dy}{dx}=p$, becomes

$$\frac{1}{\phi'(w)}\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial y}\right)+\frac{\phi(w)}{\phi'(w)}\left(\frac{\partial \psi}{\partial x}+p\frac{\partial \psi}{\partial y}\right)+\left(\frac{\partial w}{\partial x}+p\frac{\partial w}{\partial y}\right)\psi(x, y)=0.$$

The condition (IX) is therefore

$$\frac{1}{\phi'(w)}\frac{\partial f}{\partial y}+\frac{\phi(w)}{\phi'(w)}\frac{\partial \psi}{\partial y}+\psi(x, y)\frac{\partial w}{\partial y}=0,$$

which, when $w=0$, reduces to

$$\psi(x, y)\frac{\partial w}{\partial y}=0.$$

But, by hypothesis, neither $\psi(x, y)$ nor $\frac{\partial w}{\partial y}$ vanishes when $w = 0$.

Hence the condition $\frac{D\phi}{Dp} = 0$ cannot be satisfied.

[Let us see whether the condition $\frac{D\phi}{Dp}\Big/\frac{D\phi}{Dy} = 0$ can be satisfied. If we calculate $\frac{D\phi}{Dy}$, then one of the terms contains the expression

$$\phi''(w)/\{\phi'(w)\}^2.$$

It will be seen in § 16 that, in all the cases which I have been able to construct, this expression becomes infinite when $w = 0$. So that it may be expected that $\frac{D\phi}{Dy}$ will be infinite, and therefore that the condition $\frac{D\phi}{Dp}\Big/\frac{D\phi}{Dy} = 0$ will be satisfied.]

For the primitive (XVIII), the condition (V) reduces to the equation $-1 = 0$, which cannot be satisfied. But, in all the cases which I have been able to construct, I have found it possible, after transforming the primitive, and taking Lagrange's form of the complete primitive, to derive the singular solution from the equation

$$\frac{\delta y}{\delta c} = 0.$$

15. It will next be shown that the solution $w = 0$ possesses the envelope property.

For writing $$u = f(x, y) + \phi(w)\,\psi(x, y),$$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \phi(w)\frac{\partial \psi}{\partial x} + \psi(x, y)\,\phi'(w)\,\frac{\partial w}{\partial x},$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} + \phi(w)\frac{\partial \psi}{\partial y} + \psi(x, y)\,\phi'(w)\,\frac{\partial w}{\partial y}.$$

Now, when $w = 0$, $$\phi'(w) = \infty.$$

Hence at a point at which $w = 0$ meets one of the primitives $u = \text{const.}$, we have

$$\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y} = \frac{\partial w}{\partial x}\Big/\frac{\partial w}{\partial y} \qquad \text{(XXI)}$$

so that, at a point where a complete primitive $u = \text{const.}$ meets $w = 0$, these two curves have the same tangent.

It might at first sight appear likely that it would follow from (XXI) that

$$\frac{\partial(u, w)}{\partial(x, y)}$$

would vanish where $w = 0$, but that is not the case, for

$$\frac{\partial(u, w)}{\partial(x, y)} = \frac{\partial(f, w)}{\partial(x, y)} + \phi(w)\frac{\partial(\psi, w)}{\partial(x, y)}.$$

Hence, when $w = 0$, $$\frac{\partial(u, w)}{\partial(x, y)} = \frac{\partial(f, w)}{\partial(x, y)}.$$

Now, if $\dfrac{\partial(f, w)}{\partial(x, y)} = 0$, when $w = 0$, then, by § 10, a constant k exists such that $f(x, y) = k$ when $w = 0$; but this is contrary to the hypothesis in § 14 regarding $f(x, y)$.

Hence $\dfrac{\partial(u, w)}{\partial(x, y)}$ does not vanish when $w = 0$; and therefore $w = 0$ is a singular solution, according to the test of § 12.

16. Let us now consider the conditions $\phi(w) = 0$, and $\phi'(w) = \infty$, when $w = 0$.

(i) If $\phi(w)$ be an algebraic function of w, these conditions require that $\phi(w)$ must contain a factor w^n, where $0 < n < 1$.

Let us take* $$\phi(w) = w^n.$$

Then the equation (XVIII) becomes

$$w^n = \{c - f(x, y)\} / \psi(x, y):$$

therefore $$w = \{c - f(x, y)\}^{1/n} / \{\psi(x, y)\}^{1/n}.$$

If now we differentiate the primitive in this form with regard to c, then, since $1/n$ is greater than 1, the resulting equation can be satisfied by giving to c the value $f(x, y)$, leading to the singular solution $w = 0$.

Thus the primitive can in this case be transformed into one in which the ordinary method for deducing the singular solution from the primitive is applicable.

* The remaining factors of $\phi(w)$ may be included in $\psi(x, y)$.

The cases $$\phi(w) = \sin(w^n) \quad (0 < n < 1),$$

and $$\phi(w) = \tan(w^n) \quad (0 < n < 1),$$

are not essentially different from this case.

(ii) Taking next the case

$$\phi(w) = (\log w)^{-r} \quad (r > 0),$$

then $$\phi'(w) = -r(\log w)^{-r-1}\, w^{-1}.$$

In this case the equation (XVIII) becomes

$$(\log w)^{-r} = \{c - f(x, y)\}/\psi(x, y).$$

Therefore $$\log w = \{\psi(x, y)\}^{1/r}/\{c - f(x, y)\}^{1/r};$$

and therefore $$w = \exp\left[\{\psi(x, y)\}^{1/r}/\{c - f(x, y)\}^{1/r}\right].$$

If we differentiate the primitive in this form with regard to c, the condition to be satisfied is that the exponential last written, multiplied by

$$-\frac{1}{r}\{\psi(x, y)\}^{1/r}\{c - f(x, y)\}^{-1/r-1},$$

shall tend to zero as c is made to approach $f(x, y)$.

Examples are given below where this is the case: see Examples 1, 4, 6.

In these examples only one curve of the family of complete primitives passes through each point of the plane. Cases of exception may arise in consequence of an indeterminacy (see Example 7).

(iii) Consider next the case,

$$\phi(w) = w(\log w)^n \quad (n > 0).$$

Here $$\phi'(w) = (\log w)^n + n(\log w)^{n-1}.$$

Taking the primitive in the form

$$w(\log w)^n = \{c - f(x, y)\}/\psi(x, y),$$

it is apparently impossible to satisfy the condition (V). Bearing in mind the remark at the end of § 1, we take in place of (V) the condition

$$\frac{\delta y}{\delta c} = 0.$$

Differentiating the primitive with regard to c, treating x as a constant, it follows that

$$[(\log w)^n + n(\log w)^{n-1}]\frac{\partial w}{\partial y}\frac{\delta y}{\delta c}$$

$$= \left\{\left(1-\frac{\partial f}{\partial y}\frac{\delta y}{\delta c}\right)\Big/\psi(x,y)\right\} - \{[c-f(x,y)]/[\psi(x,y)]^2\}\frac{\partial \psi}{\partial y}\frac{\delta y}{\delta c}.$$

Therefore

$$\frac{\delta y}{\delta c} = 1\Big/\left[\{(\log w)^n + n(\log w)^{n-1}\}\frac{\partial w}{\partial y}\psi(x,y) + \frac{\partial f}{\partial y} + \{c-f(x,y)\}\psi^{-1}\frac{\partial \psi}{\partial y}\right].$$

Since neither $\frac{\partial w}{\partial y}$ nor $\psi(x, y)$ vanish when w vanishes, the great square bracket becomes infinite when $w = 0$; and therefore the condition $\frac{\delta y}{\delta c} = 0$ leads to the singular solution $w = 0$ (see Example 3).

Cases of the same kind can be constructed by taking

$$\phi(w) = w(\log w)\{\log(\log w)\}^n,$$

$$\phi(w) = w(\log w)\{\log(\log w)\}[\log\{\log(\log w)\}]^n, \ldots,$$

where $n > 0$.

In cases (ii) and (iii) the locus $w = 0$ is such that, as a point on any one of the curves represented by the complete primitive approaches $w = 0$, the tangent line to the curve tends to coincide with the tangent line to $w = 0$, but the curve usually ends abruptly at the point at which it meets $w = 0$, because if w be negative, $\log w$ is imaginary.

For cases of exception see Examples 6 and 7.

17. A similar phenomenon to that noticed in parts (ii) and (iii) of the preceding article occurs if a certain arbitrary restriction be made, in the case of complete primitives which are algebraic.*

Suppose that, instead of considering the equation

$$p^2x - py + 1 = 0,$$

we consider the equation obtained by solving it for p, but that we make the restriction that the radical shall always be taken with the positive sign, so that we consider

$$p = \{y + (y^2 - 4x)^{\frac{1}{2}}\}/(2x).$$

* Boole, *Finite Differences*, 2nd edition, ch. x, § 22.

Then the integral is $c = \{y + (y^2 - 4x)^{\frac{1}{2}}\}/(2x)$,

and this represents only that portion of a tangent line to the parabola

$$y^2 - 4x = 0,$$

which lies on one side of the point of contact.

18. A noteworthy result is obtained by taking the same form for the complete primitive as in § 14, but with a different hypothesis as to the nature of $\phi(w)$.

Suppose that $\phi(w) = e^{-1/w^2}$,

so that $\phi'(w) = 2w^{-3}e^{-1/w^2}$.

In this case both $\phi(w)$ and $\phi'(w)$ tend to zero as w tends to 0.

The differential equation is now

$$\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right) + e^{-1/w^2}\left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}\right) + 2w^{-3}e^{-1/w^2}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\frac{dy}{dx}\right)\psi(x, y) = 0. \tag{XXII}$$

This equation is not satisfied by $w = 0$.

If, however, the equation is written in the form

$$w^3\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right) + w^3 e^{-1/w^2}\left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}\right)$$
$$+ 2e^{-1/w^2}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\frac{dy}{dx}\right)\psi(x, y) = 0, \tag{XXIII}$$

then $w = 0$ satisfies this equation.

The question arises whether $w = 0$ is to be regarded as an integral of the equation (XXIII) or not.

Goursat, in his *Leçons sur l'intégration des équations aux dérivées partielles du premier ordre*, has on p. 35, ll. 8–15, a passage from which it seems to follow that he would regard a solution of this kind as an integral of the equation.

As against this point of view it may be observed—

(i) That $w = 0$ does not satisfy the equation in virtue of the value it gives for dy/dx, for it causes the term independent of dy/dx and the coefficient of dy/dx each to vanish.

(ii) That the curve $w = 0$ does not touch the curves of the complete primitive where it meets them.

In fact the value of dy/dx, at a point on a complete primitive where it meets $w = 0$, is

$$-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y},$$

which is not equal to

$$-\frac{\partial w}{\partial x}\Big/\frac{\partial w}{\partial y}.$$

These are two reasons for not regarding $w = 0$ as an integral of the differential equation (XXIII).

It seems to me that equation (XXIII) should be regarded as resoluble into (XXII) and $w = 0$; and that $w = 0$ ought not to be regarded as an integral of (XXIII).

19. A more general form of complete primitive than the one in § 14 is the following:—

$$f(x, y, c)+\phi(w)\psi(x, y, c) = 0, \qquad \text{(XXIV)}$$

where $\phi(w)$ has the same properties as in that section, whilst $f(x, y, c)$ and $\psi(x, y, c)$ only differ from $f(x, y)$ and $\psi(x, y)$ by containing c *algebraically*, and $f(x, y, c)$ does not vanish when w vanishes.

The result of differentiating (XXIV) becomes, after division by $\phi'(w)$,

$$\frac{1}{\phi'(w)}\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}\right)+\frac{\phi(w)}{\phi'(w)}\left(\frac{\partial \psi}{\partial x}+\frac{\partial \psi}{\partial y}\frac{dy}{dx}\right)$$
$$+\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\frac{dy}{dx}\right)\psi(x, y, c) = 0. \qquad \text{(XXV)}$$

If we find the values of c which satisfy (XXIV), and put each of them in the left-hand side of (XXV), then the result in each case vanishes when $w = 0$, and therefore the differential equation obtained by eliminating c between (XXIV) and (XXV) will be satisfied by $w = 0$.

Now $w = 0$ is not obtainable from (XXIV) by giving to c any particular constant value, since $f(x, y, c)$ does not vanish when we put $w = 0$.

Consequently $w = 0$ is a singular solution of the differential equation of which (XXIV) is the complete primitive. A particular example of this case is Example 5.

EXAMPLE 1.

Illustration of §§ 7, 14, *and* 16 (ii).

Consider the differential equation

$$p = y\,(\log y)^2.$$

The complete primitive can be expressed in either of the forms

$$x+(1/\log y)-c = 0,$$

or

$$y = \exp[-1/(x-c)].$$

It represents the family of curves obtained by moving the curve $y = \exp(-1/x)$ parallel to the axis of x.

This curve has a salient point at the origin, and touches the axis of x there. Hence $y = 0$ is an envelope of the family.

It is not possible to derive the solution $y = 0$ from the complete primitive by giving to c any constant value.

If we put $f(x, y, c) \equiv x+(1/\log y)-c$, we cannot obtain the singular solution from the condition (V); but if we put

$$f(x, y, c) \equiv y-\exp[-1/(x-c)],$$

the condition (V) is satisfied by putting $c = x$. As x tends to $c+0$, y tends to 0.

If we apply the test of § 12, and put

$$u = x+(1/\log y), \qquad v = y,$$

then

$$\frac{\partial(u, v)}{\partial(x, y)} = 1,$$

so that $y = 0$ ranks as a singular solution.

[It should be observed further that as x tends to $c-0$, $\log y$ tends to infinity, and that we may have not only $y = 0$ but $y = \infty$ as a singular solution. For, if we put $y = 1/Y$ and $x = -X$, the differential equation becomes

$$\frac{dY}{dX} = Y(\log Y)^2,$$

so that its form is unaltered. Hence $Y = 0$ is a singular solution of this equation. Consequently $y = \infty$ should rank as a singular solution of the original equation. The fact that $y = \infty$ is a part of the envelope is perhaps more readily seen by transforming the primitive by replacing y by $(y+1)/(y-1)$. See Ex. 1a.]

If we put $\phi(x, y, p) \equiv p - y(\log y)^2$,

it is impossible to satisfy the condition

$$\frac{D\phi}{Dp} = 0.$$

[In this case $\frac{D\phi}{Dy} = -(\log y)^2 - 2(\log y)$.

Hence the condition $\frac{D\phi}{Dp} \Big/ \frac{D\phi}{Dy} = 0$,

leads to the condition that $\log y = \infty$; and therefore $y = 0$ or $y = \infty$, both of which are parts of the envelope.]

In this case Laplace's Test for a Singular Solution fails. All the differential coefficients of y with regard to x as determined from the differential equation vanish when $y = 0$. But $y = 0$ is not a particular integral as required by Laplace's test. The explanation is that the point in which a complete primitive meets the singular integral is a point of discontinuity on the complete primitive, and the expansion which must be possible if Laplace's argument is to apply is here impossible.

EXAMPLE 1a.

[Consider the differential equation

$$p + \tfrac{1}{2}(y^2 - 1)(\log v)^2 = 0,$$

where $v = (y+1)/(y-1)$.

The integral can be expressed in either of the forms

$$x + (1/\log v) = c,$$

or $$y = [e^{1/(c-x)} + 1] / [e^{1/(c-x)} - 1].$$

Now $y = 1$ and $y = -1$ are both integrals of the differential equation. They are singular integrals, as they cannot be obtained from the complete primitive by giving to c any constant value.

The integral $y = 1$ is obtained by making $c - x = +0$, and the integral $y = -1$ is obtained by making $c - x = -0$.

The primitive has two salient points, viz., at $x = c$, $y = \pm 1$.

At each of these points the tangent is parallel to the axis of x. Hence $y = +1$ and $y = -1$ are parts of the envelope.

The condition $\dfrac{\delta y}{\delta c} = 0$ is satisfied by $c-x = \pm 0$, and so gives the singular integrals.

The condition $\dfrac{D\phi}{Dp} = 0$ cannot be satisfied, but the condition

$$\frac{D\phi}{Dp}\Big/\frac{D\phi}{Dy} = 0$$

is satisfied by $y = \pm 1$.]

EXAMPLE 2.

Illustration of § 7.

Consider the equation $p = y \log y$.

The integral is $\log(\log y) = x-c, \quad \text{if} \quad y > 1,$

but $\log[\log(y^{-1})] = x-c, \quad \text{if} \quad 0 < y < 1.$

If $y < 0$ the integral is imaginary.

Now $y = 0$ is obviously a solution of the differential equation.

Let us examine its relation to the integral

$$\log[\log(y^{-1})] = x-c.$$

This integral can be expressed in the form

$$e^x/[\log(y^{-1})] = \text{const.},$$

and so $y = 0$ is the particular integral given by making the constant vanish.

In this case the second and all higher differential coefficients of y with respect to x vanish when $y = 0$.

Thus Laplace's method furnishes a correct result in this case.

If we put $u = e^x/[\log(y^{-1})], \quad v = y,$

then
$$\frac{\partial(u, v)}{\partial(x, y)} = u;$$

and therefore vanishes when $y = 0$. Therefore

$$\frac{\partial(u, v)}{\partial(x, y)} = 0,$$

when $v = 0$, and so the test of § 12 makes $y = 0$ a particular integral.

If we had taken for u the form

$$u = \log[\log(y^{-1})] - x, \quad v = y,$$

we should have found
$$\frac{\partial(u, v)}{\partial(x, y)} = -1,$$

and the integral would have appeared to be a singular integral. But with this form of u, the condition that u must be taken in a form which is finite when $v = 0$ is not fulfilled, and therefore the Jacobian has not been calculated in accordance with the directions in § 12.

EXAMPLE 3.

Illustration of §§ 7, 14, *and* 16 (iii).

Consider the equation $p = (1 + \log y)^{-1}$.

This is a particular case of the equation noted by Laplace,

$$\frac{dy}{dx} = q(\log y)^{-r},$$

where $r > 0$ and q is neither zero nor infinite when $y = 0$.

The integral is $y \log y - x + c = 0$.

Now $y = 0$ is an integral of the differential equation.

It can only be derived from the complete primitive by putting $c = x$. It is therefore a singular solution.

If we take the primitive in the form

$$f(x, y, c) \equiv y \log y - x + c = 0,$$

we cannot satisfy the condition (V).

If, however, we calculate $\frac{\delta y}{\delta c}$ from the primitive, we get

$$\frac{\delta y}{\delta c} = -(1 + \log y)^{-1},$$

and therefore the condition $\frac{\delta y}{\delta c} = 0$ requires $\log y$ to be infinite, and consequently y to be zero or infinite.

The former gives the singular solution; the latter can be regarded as the particular integral corresponding to an infinite value of c.

It is impossible to satisfy the condition

$$\frac{D\phi}{Dp} = 0.$$

[But $$\frac{D\phi}{Dy} = y^{-1}(1+\log y)^{-2}.$$

therefore $$\frac{D\phi}{Dp}\bigg/\frac{D\phi}{Dy} = y(1+\log y)^{2}.$$

Hence the condition $\dfrac{D\phi}{Dp}\bigg/\dfrac{D\phi}{Dy} = 0$ requires that $y = 0$, and thus gives the singular solution.]

It may be observed that in this case

$$\frac{D\phi}{Dx} + p\frac{D\phi}{Dy} = py^{-1}(1+\log y)^{-2},$$

and is therefore indeterminate when $y = 0$, $p = 0$. It would not be permissible before examining the value of this expression to put

$$p = (1+\log y)^{-1},$$

because this value of p does not arise directly from the solution under consideration, viz., $y = 0$.

EXAMPLE 4.

Illustration of §§ 14 *and* 16 (ii).

Consider the equation

$$p[1+y(\log y)^{2}] = y(\log y)^{2}.$$

The complete primitive is

$$x-y+(1/\log y) = c.$$

There is a salient point at $x = c$, $y = 0$. The tangent at this point is the axis of x.

The family of curves is obtained by moving any one of their number parallel to the axis of x.

The curve has two asymptotes, viz.,

$$y = 1, \quad y = x-c.$$

Starting from the salient point, y increases towards 1, while x tends to $+\infty$.

The curve then passes to the other end and to the upper side of the asymptote $y = 1$.

As x increases from $-\infty$ to $+\infty$, y increases from $+1$ to $+\infty$, and the curve approaches the asymptote $y = x - c$.

Since the solution $y = 0$ can only be obtained from the primitive by putting $c = x$, it is a singular solution.

The same result follows from the test of § 12.

It is not possible to satisfy the condition (V) with the primitive in the form taken above.

But if we calculate $\frac{\delta y}{\delta c}$, we get

$$\frac{\delta y}{\delta c} = -y(\log y)^2/[1+y(\log y)^2],$$

so that $\frac{\delta y}{\delta c} = 0$ gives the singular solution $y = 0$.

The solution $y = 0$ will not satisfy the condition

$$\frac{D\phi}{Dp} = 0.$$

[But $\frac{D\phi}{Dp}\Big/\frac{D\phi}{Dy} = [1+y(\log y)^2]/[(p-1)\{(\log y)^2+2(\log y)\}]$,

so that the condition $\frac{D\phi}{Dp}\Big/\frac{D\phi}{Dy} = 0$ leads to $y = 0$, and thus gives the singular solution.]

EXAMPLE 5.

Illustration of § 19.

$$p^2x + py\log y - y^2(\log y)^4 = 0.$$

The complete primitive is

$$x + (c/\log y) - c^2 = 0,$$

and $y = 0$ is an integral of the differential equation.

But it is a singular integral because to obtain it from the complete primitive it is necessary to put $c = x^{\frac{1}{2}}$. So also is $y = \infty$.

The complete primitive can be written

$$y = \exp[c/(c^2 - x)].$$

If the curve be transformed to the point $x = c^2$, $y = 0$ as origin, it becomes

$$Y = \exp(-c/X),$$

a curve of the same nature as that of Example 1.

The curves obtained by taking $c = +a$, $c = -a$ are reflexions of each other in the line $x = a^2$.

Thus in this case the salient point is confined to the positive part of the axis of x.

There is, however, another singular solution obtainable by making the values of c or of p equal. To obtain it we observe that the point in which the curve

$$x + (c/\log y) = c^2$$

meets the curve

$$x + (a/\log y) = a^2,$$

is given by

$$x = -ac, \quad y = \exp[1/(a+c)].$$

Hence the point at which the curve

$$x + (c/\log y) = c^2$$

meets the envelope is

$$x = -c^2, \quad y = \exp[1/(2c)].$$

The equation of the envelope is

$$1 + 4x(\log y)^2 = 0.$$

This curve satisfies Lagrange's conditions.

On the other hand the singular integral $y = 0$ will satisfy the conditions

$$\phi = 0, \quad \frac{D\phi}{Dp} = 0.$$

But

$$\frac{D\phi}{Dx} + p\frac{D\phi}{Dy} = p^2 \log y + 2p^2 - 2py(\log y)^3(\log y + 2),$$

which contains the indeterminate term $p^2 \log y$, since $p = 0$, $\log y = \infty$, when $y = 0$.

Both singular integrals can be obtained from the primitive, taken in the form

$$y = \exp[c/(c^2 - x)],$$

by using the equations (IV) and (V).

The reason why $\dfrac{D\phi}{Dp} = 0$ is satisfied in this case is that the two primitives through the point $x = a^2$, $y = 0$, for which $c = +a$ and $c = -a$, meet and touch the same line $y = 0$. In this case $y = 0$ is not only an envelope, but seems to rank also as a tac-locus, inasmuch as the tangent lines to the two curves tend to coincide with the same line, as their points of contact approach the salient points.

EXAMPLE 6.

Illustration of §§ 9 and 16 (ii).

$$(v+2)p = y(v+1)(\log v)^2,$$

$$\textit{where } v = (1+y^2)^{\frac{1}{2}}-1.$$

The complete primitive is

$$x+(1/\log v) = c.$$

There is a cusp at $x = c$, $y = 0$; the tangent at the cusp being the axis of x.

There are three asymptotes

$$x = c, \quad y = 3^{\frac{1}{2}}, \quad y = -3^{\frac{1}{2}}.$$

The family of curves is formed by moving any one of their number parallel to the axis of x, which touches each of them, so that it is their envelope.

The figure when $c = 0$ is very similar to that which would be obtained by taking the curve

$$y = \exp(-1/x),$$

together with its reflexion in the axis of x, and then multiplying each ordinate by $3^{\frac{1}{2}}$.

The function v is of the kind considered in § 9, and therefore, if we take

$$u = x+(1/\log v),$$

then $\dfrac{\partial(u, v)}{\partial(x, y)} = 0$ when $v = 0$, for both $\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y}$ vanish when $v = 0$, and it would seem that $v = 0$ is a particular integral.

But the only values of x and y which make $v = 0$ are given by

$$(1+y^2)^{\frac{1}{2}}-1 = 0,$$

and therefore by

$$y = 0.$$

Suppose therefore we take

$$u = x+[1/\log\{(1+y^2)^{\frac{1}{2}}-1\}], \quad v = y,$$

then $$\frac{\partial(u, v)}{\partial(x, y)} = 1,$$

showing that $y = 0$ is a singular solution.

If the complete primitive be thrown into the form

$$y^2 = 2e^{1/(c-x)} + e^{2/(c-x)},$$

then the singular solution can be obtained from the equations (IV) and (V).

If the differential equation be written in the form

$$p = y(v+1)(\log v)^2/(v+2),$$

$$\frac{D\phi}{Dp} = 1,$$

$$-\frac{D\phi}{Dy} = 2\log v + (\log v)^2 \{(v+1)^2 + v\} / \{(v+1)(v+2)\}.$$

Thus the condition $$\frac{D\phi}{Dp} \Big/ \frac{D\phi}{Dy} = 0$$

requires that $\log v$ should be infinite, and therefore gives the singular solution $y = 0$. And $y = \infty$ is also a singular solution.

EXAMPLE 7.

Illustration of §§ 9 *and* 16 (ii).

$$(v+2)(xp-y) = xy(v+1)(\log v)^2,$$

$$where \quad v = \left(1 + \frac{y^2}{x^2}\right)^{\frac{1}{2}} - 1.$$

The primitive is $$x + (1/\log v) = c,$$

and can be put into the form

$$y^2 = x^2 \{2e^{1/(c-x)} + e^{2/(c-x)}\}.$$

The point $x = c$, $y = 0$ is a cusp on every curve of the family, the tangent at the cusp being the axis of x.

The origin is a double point on every curve of the family (a cusp when $c = 0$).

There are three asymptotes,

$$x = c, \quad y = \sqrt{3}\,[x - (2/3)], \quad y = -\sqrt{3}\,[x - (2/3)].$$

Since $y = 0$ makes $p = 0$, and $v = 0$ so that $y(\log v)^2 = 0$, it is an integral. Moreover as $y=0$ touches the curve at $x=c$, it is an envelope, and it is also a cusp-locus.

The function v belongs to the class of functions discussed in § 9.

If we take
$$u = x+(1/\log v),$$

then
$$\frac{\partial(u, v)}{\partial(x, y)} = 0,$$

when $v=0$, because both $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ vanish when $v=0$; but $v=0$ is not a particular integral.

When $v=0$, it is necessary to have $y=0$ or $x=\infty$. If now we take
$$u = x+\left[1/\log\left\{\left(1+\frac{y^2}{x^2}\right)^{\frac{1}{2}}-1\right\}\right],$$
$$v = y,$$

then
$$\frac{\partial(u, v)}{\partial(x, y)} = 1+x^{-1}(v+2)(v+1)^{-1}(\log v)^{-2},$$

which becomes equal to 1 when $v=0$, and so indicates that $y=0$ is a singular solution.

In this case, putting
$$\phi(x, y, p) = (v+2)(xp-y)-xy(v+1)(\log v)^2,$$

we get
$$\frac{D\phi}{Dp} = x(v+2),$$

which does not vanish when $y=0$.

[But
$$\frac{D\phi}{Dy} = (xyp-y^2)\,x^{-2}(v+1)^{-1}-(v+2)$$
$$-x(\log v)^2\left(\frac{2v^2+4v+1}{v+1}\right)-2x(v+2)\log v,$$

which becomes infinite when $y=0$, so that the condition
$$\frac{D\phi}{Dp}\Big/\frac{D\phi}{Dy} = 0,$$

gives the singular solution $y=0$. And $y=\infty$ is also a singular solution.]

EXAMPLE 8.

Illustration of § 16 (i).

$$r\cos(y^r)\,p = y^{1-r} \quad (0 < r < 1).$$

The primitive is $$x - \sin(y^r) = c,$$

and can be written $$y = [\arc\sin(x-c)]^{1/r};$$

and since the exponent $1/r$ is greater than 1, the condition $\delta y/\delta c = 0$ gives

$$c = x,$$

and therefore leads to the singular solution $y = 0$.

The condition $$\frac{D\phi}{Dp} = 0$$

requires that $r\cos(y^r) = 0$ when $y = 0$, which cannot be satisfied.

[On the other hand the condition $\dfrac{D\phi}{Dp}\Big/\dfrac{D\phi}{Dy} = 0$ is satisfied by $y = 0$.]

Taking the case $r = \frac{1}{2}$ as the simplest, we have

$$(c-x)^2 = \tfrac{1}{2}[1-\cos(2y^{\frac{1}{2}})] = \tfrac{1}{2}\left[\frac{4y}{2!} - \frac{(4y)^2}{4!} + \frac{(4y)^3}{6!} - \dots\right].$$

Treating this equation as $f(x, y, c) = 0$, then $\dfrac{Df}{Dc} = 0$ gives

$$\cos(2y^{\frac{1}{2}}) = 1;$$

therefore $$y^{\frac{1}{2}} = s\pi,$$

where s is zero or a positive or negative integer.

If $s = 0$, we get the envelope $y = 0$.

If we take any other value of s we do not get a solution of the equation at all, but a node-locus.

The equation is, if $r = \frac{1}{2}$,

$$\cos(y^{\frac{1}{2}})\,p = 2y^{\frac{1}{2}}.$$

Squaring to remove the irrational powers of y, we get

$$\left[2 - \frac{1}{2!}(4y) + \frac{1}{4!}(16y^2) - \dots\right]p^2 = 8y.$$

If we now make $\dfrac{D\phi}{Dp} = 0$, we get

$$\left[2 - \frac{1}{2!}(4y) + \frac{1}{4!}(16y^2) - \ldots\right] p = 0.$$

Thus (i) $p = 0$ gives the singular solution $y = 0$. And (ii) we have also

$$2 - \frac{1}{2!}(4y) + \frac{1}{4!}(16y^2) - \ldots = 0,$$

i.e. $$1 + \cos(2y^{\frac{1}{2}}) = 0,$$

whence $$\cos(y^{\frac{1}{2}}) = 0;$$

therefore $$y^{\frac{1}{2}} = (2t+1)\frac{\pi}{2},$$

which gives $p = 0$, but, on putting this value of y in the differential equation, we get

$$p = \infty.$$

In this case the differential equation is not satisfied.

We have, in fact, a tac-locus, the difference between the values of the parameters of two touching curves being 2.

If $r = \frac{1}{2}$, the curve has a simple contact with the envelope.

If $r = \frac{1}{3}$, the envelope coincides with an inflexional tangent to each curve.

If $r = \frac{2}{3}$, the envelope coincides with the tangent at a cusp on each curve. In this case the envelope is at the same time a cusp-locus.

If r be negative, $y = 0$ is still a solution of the differential equation, but in this case the complete primitive has a point of indetermination at the point where it meets the singular solution.

EXAMPLE 9.

Illustration of § 18.

Let $$2p + y^3 e^{-1/y^2} = 0.$$

The integral is $$y^{-2} = \log(c+x).$$

For y to be real, we must have $x \geqslant 1-c$. When $x = 1-c$, $y = \pm\infty$.

If $y > 0$, $p < 0$, and as y tends to $+0$, p tends to -0 and x to $+\infty$. If $y < 0$, $p > 0$, and as y tends to -0, p tends to $+0$ and x to $+\infty$.

The curve is symmetrical with regard to the axis of x. As x increases from $1-c$ to $+\infty$, y decreases from $+\infty$ to $+0$ for one branch, and increases from $-\infty$ to -0 for the other branch.

One asymptote is $x = 1-c$. The curve tends to opposite ends of the side of this asymptote for which $x > 1-c$.

Another asymptote is $y = 0$. The curve tends to opposite sides of the end of this asymptote for which $x = +\infty$.

$y = 0$ is a solution of the differential equation. It is a particular integral, obtainable by putting $c = +\infty$.

Since $$y = [\log(c+x)]^{-\frac{1}{2}},$$

$$\frac{\delta y}{\delta c} = -\tfrac{1}{2}[\log(c+x)]^{-\frac{3}{2}}(c+x)^{-1},$$

so that $\dfrac{\delta y}{\delta c} = 0$ when $c = +\infty$.

It might seem therefore that $y = 0$ was a singular solution. But $y = 0$, though it touches all the curves of the system, touches them all at infinity, for $p = 0$ when $y = \pm 0$, $x = +\infty$.

It is not however an envelope in the usual sense, viz., that in which every point of the envelope is the point of contact of one curve of the system.

EXAMPLE 10.

$$p = y^2 e^{-1/y}.$$

The integral is $$x + e^{1/y} = c.$$

The family of curves is obtained by moving the curve $x+e^{1/y} = 0$ parallel to the axis of x. If we change x into $-Y$, and y into $-X$, we get the curve of Example 1, viz.:—

$$Y = e^{-1/X}.$$

As x passes from $-\infty$ to -1, y increases from $+0$ to $+\infty$, and approaches the upper end of the asymptote $x = -1$. The curve then passes to the other side of the lower end of this asymptote. Then, as x increases from -1 to -0, y increases from $-\infty$ to -0.

The tangent at the origin is the axis of y.

Consequently $y = 0$ is not a singular solution of the differential equation.

But $y = 0$ is an asymptote to every curve of the family.

The question noticed in § 18, as to whether $y = 0$ can be regarded as a solution of the equation

$$p = y^2 e^{-1/y}$$

at all, presents considerable difficulty.

Since the integral is $$x + e^{1/y} = c,$$

as y tends to $+0$, x tends to $-\infty$, and p to zero.

The only point, however, at which the curve actually reaches the axis of x is at $x = c$, when y tends to -0, but in this case $p = +\infty$.

To obtain the solution $y = +0$ from the primitive, it is necessary to put $c = +\infty$.

And so, if $y = 0$ is to be regarded as a solution, it must be regarded as a particular integral.

But since the family of curves is obtained by moving the curve

$$x + e^{1/y} = 0$$

parallel to the axis of x, there is no point of that axis, at a finite distance from the origin, which both lies on a complete primitive and is such that the complete primitive touches the axis at it.

The equation considered in this example is the most similar in form to that mentioned by Laplace (viz.

$$\frac{dy}{dx} = q e^{-1/y},$$

where q is neither zero nor infinite when $y = 0$) which I have been able to integrate.

Both in the equation considered by Laplace and in the one here studied, where q is zero when $y = 0$, the values of all the differential coefficients tend to 0 as y tends to $+0$.

As has been seen in the study of Example 1, this does not make it certain that $y = 0$ is a particular integral.

EXAMPLE 11.

$$(y^2 + e^{-1/y})\, p = y^2.$$

The complete primitive is

$$x - y - e^{-1/y} = c.$$

There is a salient point at $x = c$, $y = 0$.

The tangent here is $y = x - c$.

There are two asymptotes $y = 0$ and $y = x-c-1$.

Starting from the salient point, as x increases to $+\infty$, y increases to $+\infty$, and the curve approaches the upper end of the asymptote

$$y = x-c-1.$$

The curve then passes to the lower end of this asymptote, and then, as x increases from $-\infty$ to $+\infty$, y increases to -0 and tends to approach the asymptote $y = 0$.

The question noticed in § 18 arises.

Is $y = +0$ to be regarded as an integral?

It satisfies the equation $(y^2+e^{-1/y})\, p = y^2$,

but does not satisfy the equation

$$(1+y^{-2}e^{-1/y})\, p = 1.$$

Moreover $y = +0$ is certainly not an envelope, for each member of the complete primitive meets it at an angle of 45°.

My view is that the equation

$$(y^2+e^{-1/y})\, p = y^2$$

should be regarded as being resoluble into

$$(1+y^{-2}e^{-1/y})\, p = 1,$$

and $y = 0$.