

XLI. *On the Self-induction of Wires.*—Part III.*By* OLIVER HEAVISIDE*.

THE subject of the decomposition of an arbitrary function into the sum of functions of special types has many fascinations. No student of mathematical physics, if he possess any soul at all, can fail to recognize the poetry that pervades this branch of mathematics. The great work of Fourier is full of it, although there only the mere fringe of the subject is reached. For that very reason, and because the solutions can be fully realized, the poetry is more plainly evident than in cases of greater complexity. Another remarkable thing to be observed is the way the principle of conservation of energy and its transfer, or the equation of activity, governs the whole subject, in dynamical applications, as regards the possibility of effecting certain expansions, the forms of the functions involved, the manner of effecting the expansions, and the possible nature of the "terminal conditions" which may be imposed.

Special proofs of the possibility of certain expansions are sometimes very vexatious. They are frequently long, complex, difficult to follow, unconvincing, and, after all, quite special; whilst there are infinite numbers of functions equally deserving. Something of a quite general nature is clearly wanted, and simple in its generality, to cover the whole field. This will, I believe, be ultimately found in the principle of energy, at least as regards the functions of mathematical physics. But in the present place only a small part of the question will be touched upon, with special reference to the physical problem of the propagation of electromagnetic disturbances through a dielectric tube, bounded by conductors.

It will be, perhaps, in the recollection of some readers that Professor Sylvester, a few years since, in the course of his learned paper on the Bipotential, poked fun at Professor Maxwell for having, in his investigation of the conjugate properties possessed by complete spherical-surface harmonics, made use of Green's Theorem concerning the mutual energy of two electrified systems. He said (in effect, for the quotation is from memory) that one might as well prove the rule of three by the laws of hydrostatics, or something similar to that. In the second edition of his treatise, Prof. Maxwell made some remarks that appear to be meant for a reply to this; to the effect that although names, involving physical ideas, are given to certain quantities, yet as the reasoning is purely mathematical, the physicist has a right to assist himself by the physical ideas.

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Certainly; but there is much more in it than that. For not only the conjugate properties of spherical harmonics, but those of all other functions of the fluctuating character, which present themselves in physical problems, including the infinitely undiscoverable, are involved in the principle of energy, and are most simply and immediately proved by it, and predicted beforehand. We may indeed get rid of the principle of energy, and treat the matter as a question of the properties of quadratic functions; a method which may commend itself to the pure mathematician. But by the use of the principle of energy, and assisted by the physical ideas involved, we are enabled to go straight to the mark at once, and avoid the unnecessary complexities connected with the use of the special functions in question, which may be so great as to wholly prevent the recognition of the properties which, through the principle of energy, are necessitated.

Considering only a dynamical system in which the forces of reaction are proportional to displacements, and the forces of resistance to velocities, there are three important quantities—the potential energy, the kinetic energy, and the dissipativity, say U , T , and Q , which are quadratic functions of the variables or their velocities. When there is no kinetic energy, the conjugate properties of normal systems are $U_{12}=0$ and $Q_{12}=0$; these standing for the mutual potential energy and the mutual dissipativity of a pair of normal systems. When there is no potential energy, we have $T_{12}=0$ and $Q_{12}=0$. When there is no dissipation of energy, $U_{12}=0$ and $T_{12}=0$. And in general, $U_{12}=T_{12}$, which covers all cases, and has two equivalents, $\frac{1}{2} Q_{12} + \dot{U}_{12}=0$, and $\frac{1}{2} Q_{12} + \dot{T}_{12}=0$; for, as the mutual potential and kinetic energies are equal, the mutual dissipativity is derived half from each.

Let the variables be x_1, x_2, \dots , their velocities $v_1 = \dot{x}_1, \dots$, and the equations of motion

$$\left. \begin{aligned} F_1 &= (A_{11} + B_{11}p + C_{11}p^2)x_1 + (A_{12} + B_{12}p + C_{12}p^2)x_2 + \dots, \\ F_2 &= (A_{21} + B_{21}p + C_{21}p^2)x_1 + (A_{22} + B_{22}p + C_{22}p^2)x_2 + \dots, \\ &\dots \dots \dots \end{aligned} \right\} \quad (88)$$

where F_1, F_2, \dots , are impressed forces, and p stands for d/dt . Forming the equation of total activity we obtain

$$\Sigma Fv = Q + \dot{U} + \dot{T}; \quad \dots \dots (89)$$

where

$$\left. \begin{aligned} 2U &= A_{11}x_1^2 + 2A_{12}x_1x_2 + A_{22}x_2^2 + \dots, \\ Q &= B_{11}v_1^2 + 2B_{12}v_1v_2 + B_{22}v_2^2 + \dots, \\ 2T &= C_{11}v_1^2 + 2C_{12}v_1v_2 + C_{22}v_2^2 + \dots \end{aligned} \right\} \quad \dots (90)$$

So far will define in the briefest manner, U , T , Q , and activity.

Now let the F 's vanish, so that no energy can be communicated to the system, whilst it can only leave it irreversibly, through Q . Then let p_1, p_2 be any two values of p satisfying (88) regarded as algebraic. Let Q_1, U_1, T_1 belong to the system p_1 existing alone; then, by (89) and (90),

$$0 = Q_1 + \dot{U}_1 + \dot{T}_1, \text{ or } 0 = Q_1 + 2p_1(U_1 + T_1);$$

$$0 = Q_2 + \dot{U}_2 + \dot{T}_2, \text{ ,, } 0 = Q_2 + 2p_2(U_2 + T_2).$$

But when existing simultaneously, so that

$$Q = Q_1 + Q_2 + Q_{12}, \quad U = U_1 + U_2 + U_{12}, \quad T = T_1 + T_2 + T_{12},$$

where U_{12}, T_{12}, Q_{12} depend upon products from both systems, thus:—

$$Q_{12} = 2 \{ B_{11}v_1v_1' + B_{22}v_2v_2' + B_{12}(v_1v_2' + v_2v_1') + \dots \},$$

$$U_{12} = A_{11}x_1x_1' + A_{22}x_2x_2' + A_{12}(x_1x_2' + x_2x_1') + \dots,$$

$$T_{12} = C_{11}v_1v_1' + C_{22}v_2v_2' + C_{12}(v_1v_1' + v_2v_1') + \dots,$$

the accents distinguishing one system from the other, we shall find, by forming the equations of mutual activity $\Sigma Fv' = \dots$, and $\Sigma F'v = \dots$, that is, with the F 's of one system, and the v 's of the other, in turn,

$$\begin{aligned} 0 &= \frac{1}{2} Q_{12} + p_2 U_{12} + p_1 T_{12}, \\ 0 &= \frac{1}{2} Q_{12} + p_1 U_{12} + p_2 T_{12}; \end{aligned} \quad \}$$

adding which, there results the equation of mutual activity,

$$0 = Q_{12} + (p_1 + p_2)(U_{12} + T_{12}), \text{ or } 0 = Q_{12} + \dot{U}_{12} + \dot{T}_{12};$$

and, on subtraction, there results

$$0 = (p_1 - p_2)(U_{12} - T_{12}), \quad . \quad . \quad . \quad (91)$$

giving $U_{12} = T_{12}$, if the p 's are unequal. But this property is true whether the p 's be equal or not; that is, $U_{11} = T_{11}$ when p_1 is a repeated root. Various cases of the above are discussed in 'The Electrician,' November 27 and December 11, 1885, with special reference to the dynamical system expressed by Maxwell's electromagnetic equations.

The following applies to Maxwell's system, using the equations (4) to (10) of Part I. (Phil. Mag. August 1886). A comparison with the above is instructive. Let E_1, H_1 and E_2, H_2 be any two systems satisfying these equations, with no impressed forces, or $e=0, h=0$. Then the energy entering the unit volume per second by the action of the first system

on the second is

$$\begin{aligned}\text{conv. } \mathbf{V}\mathbf{E}_1\mathbf{H}_2/4\pi &= (\mathbf{E}_1 \text{ curl } \mathbf{H}_2 - \mathbf{H}_2 \text{ curl } \mathbf{E}_1)/4\pi, \\ &= \mathbf{E}_1\mathbf{F}_2 + \mathbf{H}_2\mathbf{G}_1, \\ &= \mathbf{E}_1\mathbf{C}_2 + \mathbf{E}_1\dot{\mathbf{D}}_2 + \mathbf{H}_2\dot{\mathbf{B}}_1/4\pi. \quad \dots (92)\end{aligned}$$

Similarly, by the action of the second system on the first,

$$\text{conv. } \mathbf{V}\mathbf{E}_2\mathbf{H}_1/4\pi = \mathbf{E}_2\mathbf{C}_1 + \mathbf{E}_2\dot{\mathbf{D}}_1 + \mathbf{H}_1\dot{\mathbf{B}}_2/4\pi. \quad \dots (93)$$

Addition gives the equation of mutual activity. And, subtracting (93) from (92), we find

$$\begin{aligned}\text{conv. } (\mathbf{V}\mathbf{E}_1\mathbf{H}_2 - \mathbf{V}\mathbf{E}_2\mathbf{H}_1)/4\pi &= (\mathbf{E}_1\dot{\mathbf{D}}_2 - \mathbf{E}_2\dot{\mathbf{D}}_1) \\ &\quad - (\mathbf{H}_1\dot{\mathbf{B}}_2 - \mathbf{H}_2\dot{\mathbf{B}}_1)/4\pi; \quad \dots (94)\end{aligned}$$

since $\mathbf{E}_1\mathbf{C}_2 = \mathbf{E}_1k\mathbf{E}_2 = \mathbf{E}_2k\mathbf{E}_1 = \mathbf{E}_2\mathbf{C}_1$, if there be no rotatory power, or \mathbf{C} be a symmetrical linear function of \mathbf{E} . Similarly for \mathbf{D} and \mathbf{E} , and \mathbf{B} and \mathbf{H} . Hence, if the systems are normal, making $d/dt = p_1$ in one, and p_2 in the other, (94) becomes

$$\text{conv. } (\mathbf{V}\mathbf{E}_1\mathbf{H}_2 - \mathbf{V}\mathbf{E}_2\mathbf{H}_1)/4\pi = (p_2 - p_1)(\mathbf{E}_1\mathbf{D}_2 - \mathbf{H}_1\mathbf{B}_2/4\pi). \quad (95)$$

Therefore, by the well-known theorem of Convergence, if we integrate through any region, and U_{12} , T_{12} be the mutual electric energy and the mutual magnetic energy of the two systems in that region, we obtain

$$U_{12} - T_{12} = \frac{\sum \mathbf{N}(\mathbf{V}\mathbf{E}_2\mathbf{H}_1 - \mathbf{V}\mathbf{E}_1\mathbf{H}_2)/4\pi}{p_1 - p_2}, \quad \dots (96)$$

where \mathbf{N} is the unit normal drawn inward from the boundary of the region, over which the summation extends. And if the region include the whole space through which the systems extend, the right member will vanish, giving $U_{12} = T_{12}$, when these are complete.

From (96) we obtain, by differentiation, the value of twice the excess of the electric over the magnetic energy of a single normal system in any region; thus

$$2(U - T) = \sum \mathbf{N} \left(\mathbf{V}\mathbf{E} \frac{d\mathbf{H}}{dp} - \mathbf{V} \frac{d\mathbf{E}}{dp} \mathbf{H} \right) / 4\pi. \quad \dots (97)$$

This formula, or special representatives of the same, is very useful in saving labour in investigations relating to normal systems of subsidence.

The quantity that appears in the numerator in (96) is the excess of the energy entering the region through its boundary per second by the action of the second system on the first, over that similarly entering due to the action of the first on

the second system. Bearing this in mind, we can easily form the corresponding formula in a less general case. Suppose, for example, we have two fine wire terminals, a and b , that are joined through any electromagnetic and electrostatic combination which does not contain impressed forces, nor receive energy from without except by means of the current, say C , entering it at a and leaving it at b . Let also V be the excess of the potential of a over that of b . Then VC is the energy-current, or the amount of energy added per second to the combination through the terminal connections with, necessarily, some other combination. (In the previous thick-letter vector investigation V was the symbol of vector product. There will, however, be no confusion with the following use of V , as in Part II., to express the line-integral of an electric force. One of the awkward things about the notation in Prof. Tait's 'Quaternions' is the employment of a number of most useful letters, as S, T, U, V , wanted for other purposes, as mere symbols of operations, putting another barrier in the way of practically combining vector methods with ordinary scalar methods, besides the perpetual negative sign before scalar products.) The combination need not be of mere linear circuits, in which differences of current-density are insensible; there may, for example, be induction of currents in a mass of metal not connected conductively with a and b , or the same mass may be in connection; but in any case it is necessary that the arrangement should terminate in fine wires at a and b , in order that the two quantities V and C may suffice to specify, by their product, the energy-current at the terminals. Even in this we completely ignore the dielectric currents and also the displacement, in the neighbourhood of the terminals, *i. e.* we assume $c=0$, to stop displacement. This is, of course, what is always done, unless specially allowed for.

Now supposing the structure of the combination to be given, we can always, by writing out the equations of its different parts, arrive at the characteristic equation connecting the terminal V and C . For instance,

$$V = ZC, \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

where Z is a function of d/dt . In the simplest case Z is a mere resistance. A common form of this equation is

$$f_0 V + f_1 \dot{V} + f_2 \ddot{V} + \dots = g_0 C + g_1 \dot{C} + g_2 \ddot{C} + \dots,$$

where the f 's and g 's are constants. But there is no restriction to such simple forms. All that is necessary is that the equa-

tion should be linear, so that Z may be a function of p . If, for example, $(dC/dt)^2$ occurred, we could not do it.

Now this combination must necessarily be joined on to another, however elementary, to make a complete system, unless V is to be zero always. The complete system, without impressed forces in it, has its proper normal modes of subsidence, corresponding to definite values of p . Consequently, by (96),

$$U_{12} - T_{12} = (V_2 C_1 - V_1 C_2) \div (p_1 - p_2), \quad \dots (99)$$

if V_1, C_1 belong to p_1 , and V_2, C_2 to p_2 , whilst the left member refers to the combination given by $V = ZC$. Or

$$U_{12} - T_{12} = C_1 C_2 \left(\frac{V_1}{C_1} - \frac{V_2}{C_2} \right) \div (p_2 - p_1) = C_1 C_2 \frac{Z_1 - Z_2}{p_2 - p_1}, \quad (100)$$

and the value of $2(U - T)$ in a single normal system is

$$2(U - T) = V \frac{dC}{dp} - C \frac{dV}{dp} = -C^2 \frac{d}{dp} \frac{V}{C} = -C^2 \frac{dZ}{dp}. \quad (101)$$

In a similar manner we can write down the energy-differences for the complementary combination, whose equation is, say, $V = YC$; remembering that $-VC$ is the energy entering it per second, we get

$$C_1 C_2 \frac{Y_1 - Y_2}{p_1 - p_2} \text{ and } C^2 \frac{dY}{dp} \text{ respectively.}$$

By addition, the complete $U_{12} - T_{12}$ is

$$C_1 C_2 \frac{Y_1 - Y_2 - Z_1 + Z_2}{p_1 - p_2} = 0 = C_1 C_2 \frac{\phi_1 - \phi_2}{p_1 - p_2}; \quad \dots (102)$$

and the complete $2(U - T)$ is

$$C^2 \frac{d}{dp} (Y - Z), \text{ or } C^2 \frac{d\phi}{dp}, \quad \dots (103)$$

where $\phi = 0$, or $Y - Z = 0$, is the determinantal equation of the complete system (both combinations which join on at a and b , where V and C are reckoned), expressed in such a form that every term in ϕ is of the dimensions of a resistance.

If the complete system depends only upon a finite number of variables, it is clear that the number of independent normal systems is also finite, and there is no difficulty whatever in understanding how any possible initial state is decomposable into the finite number of normal states; nor is any proof needed that it is possible to do it. The constant A_1 , fixing

the size of a particular normal system p_1 , will be given by

$$A_1 = \frac{U_{01} - T_{01}}{U_{11} - T_{11}} = \frac{U_{01} - T_{01}}{2(U_1 - T_1)} = \frac{U_{01} - T_{01}}{C_1 \frac{d\phi}{dp_1}} \quad . \quad . \quad (104)$$

by the previous, if U_{01} be the mutual electric energy of the given initial state and the normal system, and T_{01} similarly the mutual magnetic energy.

And when we increase the number of variables infinitely, and pass to partial differential equations and continuously varying normal functions, it is, by continuity, equally clear that the decomposition of the initial state into the now infinite series of normal functions is not only possible, but necessary. Provided always that we have the whole series of normal functions at command. Therein lies the difficulty, when there is any.

In such a case as the system (71) of Part II., involving the partial differential equation

$$\frac{d^2 V}{dz^2} = RS \frac{dV}{dt} + LS \frac{d^2 V}{dt^2}, \quad . \quad . \quad . \quad (105)$$

wherein R , S , and L are constants, to hold good between the limits $z=0$ and $z=l$, subject to

$$V = Z_0 C \text{ at } x=0, \text{ and } V = Z_1 C \text{ at } x=l,$$

there is no possible missing of the true normal functions which arise by treating d/dt as a constant; so that we can be sure of the possibility of the expansions. Thus, denoting $RSp + LSp^2$ by $-m^2$, we may take the normal V function as

$$u = \sin(mz + \theta), \quad . \quad . \quad . \quad (106)$$

and the corresponding normal C function as

$$w = + \frac{Sp}{m^2} \frac{du}{dz} = + \frac{Sp}{m} \cos(mz + \theta). \quad . \quad . \quad (107)$$

Here θ will be determined by the terminal conditions

$$\frac{u}{w} = Z_0 \text{ at } z=0, \quad \frac{u}{w} = Z_1 \text{ at } z=l, \quad . \quad . \quad . \quad (108)$$

and the complete V and C solutions are

$$V = \Sigma A u e^{pt}, \quad C = \Sigma A w e^{pt} \quad . \quad . \quad . \quad (109)$$

at time t ; where any A is to be found from the initial state,

say V_0, C_0 , functions of z , by

$$A = \frac{\int_0^1 (SV_0 u - LC_0 w) dz}{\left[w^2 \frac{d}{dp} \left(\frac{u}{w} - Z \right) \right]_0^1}, \quad \dots \quad (110)$$

provided there be no energy initially in the terminal arrangements. If there be, we must make corresponding additions to the numerator, without changing the denominator of A . The expression to be used for u/w is, by (106) and (107),

$$\frac{u}{w} = \frac{m}{Sp} \tan (mz + \theta), \quad \dots \quad (111)$$

remembering that m is a function of p . There are four components in the denominator of (110), as there are three electrical systems; viz. the terminal arrangements, which can only receive energy from the "line," and the line itself, which can receive or part with energy at both ends.

In a similar manner, if we make R, S , and L any single-valued functions of z , subject to the elementary relations of (71), Part II., or

$$-\frac{dV}{dz} = RC + L\dot{C}, \quad -\frac{dC}{dz} = S\dot{V}, \quad \dots \quad (112)$$

getting this characteristic equation of C ,

$$\frac{d}{dz} \left(S^{-1} \frac{dC}{dz} \right) = \left(R + L \frac{d}{dt} \right) \frac{dC}{dt}, \quad \dots \quad (113)$$

and, putting w for C and p for $\frac{d}{dt}$, this equation for the current function,

$$\frac{d}{dz} \left(S^{-1} \frac{dw}{dz} \right) = (R + Lp)pw, \quad \dots \quad (114)$$

and finding the u functions by the second of (112), giving

$$-Spw = \frac{dw}{dz}, \quad \dots \quad (115)$$

we see that the expansions of the initial states V_0 and C_0 can be effected, subject to the terminal conditions (108). For the normal potential and current functions will be perfectly definite (singularities, of course, to receive special attention), given by (113) and (114), as each the sum of two independent functions, and the terminal conditions will settle in what ratio they must be taken. (109) and (110) will constitute

the solution, except as regards the initial energy beyond the terminals.

It is, however, remarkable, that we can often, perhaps universally, find the expression for the part of the numerator of (110) to be added for the terminal arrangements, except as regards arbitrary multipliers, from the mere form of the Z functions, without knowing in detail what electrical combinations they represent. This is to be done by first decomposing the expression for $C^2(dZ/dp)$ into the sum of squares, for instance,

$$C^2 \frac{dZ}{dp} = r_1 \{f_1(p)\}^2 + r_2 \{f_2(p)\}^2 + \dots, \quad \dots \quad (116)$$

where r_1, r_2, \dots are constants. The terminal arbitraries are then $\Sigma A f_1(p)$, $\Sigma A f_2(p)$, &c. : calling these E_1, E_2, \dots , the additions to the numerator of (110) are

$$- \{E_1 r_1 f_1(p) + E_2 r_2 f_2(p) + \dots\}, \quad \dots \quad (117)$$

wherein the E 's may have any values. This must be done separately for each terminal arrangement. The matter is best studied in the concrete application, which I may consider under a separate heading.

It is also remarkable that, as regards the obtaining of correct expansions of functions, there is no occasion to impose upon R , S , and L the physical necessity of being positive quantities, or real. This will be understandable by going back to a finite number of variables, and then passing to continuous functions.

Let us now proceed to the far more difficult problems connected with propagation along a dielectric tube bounded by concentric conducting tubes, and examine how the preceding results apply, and in what cases we can be sure of getting correct solutions. Start with the general system, equations (11) to (14), Part I., with the extension mentioned at the commencement of Part II. from a solid to a tubular inner conductor. Suppose that the initial state is of purely longitudinal electric force, independent of z , so that the longitudinal E and circular H are functions of r only. How can we secure that they shall, in subsiding, remain functions of r only, so that any short length is representative of the whole? Since E is to be longitudinal, there must be no longitudinal energy-current, or it must be entirely radial. Therefore no energy must be communicated to the system at $z=0$ or $z=l$, or leave it at those places. This seems to be securable in only five cases. Put infinitely conducting plates across the section at either or both ends of the line. This will make $V=0$ there,

if V is the line-integral of the radial electric force across the dielectric. Or put non-conducting and non-dielectric plates there similarly. This will make $C=0$. Or, which is the fifth case, let the inner and the outer conductors be closed upon themselves. In any of these cases, the electric force will remain longitudinal during the subsidence, which will take place similarly all along the line. By (14), the equation of H will be

$$\frac{d}{dr} \frac{1}{r} \frac{d}{dr} rH = 4\pi k\mu \dot{H} + \mu c \ddot{H};$$

and it is clear that the normal functions are quite definite, so that the expansion of the initial state of E and H can be truly effected. In the already given normal functions take $m=0$.

But if we were to join the conductors at one end of the line through a resistance, we should, to some extent, upset this regular subsidence everywhere alike. For energy would leave the line; this would cause radial displacement, first at the end where the resistance was attached, and later all along the line. (By "the line" is meant, for brevity, the system of tubes extending from $z=0$ to $z=l$.)

Now in short-wire problems the electric energy is of insignificant importance, as compared with the magnetic. It is usual to ignore it altogether. This we can do by assuming $c=0$. This necessitates equality of wire and return current, for one thing; but, more importantly, it prevents current leaving the conductors, so that C and \dot{H} and Γ the current-density, are independent of z . There will be no radial electric force in the conductors, in which therefore the energy-current will be radial. But there will be radial force in the dielectric, and therefore longitudinal energy-current. Since the radial electric force and also the magnetic force in the dielectric vary inversely as the distance from the axis, the longitudinal energy-current density will vary inversely as the square of the distance. But, on account of symmetry, we are only concerned with its total amount over the complete section of the dielectric. This is

$$\frac{1}{4\pi} \int_{a_1}^{a_2} \frac{2C}{r} \cdot E_r \cdot 2\pi r dr = VC, \quad \dots \quad (118)$$

if V is the line-integral of E_r the radial force, and C the wire-current. It is clear, then, that we can now allow terminal connections of the form $V/C=Z$ before used, and still have correct expansions of the initial magnetic field, giving correct subsidence solutions.

But it is simpler to ignore V altogether. For the equation

of E.M.F. will be

$$e_0 = (Z_0 + Z_1 + lL_0p + lR_1'' + lR_2'')C, \quad \dots \quad (119)$$

if e_0 is the total impressed force in the circuit, R_1'' and R_2'' the wire and sheath functions of equations (55) and (56), Part II., on the assumption $m=0$, and Z_0, Z_1 the terminal functions, such that $V/C = Z_1$ at $z=l$, and $= -Z_0$ at $z=0$. It does not matter how e_0 is distributed so far as the magnetic field and the current is concerned. Let it then be distributed in such a way as to do away with the radial electric field, for simplicity of reasoning. The simple-harmonic solution of (119) is obviously to be got by expanding Z_0 and Z_1 in the form $R + Lp$, where R and L are functions of p^2 , and adding them on to the $l(R' + L'p)$ equivalent of $l(L_0p + R_1'' + R_2'')$, as in equation (66), Part II.

Regarding the free subsidence, putting $e_0=0$ in (119) gives us the determinantal equation of the p 's; and as the normal H functions are definitely known, the expansion of the magnetic field can be effected. The influence of the terminal arrangements must not be forgotten in reckoning A .

In coming, next, to the more general case of equation (56), but without restriction to exactly longitudinal current in the conductors, it is necessary to consider the transfer of energy more fully. In the dielectric the longitudinal energy-current is still VC . The rate of decrease of this quantity with z is to be accounted for by increase of electric and magnetic energy in the dielectric, and by the transfer of energy into the conductors which bound it. Thus,

$$-\frac{d}{dz}VC = -\frac{dV}{dz}C - \frac{dC}{dz}V.$$

But here,

$$-\frac{dC}{dz} = S\dot{V}, \text{ and } -\frac{dV}{dz} = L_0\dot{C} + E - F, \quad \dots \quad (120)$$

by (59) and (56), Part II., E and F being the longitudinal electric forces at the inner and outer boundaries of the dielectric (when there is no impressed force). So

$$-\frac{d}{dz}VC = SV\dot{V} + L_0\dot{C}C + EC - FC. \quad \dots \quad (121)$$

The first term on the right side is the rate of increase of the electric energy, the second term the rate of increase of the magnetic energy in the dielectric, the third is the energy entering the inner conductor per second, the fourth that entering the outer conductor; all per unit length.

If the electric current in the conductors were exactly lon-

gitudinal, the energy-transfer in them would be exactly radial, and EC and -FC would be precisely equal to the Joule heat per second *plus* the rate of increase of the magnetic energy, in the inner and the outer conductor, respectively. But as there is a small radial current, there is also a small longitudinal transfer of energy in the conductors. Thus, E_r and E_z being the radial and longitudinal components of the electric force, in the inner conductor, for example, the longitudinal and the radial components of the energy-current per unit area are

$$E_r H / 4\pi \text{ and } E_z H / 4\pi,$$

the latter being inward. Their convergences are

$$-\frac{d}{dx} \frac{E_r H}{4\pi}, \text{ and } \frac{1}{r} \frac{d}{dr} r \frac{E_z H}{4\pi},$$

or

$$\frac{E_r}{4\pi} \left(-\frac{dH}{dz} \right) - \frac{H}{4\pi} \frac{dE_r}{dz}, \text{ and } \frac{E_z H}{4\pi} + \frac{E_z}{4\pi} \frac{dH}{dr} + \frac{H}{4\pi} \frac{dE_z}{dr},$$

or

$$E_r \Gamma_r - \frac{H}{4\pi} \frac{dE_r}{dz}, \text{ and } E_z \Gamma_z + \frac{H}{4\pi} \frac{dE_z}{dr},$$

if Γ_r and Γ_z are the components of the electric current-density. The sum of the first terms is clearly the dissipativity per unit volume; and that of the second terms is, by equation (13), Part I., $H\mu\dot{H}/4\pi$, the rate of increase of the magnetic energy.

The longitudinal transfer of energy in either conductor per unit area is also expressed by $-(4\pi k)^{-1} H(dH/dz)$; or, by $-(4\pi k\mu)^{-1}(dT_1/dz)$ across the complete section, if T_1 temporarily denote the magnetic energy in the conductor per unit length.

Now let E_1, F_1, C_1, V_1 , and E_2, F_2, C_2, V_2 refer to two distinct normal systems. Then, if we could neglect the longitudinal transfer in the conductors, we should have

$$U_{12} - T_{12} = \frac{d}{dz} (V_1 C_2 - V_2 C_1) \div (p_1 - p_2), \quad \dots \quad (122)$$

the left side referring to unit length of line; and, in the whole line,

$$U_{12} - T_{12} = [V_1 C_2 - V_2 C_1]_0^l \div (p_1 - p_2). \quad \dots \quad (123)$$

Similarly, for a single normal system,

$$2(U - T) = \frac{d}{dz} C^2 \frac{d}{dp} \frac{V}{C}, \quad \dots \quad (124)$$

the first two of these being the terminal conditions, and $R'_m + L'_m p$ being merely a convenient way of writing the real complex expressions; (equation (68), with $e_m = 0$). It is clear that the only cases in which the m 's become clear of the p 's are the before-mentioned five cases, equivalent to Z_0 and Z_1 being zero or infinite, and the line closed upon itself, which is a sort of combination of both. Considering only the four, they are summed up in this, $VC=0$ at the terminals, or the line cut off from receiving or losing energy at the ends. We have then the series of m 's, $0, \pi/l, 2\pi/l, \&c.$; or $\frac{1}{2}\pi/l, \frac{3}{2}\pi/l, \frac{5}{2}\pi/l, \&c.$; and every m^2 has its own infinite series of p 's through the third equation (128). These, though very special, are certainly important cases, as well as being the most simple. We can definitely effect the expansions of the initial states in the normal functions, and obtain the complete solutions in every particular.

Although rather laborious, it is well to verify the above results by direct integration of the proper expressions for the electric and magnetic energies of normal systems throughout the whole line. Thus, let

$$\frac{d}{dr} \frac{1}{r} \frac{d}{dr} r H_1 + s_1^2 H_1 = 0, \text{ where } -s_1^2 = 4\pi\mu_1 k_1 p_1 + m_1^2,$$

$$\frac{d}{dr} \frac{1}{r} \frac{d}{dr} r H_2 + s_2^2 H_2 = 0, \text{ where } -s_2^2 = 4\pi\mu_1 k_1 p_2 + m_2^2,$$

in the inner conductor. We shall find

$$(s_1^2 - s_2^2) \int_{a_0}^{a_1} H_1 H_2 r dr = 8\pi(C_1 \Gamma_2 - C_2 \Gamma_1),$$

as $H_1 = 0 = H_2$ at $r = a_0$; Γ_1 and Γ_2 being the longitudinal current-densities at $r = a_1$. Similarly for the outer conductor,

$$(s_1'^2 - s_2'^2) \int_{a_2}^{a_3} H_1' H_2' r dr = -8\pi(C_1 \Gamma_2' - C_2 \Gamma_1')$$

if C_1, C_2 still be the currents in the inner conductor; the accents merely meaning changes produced by the altered μ and k in the outer conductor. $H_1' = 0 = H_2'$ at $r = a_3$ in this case. Then, thirdly, for the intermediate space,

$$\int_{a_1}^{a_2} H_1'' H_2'' r dr = C_1 C_2 \times 4 \log \frac{a_2}{a_1}.$$

Therefore the total mutual magnetic energy of the two distri-

butions per unit length is

$$\frac{\mu_1}{4\pi} \int_{a_0}^{a_1} H_1 H_2 \cdot 2\pi r dr + \frac{\mu_2}{4\pi} \int_{a_1}^{a_2} H_1'' H_2'' \cdot 2\pi r dr \\ + \frac{\mu_3}{4\pi} \int_{a_2}^{a_3} H_1' H_2' \cdot 2\pi r dr,$$

which, by using the above expressions, becomes, provided $m_1^2 = m_2^2$,

$$L_0 C_1 C_2 - \frac{C_1(E_2 - F_2)}{p_1 - p_2} + \frac{C_2(E_1 - F_1)}{p_1 - p_2}, \quad . \quad (126a)$$

E and F being Γ/k or the longitudinal electric forces at $r = a_1$ or $r = a_2$. But

$$E - F = R''C,$$

where $R'' =$ the $R_1'' + R_2''$ of equation (56), Part II. ; and

$$0 = \frac{m^2}{Sp} + L_0 p + R'' = \frac{m^2}{Sp} + R' + L'p,$$

so (126) becomes

$$\left(C_1 \frac{dV_2}{dz} - C_2 \frac{dV_1}{dz} \right) \div (p_1 - p_2), \text{ or } \frac{m^2 C_1 C_2}{Sp_1 p_2}. \quad . \quad (127a)$$

The mutual electric energy is obviously $SV_1 V_2$ per unit length. By summation with respect to z from 0 to l , subject to $VC=0$ at both ends, we verify that the total mutual magnetic energy equals the total mutual electric energy. The value of $2T$ in a single normal system is, by (126a), and the next equation,

$$L_0 C^2 + C^2 \frac{dR''}{dp} = C^2 \frac{d}{dp} (R' + L'p) \quad . \quad (128a)$$

per unit length ; and that of $2U$ is SV^2 . Hence, per unit length,

$$2(U - T) = SV^2 - C^2 \frac{d}{dp} (R' + L'p). \quad . \quad (129)$$

In this use $V = u$ and $C = w$, equations (126), and we shall obtain, for the complete energy-difference in the whole line,

$$- \left\{ \frac{a_1}{2} J_1(s_1 a_1) - \dots \right\}^2 \frac{l}{2} \frac{d}{dp} \left(\frac{m^2}{Sp} + R' + L'p \right) = M \text{ say,} \quad (130)$$

which is the expanded form of

$$\left[w \frac{du}{dp} - u \frac{dw}{dp} \right]_0^l \text{ or } \left[w^2 \frac{d}{dp} \left(\frac{u}{w} - Z \right) \right]_0^l,$$

as may be verified by performing the differentiations, using the expression for u/w in (127), remembering that m^2 in it is

a function of p ; or more explicitly, put $\sqrt{-Sp(R' + L'p)}$ for m , and then differentiate to p .

Given, then, the initial state to be $V = V_0$, a function of z , and $H = H_{01}$ in the inner conductor, H_{02} in the dielectric, and H_{03} in the outer conductor, functions of r and z , and that this system is left without impressed force, subject to $VC = 0$ at both ends, the state at time t later will be given by

$$V = \Sigma Aue^{pt}, \quad C = \Sigma Awe^{pt};$$

the summations to include every p , with similar expressions for H , Γ , γ , &c., the magnetic force and two components of current, by substituting for u or w the proper corresponding normal functions; the coefficient A being given by the fraction whose denominator is the expression M in (130), and whose numerator is the excess of the mutual electric energy of the initial and the normal system over their mutual magnetic energy, expressed by

$$\begin{aligned} & \frac{m}{p} C' \int_0^l V_0 \sin(mz + \theta) dz \\ & - \int_0^l \cos(mz + \theta) dz \left\{ \int_{a_0}^{a_1} \mu_1 H_{01} C_1' dr + \int_{a_1}^{a_2} \mu_2 H_{02} C' dr \right. \\ & \quad \left. + \int_{a_2}^{a_3} \mu_3 H_{03} C_3' dr \right\}, \quad (131) \end{aligned}$$

where

$$C' = \frac{a_1}{2} \{J_1(s_1 a_1) - (J_1/K_1)(s_1 a_0) K_1(s_1 a_1)\};$$

and C_1' is the same with r put for a_1 , and C_3' is the same with r put for a_1 , a_3 for a_0 , and s_3 for s_1 . It should not be forgotten that in the case $m=0$, the denominator (130) requires to be doubled, $\frac{1}{2}l$ becoming l . Also that R'' , or $R' + L'p$, contains m^2 , and must not be the $m=0$ expressions for the same.

To check, take the initial state to be $e_0(1 - z/l)$, with no magnetic force, and that $V=0$ at both ends. We find immediately, by (130) and (131), that at time t ,

$$V = \frac{2e_0}{l} \Sigma \frac{1}{m} \sin mz \Sigma \frac{(m^2/S p^2) e^{pt}}{-\frac{d}{dp} \left(\frac{m^2}{Sp} + R' + L'p \right)}, \quad (132)$$

where the m 's are to be π/l , $2\pi/l$, $3\pi/l$, &c.; the first summation being with respect to m , and the second for the p 's of a particular m .

But, initially,

$$V = e_0 \left(1 - \frac{z}{l} \right) = \frac{2e_0}{l} \Sigma \frac{1}{m} \sin mz.$$

2 A 2

Therefore we must have

$$1 = \Sigma \frac{m^2 / Sp^2}{- \frac{d}{dp} \left(\frac{m^2}{Sp} + R' + L'p \right)}.$$

Simplified, it makes this theorem

$$- \frac{1}{\phi(0)} = \Sigma \left(p \frac{d\phi}{dp} \right)^{-1},$$

if the p 's are the roots of $\phi(p) = 0$. This is correct.

To determine the effect of longitudinal impressed force, keeping to the case of uniform intensity over the cross section of either conductor. Let a steady impressed force of integral amount e_0 be introduced in the line at distance z_1 ; it may be partly in one and partly in the other conductor, as in Part II. By elementary methods, we can find the steady state of V , C it will set up. If, then, we remove e_0 , we can, by the preceding, find the transient state that will result. Let V_0 be the steady state of V set up, and V_1 what it becomes at time t after removal of e_0 ; then $V_0 - V_1$ represents the state at time t after e_0 is put on. So, if $\Sigma A u$ represent the V set up by the unit impressed force at z_1 ,

$$V = V_0 - e_0 \Sigma A u \epsilon^{pt}$$

will give the distribution of V at time t after e_0 is put on, being zero when $t=0$, and V_0 when $t=\infty$. No zero value of p is admissible here.

From this we deduce that the effect of e_0 lasting from $t=t_1$ to $t=t_1+dt_1$ at the later time t is

$$- \Sigma A u p e_0 dt_1 \epsilon^{p(t-t_1)};$$

therefore, by time integration, the effect due to an impressed force e_0 at one spot, variable with the time, starting at time t_0 is

$$V = - \Sigma A u p e_0 \int_{t_0}^t \epsilon^{-pt_1} dt_1,$$

in which e_0 is a function of t_1 .

By integrating along the line, we find the effect of a continuously distributed impressed force, e per unit length, to be

$$V = - \Sigma u p \epsilon^{pt} \int_0^l \int_{t_0}^t A e \epsilon^{-pt_1} dz_1 dt_1, \quad . \quad . \quad (133)$$

wherein e is a function of both z_1 and t_1 , and starts at time t_0 ; whilst A is a function of z_1 , the position of the elementary impressed force edz_1 .

To find A as a function of z_1 , we might, since ΣAu is the V set up by unit e at z_1 , expand this state by the former process of integration. But the following method, though unnecessary for the present purpose, has the advantage of being applicable to cases in which VC is not zero at the terminals, but $V=ZC$ instead. It is clear that the integration process, including the energy in the terminal apparatus, would be very lengthy, and would require a detailed knowledge of the terminal combinations. This is avoided by replacing the impressed force at z_1 by a charged condenser; when, clearly, the integration is confined to one spot. Let S_1 be the capacity and V_0 the difference of potential of a condenser inserted at z_1 . If we increase S_1 infinitely it becomes mathematically equivalent to an impressed force V_0 , without the condenser.

Suppose $\Sigma Aw'\epsilon^{pt}$ is the current at z at time t after the introduction of the condenser, of finite capacity; then, since $-S_1\dot{V}$ is the current leaving the condenser, or the current at z_1 , we have

$$-S_1\dot{V} = \Sigma Aw_1'\epsilon^{pt},$$

w_1' being the value of w' at z_1 . The expansion of V_0 is therefore

$$V_0 = -\Sigma Aw_1'/S_1 p,$$

initially; and the mutual potential energy of the initial charge of the condenser and of the normal w' corresponding to w' must be

$$S_1 V_0 (-w_1'/S_1 p) = -V_0 w_1'/p.$$

But since there is, initially, electric energy only at z_1 , and magnetic energy nowhere at all, the only term in the numerator of A will be that due to the condenser, or this $-V_0 w_1'/p$; hence

$$A = -V_0 w_1/pM,$$

where M is the $2(U-T)$ of the complete normal system, as modified by the presence of the condenser, is the value of A in $V = \Sigma Au'\epsilon^{pt}$, making

$$V = -V_0 \Sigma (w_1'/pM) u'\epsilon^{pt},$$

expressing the effect at time t after the introduction of the condenser, and due to its initial charge.

So far S_1 has been finite, and consequently u' , w' , M , and p depend on its capacity as well as on the line and terminal conditions. But on infinitely increasing its capacity, u' and w' become u and w , the same as if the condenser were non-existent. Therefore

$$V = -\Sigma V_0 (w_1/pM) u\epsilon^{pt} \quad . \quad . \quad . \quad (134)$$

expresses the effect due to the steady impressed force V_0 at z_1 at time t after it was started. This will have a term corresponding to a zero p (due to the infinite increase of S_1 in the previous problem), expressing the final state. Hence, leaving out this term, the summation (134), with sign changed, and $t=0$, expresses the final state itself. Thus, taking $V_0=1$,

$$\Sigma Au = \Sigma w_1 u / pM$$

is the expansion required to be applied to (133). Put $A = w_1 / pM$ in it, and it becomes

$$V = \Sigma (u/M) e^{pt} \int_0^t \int_{t_0}^t w_1 e e^{-pt_1} dz_1 dt_1, \quad . \quad . \quad . \quad (135)$$

fully expressing the effect at z, t , due to the impressed force e , a function of z_1 and t_1 , starting at time t_0 . To obtain the current, change u to w outside the double integral. The M , when the condition $VC=0$ at the ends is imposed, is that of (130); the u and w expressions those of (126). But if we regard S, R' , and L' as constants (or functions of z), then (135) holds good when terminal conditions $V=ZC$ are imposed, provided the impressed force be in the line only, as supposed in (135).

When the impressed force is steady, and is confined to the place $z=0$, and is of integral amount e_0 , (135) gives

$$V = e_0 \Sigma u w_0 / pM - e_0 \Sigma u w_0 e^{pt} / pM, \quad . \quad . \quad . \quad (136)$$

w_0 being the value of w at $z=0$, as the effect at time t after starting e_0 . The first summation expresses the state finally arrived at.

Again, in (135) let the impressed force be a simple harmonic function of the time. I have already given the solution in this case, so far as the formula for C is concerned, in the case $V=0$ at both ends, in equation (76), Part II., which may be derived from (135), by using in it w instead of u at its commencement, putting $e=e_0 \sin nt$, and effecting some reductions. The V formula may be got in a similar manner to that used in getting (76), but it is instructive to derive it from (135), as showing the inner meaning of that formula. Let in it $e=e_0 \sin (nt+\alpha)$, where e_0 is a function of z . Effect the t_1 integration, with $t_0=0$ for simplicity. The result is

$$V = -\Sigma \frac{u e^{pt}}{E} \left(\frac{p \sin \alpha + n \cos \alpha}{p^2 + n^2} \right) \int_0^t w_1 e_0 dz_1$$

$$\Sigma + \frac{u}{M} \left(\frac{p \sin (nt+\alpha) + n \cos (nt+\alpha)}{p^2 + n^2} \right) \int_0^t w_1 e_0 dz_1. \quad (137)$$

The first summation cancels the second at the first moment, and ultimately vanishes, leaving the second part to represent the final periodic solution. Take $\alpha=0$; and use the u, w, M expressions of (126) and (130), and let ϕ_m stand for $m^2 + Sp(R'_m + L'_m p)$, so that $\phi_m=0$ gives the p 's for a particular m^2 . Then we obtain, (with $V=0$ at both ends),

$$\begin{aligned} V &= \frac{d}{dz} \sum \frac{\cos mz \int_0^l \cos mz_1 \cdot e_0 dz_1 \cdot (p \sin nt + n \cos nt)}{\frac{1}{2} l \frac{d\phi_m}{dp} (p^2 + n^2)} \\ &= \frac{d}{dz} \frac{2}{l} \sum \frac{\cos mz \int_0^l \cos mz_1 \cdot e_0 \sin nt \cdot dz_1}{\left(\frac{d}{dt} - p\right) \frac{d\phi_m}{dp}}, \quad \dots \quad (138) \end{aligned}$$

because $d^2/dt^2 = -n^2$. But, if $e_0 = \sum e_m$, the equation of V_m is

$$-\phi_m V_m = \frac{de_m}{dz} \sin nt,$$

(by (60) and (63), Part II.), so that

$$V_m = -\phi_m^{-1} \frac{de_m}{dz} = -\frac{d}{dz} \sum \frac{e_m \sin nt}{\left(\frac{d}{dt} - p\right) \frac{d\phi}{dp}}, \quad \dots \quad (139)$$

by a well-known algebraical theorem, the summation being with respect to the p 's, which are the roots of $\phi_m=0$, considered as algebraic. We have also

$$e_0 = \frac{2}{l} \sum \cos mz \int_0^l \cos mz_1 e_0 dz_1, \quad \dots \quad (140)$$

the summation being with respect to m .

Uniting (139) and (140), there results the previous equation (138), in which the summation is with respect to all the p 's belonging to all the m 's. In the case $m=0$, the $2/l$ must be halved. In the form of a summation with respect to m , similar to (77) for C, the corresponding V solution is

$$V = -\frac{2V_0}{Snl} \sum \frac{m \sin mz \{ (L'_m - m^2/Sn^2)n \sin nt + R'_m \cos nt \}}{R_m^2 + (L_m - m^2/Sn^2)^2 n^2},$$

the impressed force being $V_0 \sin nt$, at $z=0$. This, on the assumption $R'_m=R'$, $L'_m=L'$, will be found to be the expansion of the form (80), Part II.

Now to make some remarks on the impossibility of joining on terminal apparatus without altering the normal functions, the terminal arrangements being made to impose conditions of the form $V=ZC$. It is clear, in the first place, that if the quantity VC at $z=0$ and $z=l$ really represents the energy-transfer in or out of the line at those places, then the equation

$$A_1 = \frac{U_{01} - T_{01}}{\left[w^2 \frac{d}{dp} \left(\frac{u}{w} - Z \right) \right]_0'}$$

will be valid, provided u and w be the correct normal functions. But to make VC be the energy-transfer at the ends requires us to stop the longitudinal transfer in the conductors there, or make the current in the conductors longitudinal. This condition is violated when the current function w is proportional to $\cos(mz + \theta)$, as in the previous, except in the special cases, because the radial current γ in the conductors is proportional to $\sin(mz + \theta)$, and γ has to vanish. Not in the dielectric, but merely in the conductors.

We can ensure that VC is the energy-transfer at the ends by coating the conductors over their exposed sections with infinitely conducting material and joining the terminal apparatus on to the latter. The current in the conductors will be made strictly longitudinal, close up to the infinitely conducting material, and γ will vanish in the conductors. But γ in the dielectric at the same place will be continuous with the radial surface-current on the infinitely conducting ends, due to the sudden discontinuity in the magnetic force. Thus the energy-transfer, at the ends, is confined to the dielectric.

It is clear, however, that the normal current-functions in the two conductors must be such as to have no radial components at the terminals, so that they cannot be what have been used, such that $d^2/dz^2 = \text{constant}$. They require alteration, of sensible amount may be, only near the terminals, but theoretically, all along the line. It would therefore appear that only the five cases of $V=0$ at either or both ends, or $C=0$ ditto, or the line closed upon itself, admit of full solution in the above manner. The only practical way out of the difficulty is to abolish the radial electric current in the conductors, making (66) the equation of V , and VC the longitudinal energy-transfer, with full applicability of the $V=ZC$ terminal conditions. With a further consideration of this system, and some solutions relating to it, I propose to conclude this paper.