

*On Wave-Propagation in Two Dimensions.* By HORACE LAMB,  
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1. The chief object of this paper is to work out and illustrate graphically some problems relating to the divergence of waves from a centre of disturbance in a space of two dimensions, the source being of a more or less transient character. The waves in question may be, for example, cylindrical waves of sound, or waves travelling over a uniform and uniformly tense membrane.\* In any case they are subject to an equation of the form

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right), \quad (1)$$

or in the case of symmetry about the origin, which is more especially considered,

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right). \quad (2)$$

In a space of one or of three dimensions the solution of such problems is extremely easy, owing to the existence of simple general integrals of the corresponding differential equations. Thus in one dimension we have

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2}, \quad (3)$$

with the solution  $\phi = f \left( t - \frac{x}{c} \right) + F \left( t + \frac{x}{c} \right); \quad (4)$

whilst in three dimensions we have

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r} \frac{\partial \phi}{\partial r} \right), \quad (5)$$

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\* The case of waves on a sheet of water of uniform depth may be included, provided the dimensions of the "source" be large compared with the depth.

with the solution\*

$$4\pi r\phi = f\left(t - \frac{r}{c}\right) + F\left(t + \frac{r}{c}\right). \quad (6)$$

It may be well here to remind the reader that the formal resemblance between (4) and (6) is to some extent delusive; the physical interpretation reveals important points of difference, which have been insisted upon by Stokes† and Rayleigh.‡

The theory of solitary waves in *two* dimensions has attracted less attention,§ partly no doubt because of its inferior physical interest, and also because of a certain degree of mathematical obscurity which has attached to it owing to the absence of a simple general formula analogous to (4) and (6). The general integral of (2) which was given by Poisson,|| viz.,

$$\phi = \int_0^r f(ct + r \cos \theta) d\theta + \int_0^r F(ct + r \cos \theta) \log(r \sin^2 \theta) d\theta, \quad (7)$$

is unsuitable for the purpose, as it does not discriminate between the waves which travel inwards and outwards respectively.

The proper analogue of (4) and (6) is, however, easily obtained if we start from the known solution for the case of waves diverging from a simple harmonic source ( $e^{i\omega t}$ ), viz.,¶

$$2\pi\phi = \int_0^\infty e^{i\omega\left(t - \frac{r}{c} \cosh u\right)} du. \quad (8)$$

If we generalize this by Fourier's theorem, we have at once

$$2\pi\phi = \int_0^\infty f\left(t - \frac{r}{c} \cosh u\right) du, \quad (9)$$

corresponding to an arbitrary source  $f(t)$ . It is of course implied

\* The factor  $4\pi$  is inserted for the sake of comparison with (9) below. It makes the "strength" of the source at the origin equal to  $f(t) + F(t)$ .

† *Phil. Mag.*, January, 1849; *Math. and Phys. Papers*, Vol. II., p. 82.

‡ *Theory of Sound*, Vol. II., §§ 271, 279.

§ Except in the case of cylindrical electric waves, which have been discussed to some extent by Heaviside, *Phil. Mag.*, November, 1888; *Electrical Papers*, Vol. II.

|| *Journal de l'École Polytechnique*, Vol. XII., p. 215 (1821).

¶ Cf. Rayleigh, *Sound*, Vol. II., § 312; *Proc. Lond. Math. Soc.*, Vol. XIX., p. 504 (1888); *Scientific Papers*, Vol. III., p. 44.

that the form of  $f(t)$  must be such that the integral is convergent. A sufficient (but not an essential) condition for this is

$$\lim_{z \rightarrow -\infty} z^{1+\epsilon} f(z) = 0, \quad (10)$$

$\epsilon$  being any positive quantity; this is fulfilled of course whenever the source has been in action only for a finite time.\* The complete formula, embracing both converging and diverging waves, is

$$2\pi\phi = \int_0^\infty f\left(t - \frac{r}{c} \cosh u\right) du + \int_0^\infty F\left(t + \frac{r}{c} \cosh u\right) du, \quad (11)$$

where the convergence of the second integral is assured if

$$\lim_{z \rightarrow +\infty} z^{1+\epsilon} F(z) = 0. \quad (12)$$

The expression (11) is arrived at independently if we imagine a system of point-sources, such as the origin in (6), to be distributed uniformly along a straight line,† say the axis of  $z$ , distance from which is denoted in (11) by  $r$ .

The direct verification of (11) by substitution in the differential equation (2) is a matter of some nicety. Taking the first term alone, as in (9), we find, subject to certain conditions,

$$\begin{aligned} & 2\pi \left\{ c^2 \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right) - \frac{\partial^2 \phi}{\partial t^2} \right\} \\ &= \int_0^\infty \left\{ \sinh^2 u \cdot f''\left(t - \frac{r}{c} \cosh u\right) - \frac{c}{r} \cosh u \cdot f'\left(t - \frac{r}{c} \cosh u\right) \right\} du \\ &= \frac{c^2}{r^2} \int_0^\infty \frac{\partial^2}{\partial u^2} f\left(t - \frac{r}{c} \cosh u\right) du \\ &= -\frac{c}{r} \left[ \sinh u \cdot f'\left(t - \frac{r}{c} \cosh u\right) \right]_{u=0}^{u=\infty} = 0. \end{aligned}$$

The conditions referred to are obviously satisfied whenever  $f(z)$  vanishes for negative values of  $z$  exceeding a certain limit, as well as

\* The referees point out that a result equivalent to (9) was obtained (in a different manner) by T. Levi-Civita, *Nuovo Cimento*, Ser. 4, t. vi. (1897).

† This is the method by which Rayleigh (*loc. cit.*) deduces (8) from the formula for a simple-harmonic point-source in three dimensions.

in other cases less easily defined.\* Again,

$$\begin{aligned}
 -2\pi r \frac{\partial \phi}{\partial r} &= \frac{r}{c} \int_0^\infty \cosh u \cdot f' \left( t - \frac{r}{c} \cosh u \right) du \\
 &= \frac{r}{c} \int_0^\infty (\sinh u + e^{-u}) f' \left( t - \frac{r}{c} \cosh u \right) du \\
 &= - \int_0^\infty \frac{\partial}{\partial u} f \left( t - \frac{r}{c} \cosh u \right) du + \frac{r}{c} \int_0^\infty e^{-u} f' \left( t - \frac{r}{c} \cosh u \right) du \\
 &= - \left[ f \left( t - \frac{r}{c} \cosh u \right) \right]_{u=0}^{u=\infty} + \frac{r}{c} \int_0^\infty e^{-u} f' \left( t - \frac{r}{c} \cosh u \right) du \\
 &= f \left( t - \frac{r}{c} \right) + \frac{r}{c} \int_0^\infty e^{-u} f' \left( t - \frac{r}{c} \cosh u \right) du,
 \end{aligned}$$

under similar restrictions. Hence

$$\lim_{r=0} \left( -2\pi r \frac{\partial \phi}{\partial r} \right) = f(t), \quad (13)$$

which verifies the statement previously made, as to the strength of the source in (9).†

A similar process will apply to the second term in (11), subject to certain restrictions which are obviously fulfilled when  $F(z)$  vanishes for positive values of  $z$  exceeding a certain limit.

It may be remarked that from (11) we can derive the general solution of (1), by a known analogy. In polar coordinates we have

$$\phi = \sum_0^\infty r^n \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^n \{ \Phi(r, t) \cos n\theta + \Psi(r, t) \sin n\theta \}, \quad (14)$$

where  $\Phi(r, t)$  and  $\Psi(r, t)$  are functions of the same type as the right-hand member of (11).

2. We may apply the formula (9) to trace the effect of a temporary source at the origin varying according to some simple prescribed law. The most immediate interpretation of our results will be to suppose that  $\phi$  denotes the transverse displacement at any point of an unlimited tense membrane. The waves are then due to a local

\* The verification is very similar to that given by Levi-Civita. It is to be remarked that the process may fail, owing to the form of  $f$ , even when (9) is really a solution of (2). An instance is supplied by the function (8).

† In the application to sound-waves the definition of  $\phi$  is supposed to be as in the author's *Hydrodynamics*, so that the radial velocity is  $-\partial\phi/\partial r$ . If  $s$  be the "condensation," we have on the same convention  $c^2 s = \partial\phi/\partial t$ .

application of normal force at the origin, whose aggregate amount, reckoned in the direction of  $\phi$  positive, is  $f(t)$  multiplied by the tension of the membrane. This appears at once from (13).

If we suppose that everything is quiescent until the time  $t = 0$ , so that  $f(t)$  vanishes for negative values of  $t$ , we see from (9), or from the equivalent form

$$2\pi\phi = \int_{-\infty}^{t-r/c} \frac{f(\theta) d\theta}{\sqrt{\{(t-\theta)^2 - \frac{r^2}{c^2}\}}}, \quad (15)$$

that  $\phi$  vanishes at any point so long as  $t < r/c$ . If, moreover, the source acts only for a finite time  $\tau$ , so that  $f(t) = 0$  for  $t > \tau$ , we have, for  $t > r/c + \tau$ ,

$$2\pi\phi = \int_0^{\tau} \frac{f(\theta) d\theta}{\sqrt{\{(t-\theta)^2 - \frac{r^2}{c^2}\}}}. \quad (16)$$

This expression does not as a rule vanish; the wave accordingly is not sharply defined in the rear, as it is in front, but has, on the contrary, a sort of "tail," whose form, when  $t - r/c$  is large compared with  $\tau$ , is given approximately by the formula\*

$$2\pi\phi = \frac{1}{\sqrt{(t^2 - \frac{r^2}{c^2})}} \int_0^{\tau} f(\theta) d\theta. \quad (17)$$

As a first example, let us suppose that the source is constant during the interval of its existence, say,

$$\begin{aligned} f(t) &= 0, & \text{for } t < 0; \\ &= 1, & \text{for } 0 < t < \tau; \\ &= 0, & \text{for } t > \tau. \end{aligned}$$

We easily find from (9) that

$$\left. \begin{aligned} 2\pi\phi &= 0, & \text{for } t < \frac{r}{c} \\ &= \cosh^{-1} \frac{ct}{r}, & \text{for } \frac{r}{c} < t < \frac{r}{c} + \tau \\ &= \cosh^{-1} \frac{ct}{r} - \cosh^{-1} \frac{c(t-\tau)}{r}, & \text{for } t > \frac{r}{c} + \tau \end{aligned} \right\} \quad (18)$$

\* The existence of the "tail," in the case of cylindrical electric waves, was noted by Heaviside, *loc. cit.*

To tabulate the results, we require a series of values of a variable  $u$  corresponding to equidistant values of  $\cosh u$ . For this purpose existing tables of the hyperbolic functions are not very convenient; but for the more important part of the wave it is sufficient to assume

$$\cosh u = 1 + \frac{1}{2}z^2, \quad (19)$$

where  $z^2$  is small; this makes

$$u = z \left( 1 - \frac{1}{24}z^2 + \frac{3}{640}z^4 - \dots \right). \quad (20)$$

The first two terms of the series are alone sensible within the range of the following table.

$\cosh u$	$u$	$\cosh u$	$u$	$\cosh u$	$u$
1.000	0	1.010	.1413	1.020	.1997
1	.0447	1	.1482	25	.2231
2	.0632	2	.1548	30	.2443
3	.0774	3	.1611	35	.2638
4	.0894	4	.1671	40	.2819
5	.1000	5	.1730	45	.2989
6	.1095	6	.1787	50	.3149
7	.1183	7	.1841		
8	.1264	8	.1895		
9	.1341	9	.1946		

The diagram (Fig. 1), constructed with  $\phi$  as ordinate and  $t$  as abscissa, exhibits the variation of  $\phi$  at a particular point ( $r = 100\epsilon r$ )

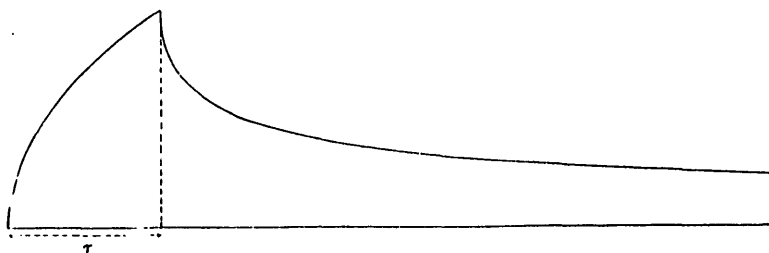


FIG. 1.

as the wave passes over it. Within the range shown, the same curve also indicates very fairly the profile of the wave after a time  $t$  which is about equal to  $100\tau$ , it being understood, of course (in this interpretation), that the direction of propagation is from right to left. The abrupt front of the wave and the indefinite prolongation in the "tail" are well marked.

The diagram exhibits certain peculiarities due to the discontinuous nature of the source. To obtain a representation of a source of a less abrupt character, we may suppose the strength to rise uniformly from zero to a maximum, and then to fall uniformly to zero again at the same rate. We assume, therefore,

$$\begin{aligned} f(t) &= 0, & \text{for } t < -\tau, \\ &= 1 + \frac{t}{\tau}, & \text{for } -\tau < t < 0; \\ &= 1 - \frac{t}{\tau}, & \text{for } 0 < t < \tau; \\ &= 0, & \text{for } t > \tau. \end{aligned}$$

We find, from (9), or (15), after some reductions,

$$\left. \begin{aligned} 2\pi\phi &= 0, & \text{for } t < \frac{r}{c} - \tau \\ &= \frac{r}{cr} \chi \left\{ \frac{c(t+\tau)}{r} \right\}, & \text{for } \frac{r}{c} - \tau < t < \frac{r}{c} \\ &= \frac{r}{cr} \left[ \chi \left\{ \frac{c(t+\tau)}{r} \right\} - 2\chi \left( \frac{ct}{r} \right) \right], & \text{for } \frac{r}{c} < t < \frac{r}{c} + \tau \\ &= \frac{r}{cr} \left[ \chi \left\{ \frac{c(t+\tau)}{r} \right\} + \chi \left\{ \frac{c(t-\tau)}{r} \right\} - 2\chi \left( \frac{ct}{r} \right) \right], & \text{for } t > \frac{r}{c} + \tau \end{aligned} \right\}.$$

where  $\chi(x) = x \cosh^{-1} x - \sqrt{(x^2 - 1)}$

(21)

This suggests the tabulation of the function  $u \cosh u - \sinh u$  for equidistant values of  $\cosh u$ . If we put

$$\cosh u = 1 + \frac{1}{2}z^2,$$

as before, we find

$$u \cosh u - \sinh u = \frac{1}{3}z^3 \left( 1 - \frac{1}{40}z^2 + \frac{9}{4480}z^4 - \dots \right); \quad (22)$$

the third term is just sensible towards the end of the following table.

$\cosh u$	$u \cosh u - \sinh u$	$\cosh u$	$u \cosh u - \sinh u$	$\cosh u$	$u \cosh u - \sinh u$
1.000	0	1.010	.0009423	1.020	.0026640
1	.0000298	1	.0010871	5	37221
2	843	2	12386	30	48917
3	.0001549	3	13966	5	61627
4	2385	4	15607	40	75275
5	3333	5	17308	5	89799
6	4380	6	19065	50	.0105148
7	5520	7	20880		
8	6743	8	22748		
9	8046	9	24668		

The relation between  $\phi$  and  $t$  at a distance  $r = 100cr$  is shown in Fig. 2, which corresponds in scale with Fig. 1, the time-integral of

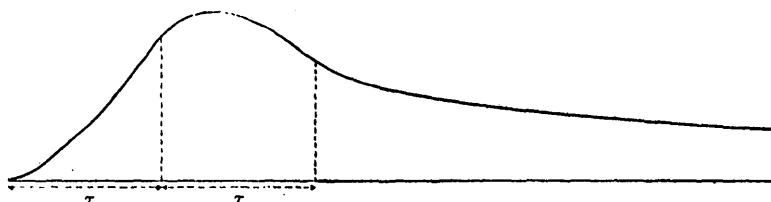


FIG. 2.

the source being the same in each case. It will be noticed that there is now no abrupt change of direction in the curve—only changes of curvature.

3. A solitary wave, free from every degree of discontinuity, is obtained if we assume a source

$$f(t) = \frac{\tau}{t^2 + \tau^2}. \quad (23)$$

This has, it is true, no definite beginning or ending, but the preceding examples enable us to understand what features in the result are to

be attributed solely to this cause. For purposes of calculation it is a little simpler to assume

$$f(t) = \frac{1}{t - ir}, \quad (24)$$

and retain, in the end, only the imaginary part.\* We have then, from (9),

$$\begin{aligned} 2\pi\phi &= \int_0^\infty \frac{du}{t - \frac{r}{c} \cosh u - ir} \\ &= 2 \int_0^1 \frac{dz}{t - \frac{r}{c} - ir - \left(t + \frac{r}{c} - ir\right) z^2}, \end{aligned} \quad (25)$$

where

$$z = \tanh \frac{1}{2}u.$$

We now write

$$t - \frac{r}{c} - ir = a^2 e^{-2i\alpha}, \quad t + \frac{r}{c} - ir = b^2 e^{-2i\beta}, \quad (26)$$

where we may suppose that  $a, b$  are positive, and that the angles  $2\alpha, 2\beta$  lie between 0 and  $\pi$ . We have

$$\left. \begin{aligned} a^4 &= \left(t - \frac{r}{c}\right)^2 + r^2, & b^4 &= \left(t + \frac{r}{c}\right)^2 + r^2 \\ \tan 2\alpha &= \frac{cr}{ct - r}, & \tan 2\beta &= \frac{cr}{ct + r} \end{aligned} \right\} \quad (27)$$

It appears that  $a \leq b$  according as  $t \geq 0$ , and that  $\alpha - \beta$  lies between 0 and  $\frac{1}{2}\pi$ . With this notation

$$\begin{aligned} 2\pi\phi &= 2 \int_0^1 \frac{dz}{a^2 e^{-2i\alpha} - b^2 e^{-2i\beta} z^2} \\ &= \frac{1}{ab} e^{i(\alpha+\beta)} \left( \int_0^1 \frac{dz}{z + \frac{a}{b} e^{-i(\alpha-\beta)}} - \int_0^1 \frac{dz}{z - \frac{a}{b} e^{-i(\alpha-\beta)}} \right) \\ &= \frac{1}{ab} e^{i(\alpha+\beta)} \left\{ \log \left( z + \frac{a}{b} e^{-i(\alpha-\beta)} \right) - \log \left( z - \frac{a}{b} e^{-i(\alpha-\beta)} \right) \right\}. \end{aligned} \quad (28)$$

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\* Cf. Lord Kelvin, *Phil. Mag.*, Feb., 1899, p. 189. The graph of the function (23) is the curve marked  $\mathcal{A}$  in Fig. 6, p. 157.

To interpret the logarithms, let us mark, in the plane of a complex variable  $z$ , the points

$$I = +1, \quad P = -\frac{a}{b} e^{-i(\alpha-\beta)}, \quad Q = \frac{a}{b} e^{-i(\alpha+\beta)}.$$

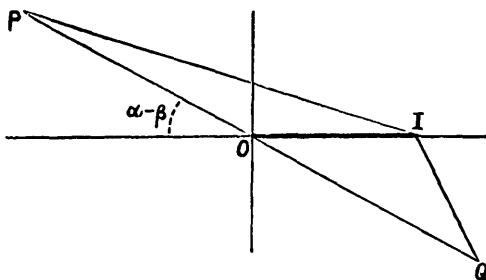


FIG. 3.

Since the integrals in the second line of (28) are to be taken along the path  $OI$ , the proper value of the third line is

$$\frac{1}{ab} e^{i(\alpha+\beta)} \left\{ \left( \log \frac{IP}{OP} + i \cdot OPI \right) - \left( \log \frac{IQ}{OQ} - i \cdot OQI \right) \right\}, \quad (29)$$

where real logarithms and positive values of the angles are to be understood. Hence, rejecting all but the imaginary part, we find

$$2\pi\phi = \frac{1}{ab} \sin(\alpha+\beta) \log \frac{IP}{IQ} + \frac{1}{ab} \cos(\alpha+\beta) (OPI + OQI), \quad (30)$$

as the expression of the disturbance due to the source (23). Since

$$\left. \begin{aligned} \tan OPI &= \frac{b \sin(\alpha-\beta)}{a+b \cos(\alpha-\beta)}, & \tan OQI &= \frac{b \sin(\alpha-\beta)}{a-b \cos(\alpha-\beta)}, \\ \frac{IP}{IQ} &= \frac{\sin OQI}{\sin OPI} \end{aligned} \right\}, \quad (31)$$

$\phi$  is uniquely determined as a function of  $a$ ,  $b$ ,  $\alpha$ ,  $\beta$ , and thence, by (27), of  $r$  and  $t$ .

Some special points may be noted. When  $t = 0$ , we have

$$a = b = \left( \frac{r^2}{c^2} + r^2 \right)^{\frac{1}{2}}, \quad \alpha + \beta = \frac{1}{2}\pi, \quad \tan 2\beta = \frac{cr}{r};$$

$$\text{and therefore} \quad 2\pi\phi = \frac{c}{\sqrt{r^2 + c^2 r^2}} \log \frac{1 + \tan \beta}{1 - \tan \beta}. \quad (32)$$

This may be confirmed by an independent calculation from (9), with  $t = 0$  throughout. If  $r$  be large compared with  $ct$ , the formula makes  $2\pi\phi = c^2\tau/r^2$ , approximately.

Again, when  $t$  is large compared with  $r/c$  (as well as with  $\tau$ ), and negative, we find

$$\alpha + \beta = \pi + \frac{\tau}{t}, \quad \alpha - \beta = \frac{\tau}{t} \cdot \frac{r}{ct},$$

$$a = \sqrt{\left(-t + \frac{r}{c}\right)}, \quad b = \sqrt{\left(-t - \frac{r}{c}\right)},$$

approximately, whence

$$OPI = \frac{1}{2} \frac{\tau}{t} \cdot \frac{r}{ct}, \quad OQI = -\frac{\tau}{t}.$$

Keeping only the most important term in (30), we obtain

$$2\pi\phi = -\frac{1}{\sqrt{\left(t^2 - \frac{r^2}{c^2}\right)}} \cdot \frac{\tau}{t} \log\left(-\frac{2ct}{r}\right). \quad (33)$$

On the other hand, when  $t$  is large compared with  $r/c$ , and positive,

$$\alpha + \beta = \frac{\tau}{t}, \quad \alpha - \beta = \frac{\tau}{t} \cdot \frac{r}{ct},$$

$$a = \sqrt{\left(t - \frac{r}{c}\right)}, \quad b = \sqrt{\left(t + \frac{r}{c}\right)},$$

approximately, and

$$OPI = \frac{1}{2} \frac{\tau}{t} \cdot \frac{r}{ct}, \quad OQI = \pi - \frac{\tau}{t}.$$

The most important part of (30) is then

$$2\pi\phi = \frac{\pi}{\sqrt{\left(t^2 - \frac{r^2}{c^2}\right)}}, \quad (34)$$

which agrees with (17) when the limits of integration are suitably modified.

The crest of the wave at any instant will be in the neighbourhood of  $r = ct$ . If we put  $r = ct$ , exactly, we have

$$\alpha = \frac{1}{4}\pi, \quad \tan 2\beta = \frac{1}{2} \frac{ct}{r}, \quad a = \sqrt{r}, \quad b = \sqrt{\left(\frac{4r^2}{c^2} + r^2\right)}.$$

If, moreover,  $r$  be large compared with  $c\tau$ , the angle  $PIQ$  will be small, and

$$2\pi\phi = \frac{\pi}{2r} \sqrt{\left(\frac{c\tau}{r}\right)}, \quad (35)$$

approximately.

To examine more closely the progress of the disturbance at a distance  $r$  which is large compared with  $c\tau$ , as the crest of the wave passes, we put

$$t = \frac{r}{c} + r \tan \eta.$$

This makes  $\alpha = \frac{1}{4}\pi - \frac{1}{2}\eta$ ,  $a = \sqrt{(r \sec \eta)}$ ,

whilst  $\beta = \frac{1}{4} \frac{c\tau}{r}$ ,  $b = \sqrt{\left(\frac{2r}{c}\right)}$ ,

approximately, provided  $r/c\tau$  be large compared with the greatest value of  $\tan \eta$  considered. The ratio  $a/b$  will then be small, so that  $IP/IQ = 1$ , nearly, whilst  $PIQ$  is a small angle. The most important part of (30) is then

$$2\pi\phi = \frac{\pi}{ab} \cos \alpha = \frac{\pi}{\sqrt{2}r} \sqrt{\left(\frac{c\tau}{r}\right)} \cos \left(\frac{1}{4}\pi - \frac{1}{2}\eta\right) \sqrt{(\cos \eta)}. \quad (36)$$

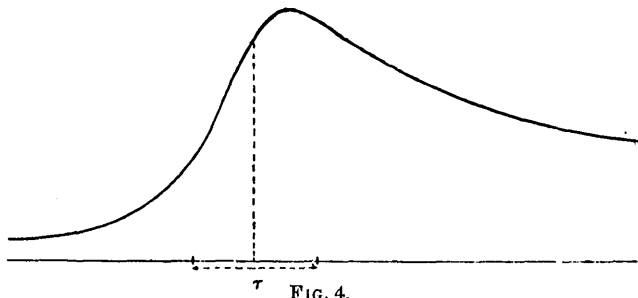
The following table gives a series of values of the functions

$$x = \tan \eta, \quad y = \cos \left(\frac{1}{4}\pi - \frac{1}{2}\eta\right) \sqrt{(\cos \eta)}.$$

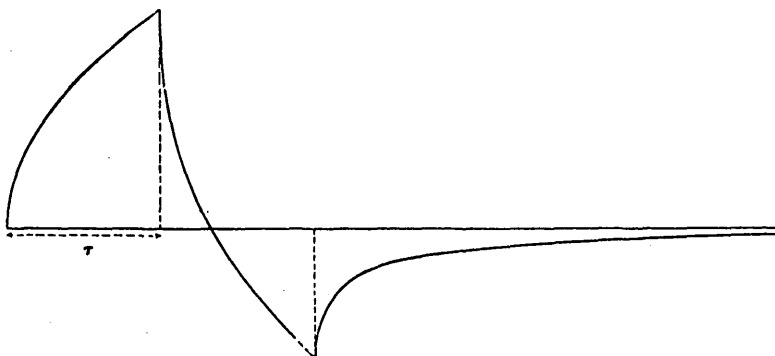
$2\eta/\pi$	$x$	$y$	$2\eta/\pi$	$x$	$y$
0	0	.707	$\pm .5$	$\pm 1.000$	$\begin{cases} .777 \\ .322 \end{cases}$
$\pm .1$	$\pm .158$	$\begin{cases} .756 \\ .645 \end{cases}$	$\pm .6$	$\pm 1.376$	$\begin{cases} .729 \\ .237 \end{cases}$
$\pm .2$	$\pm .325$	$\begin{cases} .789 \\ .573 \end{cases}$	$\pm .7$	$\pm 1.963$	$\begin{cases} .655 \\ .157 \end{cases}$
$\pm .3$	$\pm .510$	$\begin{cases} .805 \\ .493 \end{cases}$	$\pm .8$	$\pm 3.078$	$\begin{cases} .549 \\ .087 \end{cases}$
$\pm .4$	$\pm .727$	$\begin{cases} .801 \\ .408 \end{cases}$	$\pm .9$	$\pm 6.314$	$\begin{cases} .394 \\ .031 \end{cases}$

The maximum value of  $y$  is .806, corresponding to  $\eta = \frac{1}{8}\pi$ , or

$t = r/c + .577\tau$ . The curve in Fig. 4, constructed with  $x, y$  as coordinates, shows the variation of  $\phi$  with  $t$ .



4. The most interesting feature in the preceding diagrams is, of course, the unsymmetrical character of the wave, and the prolongation of its rear in the form of the "tail" already referred to. If, however, the source change its sign, and especially if its time-integral be zero, the positive and negative tails will tend to obliterate one another, although they are not without influence on the residual wave-form. The almost complete cancelling in the special case mentioned is evident from the approximate formula (17). To illustrate the matter further, we may suppose the temporary constant source of § 2 to be followed immediately by a negative source of equal strength and duration. The resulting curve, easily constructed from the numerical table given on p. 146, is shown in Fig. 5. The tail is now insignificant,



but it will be noticed that the depression which follows the primary

elevation has only about one-half its amplitude.\* It is plain, moreover, that in any case of a periodic source suddenly beginning to act the first wave will be much higher than those which follow it.

Now that we are in possession of a number of examples (which might easily be multiplied) of the propagation of a solitary wave in two dimensions, it is of interest to examine how they are related to the corresponding results in one and in three dimensions.

The equation (1) is included in the general form

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \left( \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \dots + \frac{\partial^2 \phi}{\partial x_n^2} \right), \quad (37)$$

which has been called the equation of wave-motion in  $n$  dimensions. In the case of symmetry about the origin, this reduces to

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{n-1}{r} \frac{\partial \phi}{\partial r} \right), \quad (38)$$

which includes (2) and (5) as particular cases. The equation (37) has been very fully discussed by Duhem,† who has insisted on the exceptional position occupied by the cases  $n = 1$ ,  $n = 3$ ; in particular he has shown, amongst other things, that it is only in these cases that (38) admits of a solution of the type

$$\phi = \psi(r) f\left(t - \frac{r}{c}\right). \quad (39)$$

However interesting and important such investigations may be from an analytical point of view, we must recognize that what they chiefly determine is the comparative facility of mathematical expression in the various cases. From a physical standpoint, the idea that the cases  $n = 1$ ,  $n = 3$  are closely analogous, whilst that of  $n = 2$  occupies a sort of exceptional and outlying position, would be wholly misleading. A truer account of the matter would be to say that the cases  $n = 1$ ,  $n = 2$ ,  $n = 3$  form a sequence, with a regular gradation of properties due to the increasing degree of mobility of the medium.‡

\* A further illustration is supplied accidentally by the diagrams on p. 157. The curve  $B$  shows the relation between  $\phi$  and  $t$  for a source whose graph has the form  $C$ .

† *Hydrodynamique, Élasticité, Acoustique*, Paris, 1891, Vol. II., p. 138.

‡ The one common characteristic is that the boundary of a disturbed region spreads everywhere normally with the constant velocity  $c$ . It would be interesting to have a proof of this property direct from the differential equation (37). Duhem, adopting a method of Hugoniot, endeavours to supply this. The argument is interesting, but I do not think that it amounts to a demonstration. It appears to assume tacitly that the disturbance spreads at some finite rate; if this be granted, the conclusion that the rate is constant appears to be unexceptionable.

For the sake of a definite comparison between the three cases, we may take the case of air-waves, and examine the effect (A) of a plane-source, (B) of a line-source, and (C) of a point-source, whose "strength" is given in each case by (23). The results may be conveniently expressed in terms of the condensation  $s$ , which is given by the formula

$$c^2 s = \frac{\partial \phi}{\partial t}. \quad (40)$$

(A) In the case  $n = 1$ , we find, for  $x > 0$ ,

$$s = \frac{r}{2c} \frac{1}{\left(t - \frac{x}{c}\right)^2 + r^2}. \quad (41)$$

(B) When  $n = 2$ , the value of  $s$  is to be deduced from (30). Near the "crest" of the wave a sufficient approximation is obtained by differentiating (36) with respect to  $r \tan \eta$ . We thus find

$$s = \frac{1}{4\sqrt{2}c^2 r^3} \sqrt{\left(\frac{cr}{r}\right)} \sin\left(\frac{1}{4}\pi - \frac{3}{2}\eta\right) \cos^3 \eta. \quad (42)$$

The values of the functions

$$x = \tan \eta, \quad y = \sin\left(\frac{1}{4}\pi - \frac{3}{2}\eta\right) \cos^3 \eta$$

are tabulated below.

$2\eta/\pi$	$x$	$y$	$2\eta/\pi$	$x$	$y$
0	0	+·707	$\pm 5$	$\pm 1\cdot000$	$\begin{cases} -\cdot228 \\ +\cdot549 \end{cases}$
$\pm 1$	$\pm 1\cdot58$	$\begin{cases} +\cdot513 \\ +\cdot837 \end{cases}$	$\pm 6$	$\pm 1\cdot376$	$\begin{cases} -\cdot265^\dagger \\ +\cdot365 \end{cases}$
$\pm 2$	$\pm 3\cdot25$	$\begin{cases} +\cdot287 \\ +\cdot882^* \end{cases}$	$\pm 7$	$\pm 1\cdot963$	$\begin{cases} -\cdot233 \\ +\cdot199 \end{cases}$
$\pm 3$	$\pm 5\cdot10$	$\begin{cases} +\cdot066 \\ +\cdot838 \end{cases}$	$\pm 8$	$\pm 3\cdot078$	$\begin{cases} -\cdot153 \\ +\cdot078 \end{cases}$
$\pm 4$	$\pm 7\cdot27$	$\begin{cases} -\cdot114 \\ +\cdot719 \end{cases}$	$\pm 9$	$\pm 6\cdot314$	$\begin{cases} -\cdot060 \\ +\cdot014 \end{cases}$

\* Maximum.

† Minimum.

(C) In three dimensions we have

$$4\pi r\phi = \frac{r}{\left(t - \frac{r}{c}\right)^2 + r^2}, \quad (43)$$

$$s = \frac{r}{2\pi c^2} \frac{\frac{r}{c} - t}{r \left\{ \left(t - \frac{r}{c}\right)^2 + r^2 \right\}^{\frac{3}{2}}}. \quad (44)$$

The three cases are represented, with  $s$  as ordinate and  $t$  as abscissa, on p. 157. The scale of  $t$  is the same in each case, but there is, of course, no relation between the vertical scales. In (A) we have a wave of pure condensation; in (B) the primary condensation is followed by a rarefaction of less amount, but lasting for a longer time; whilst in (C) the condensation and rarefaction are anti-symmetrical. In (B) and (C) alike we necessarily have, at any point,

$$\int_{-\infty}^{\infty} s \, dt = 0, \quad (45)$$

in virtue of (40). If the source had been strictly limited in duration, the medium, in the case of three dimensions, would have remained absolutely\* at rest after the passage of the wave, as in the case of one dimension, although for a different reason. In the intermediate case of two dimensions, there is only an asymptotic approach to rest.

5. If in a two-dimensional medium free from singular points we wish to calculate the disturbance consequent on arbitrary initial (symmetrical) conditions, it is probably simplest to adopt the method

\* The statement has reference to a *point*-source. If a source  $f(t)$  be uniformly distributed over the surface of a sphere of radius  $a$ , we have

$$\begin{aligned} 4\pi r\phi &= \frac{c}{a} e^{-[ct-(r-a)]/a} \int_{-\infty}^{t-(r-a)/c} f(\lambda) e^{c\lambda/a} d\lambda \\ &= \frac{c}{a} \int_0^{\infty} f\left(t - \frac{r-a}{c} - \theta\right) e^{-c\theta/a} d\theta, \end{aligned}$$

as may be easily verified. If  $f(t) = 0$  for  $t < 0$  and for  $t > \tau$ , we find, for  $t > (r-a)/c + \tau$ ,

$$4\pi r\phi = \frac{c}{a} e^{-[ct-(r-a)]/a} \int_0^{\tau} f(\lambda) e^{c\lambda/a} d\lambda,$$

so that  $\phi$  falls only asymptotically to 0. But when  $ct - r$  is only a moderate multiple of  $a$  the residual disturbance is already very small.

This point was noticed by Stokes (*l.c. ante*) in connection with the same problem. Reference may also be made to Kirchhoff, *Mechanik*, c. xxiii.

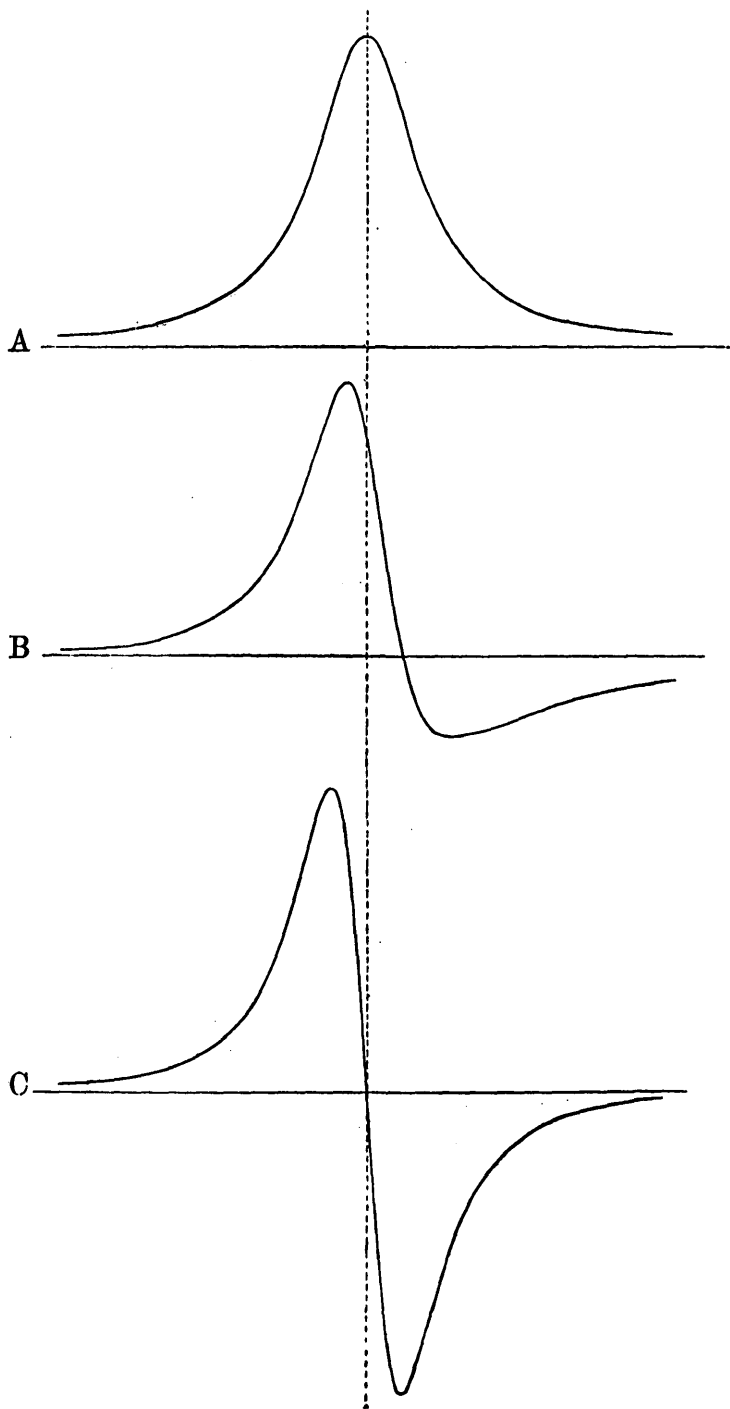


FIG. 6.

indicated by Rayleigh,\* treating the problem as a particular case of motion in three dimensions. If, however, we wish to follow a more analytical procedure, we require, in the first instance, the solution of the following subsidiary problem:—Given a function  $f$ , to find a function  $F$ , such that

$$\int_0^\infty F(r \cosh u) du = f(r). \quad (46)$$

Provided  $f(\infty) = 0$ , a condition naturally fulfilled in the applications we have in view, the solution is

$$F(z) = -\frac{2}{\pi} z \int_0^\infty f'(z \cosh v) dv. \quad (47)$$

For this makes

$$\int_0^\infty F(r \cosh u) du = -\frac{2}{\pi} \int_0^\infty r \cosh u du \int_0^\infty f'(r \cosh u \cosh v) dv. \quad (48)$$

Let us now write

$$x = r \cosh u \cosh v, \quad y = r \cosh u \sinh v,$$

so that 
$$\frac{\partial(x, y)}{\partial(u, v)} = r^2 \sinh u \cosh u.$$

Transforming to  $x, y$  as independent variables, we have

$$\int_0^\infty F(r \cosh u) du = -\frac{2}{\pi} \iint \frac{f'(x) dx dy}{\sqrt{(x^2 - y^2 - r^2)}}, \quad (49)$$

where the integrations extend over that part of the plane  $xy$  which lies above the axis of  $x$  and to the right of the positive branch of the hyperbola  $x^2 - y^2 = r^2$ . If we integrate first with respect to  $y$ , we find

$$\int_0^\infty F(r \cosh u) du = -\int_r^\infty f'(x) dx; \quad (50)$$

so that (46) is satisfied, subject to the condition stated.†

Writing (11) in the now slightly more convenient form

$$\phi = \int_0^r f(ct - r \cosh u) du + \int_0^\infty F(ct + r \cosh u) du, \quad (51)$$

\* *Sound*, § 276.

† This determination of the unknown function  $F$  in (46) is suggested by an investigation of Schlömilch, which is reproduced in Todhunter's *Functions of Laplace*, &c., § 441.

we have to determine the arbitrary functions so that

$$\varphi = \psi(r), \quad \frac{\partial \varphi}{\partial t} = \chi(r), \quad (52)$$

say, for  $t = 0$ ; that is, we must have

$$\left. \begin{aligned} \int_0^\infty \{f(-r \cosh u) + F(r \cosh u)\} du &= \psi(r) \\ \int_0^\infty \{f'(-r \cosh u) + F'(r \cosh u)\} du &= \frac{1}{c} \chi(r) \end{aligned} \right\}. \quad (53)$$

These conditions will be satisfied, in virtue of (47), provided

$$f(-r) + F(r) = -\frac{2r}{\pi} \int_0^\infty \psi'(r \cosh v) dv, \quad (54)$$

and 
$$f'(-r) + F'(r) = -\frac{2r}{\pi c} \int_0^\infty \chi'(r \cosh v) dv, \quad (55)$$

it being assumed that  $\psi(r)$  and  $\chi(r)$  vanish for  $r = \infty$ . The condition (55) may be replaced by

$$f(-r) - F(r) = \frac{2}{\pi c} \int_r^\infty r' dr' \int_0^\infty \chi'(r' \cosh v) dv, \quad (56)$$

where the lower limit of the integration with respect to  $r$  is indifferent. The solution of our problem is now virtually complete; for (54) and (56) determine  $f$  for positive and  $F$  for negative values of the argument, whilst the determination of  $f$  for positive arguments is supplied by

$$f(ct) + F(ct) = 0, \quad (57)$$

which is the condition of "continuity" at the origin.\*

As an example, suppose that initially  $\varphi = 0$ , everywhere, whilst  $\partial\varphi/\partial t = 1$  for  $r < a$ , and  $= 0$  for  $r > a$ .† We have, from (54),

$$f(-r) + F(r) = 0. \quad (58)$$

Again, in (56) we have, denoting by  $\epsilon$  an infinitely small positive constant,

$$\int_0^\infty \chi'(r \cosh v) dv = \int_{a-\epsilon}^{a+\epsilon} \frac{\chi'(z) dz}{\sqrt{(z^2 - r^2)}} = -\frac{1}{\sqrt{(a^2 - r^2)}},$$

\* Cf. Rayleigh, *Sound*, § 279.

† The corresponding problem in three dimensions is solved in *Hydrodynamics*, § 264.

provided  $r < a$ , whilst for  $r > a$  the integrand vanishes. Hence, taking  $a$  for the unindicated lower limit in (56), we find

$$\left. \begin{aligned} f(-r) - F(r) &= \frac{2}{\pi c} \sqrt{(a^2 - r^2)}, & \text{for } r < a; \\ &= 0, & \text{for } r > a. \end{aligned} \right\} \quad (59)$$

From (57), (58), (59) we infer

$$\left. \begin{aligned} f(r) &= f(-r) = -F(r) \\ &= \frac{1}{\pi c} \sqrt{(a^2 - r^2)}, & \text{for } r < a; \\ &= 0, & \text{for } r > a. \end{aligned} \right\} \quad (60)$$

To trace the course of the disturbance at a distance  $r$  which is  $> a$ , we remark, in the first place, that, since  $ct + r \cosh u > a$ , the second term in (51) disappears, and we have accordingly to deal with a diverging wave only. Again, so long as  $ct < r - a$ , we have  $|ct - r \cosh u| > a$ , and therefore  $\phi = 0$ . When  $r - a < ct < r + a$ , we find

$$\phi = \frac{1}{\pi c} \int_0^{\cosh^{-1}(ct+a)/r} \sqrt{\{a^2 - (ct - r \cosh u)^2\}} du; \quad (61)$$

whilst, for  $ct > r + a$ ,

$$\phi = \frac{1}{\pi c} \int_{\cosh^{-1}(ct-a)/r}^{\cosh^{-1}(ct+a)/r} \sqrt{\{a^2 - (ct - r \cosh u)^2\}} du. \quad (62)$$

The last result may also be written

$$\phi = \frac{a^2}{\pi c} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2 \theta d\theta}{\sqrt{\{(ct + a \sin \theta)^2 - r^2\}}}, \quad (63)$$

or, for large values of  $ct - r$ ,\*

$$\phi = \frac{a^2}{2c \sqrt{(c^2 t^2 - r^2)}}. \quad (64)$$

6. It would be of interest to examine whether the energy of a free progressive wave in two dimensions is equally divided between the kinetic and the potential forms. That this is the case in one dimension is well known, and the proof for spherical waves is easily supplied. For we have, identically,

$$r^2 \left( \frac{\partial \phi}{\partial r} \right)^2 = \left\{ \frac{\partial (r\phi)}{\partial r} \right\}^2 - \frac{\partial}{\partial r} (r\phi^2).$$

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\* This may be compared with Rayleigh, *Sound*, § 275, equation (1).

If we assume

$$r\phi = f(ct-r), \quad (65)$$

and put

$$u = -\frac{\partial\phi}{\partial r}, \quad c^2s = \frac{\partial\phi}{\partial t},$$

this may be written  $r^2u^2 = c^2r^2s^2 - \frac{\partial(r\phi^2)}{\partial r}.$  (66)

Hence 
$$\int_0^\infty \frac{1}{2}\rho u^2 \cdot 4\pi r^2 dr = \int_0^\infty \frac{1}{2}\rho c^2 s^2 \cdot 4\pi r^2 dr, \quad (67)$$

if we suppose that  $r\phi^2$  vanishes at the inner and outer boundaries of the wave.

The direct examination of the question for the case of cylindrical waves, starting from the formula (9), would not appear to be easy. If we try to adapt the indirect method given by Rayleigh\* for the case of plane waves, the argument would run somewhat as follows:—Consider at any instant (which we may conveniently take as origin of  $t$ ) an arbitrary diverging wave  $A$ , and also the converging wave  $B$  which is obtained from  $A$  by reversing the *velocity* of every particle. It is plain that an initial disturbance ( $A+B$ ) constructed by superposing these would split up into the two waves  $A$  and  $B$ , and that, *if in the subsequent motion these become wholly separate*, the sum of the energies of  $A$  and  $B$  can be asserted to be equal to the initial energy of the disturbance  $A+B$ . Now the energies of  $A$  and  $B$  alone are constant, and are initially equal, and the initial energy of  $A$  is therefore half that of  $A+B$ . Since the latter is wholly potential, and equal to four times the initial potential energy of  $A$ , the desired conclusion follows.

The proviso in italics would seem to be essential to the argument. It is satisfied in the case of spherical waves, but not in the case of cylindrical waves, at all events such as result from a limited original disturbance.

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\* *Sound*, Vol. II., § 245.

*Expansions by means of Lamé's Functions.* By A. C. DIXON.  
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I have found myself hampered lately by the want of strict proofs of the validity of expansions by means of Lamé's functions. The present paper is meant to supply this want, and to give the reader a fairly connected idea of the analytical theory, so far as it relates to the determination of a potential or harmonic function whose values on an ellipsoidal surface are given. The better known parts of the theory have only been sketched.

An important development, which I have not met with before, is the use of Lamé's functions to express functions in a double space, analogous to a Riemann surface.

The notation used for elliptic functions will, I think, be found to have advantages in working with ellipsoidal coordinates.

The works that I have chiefly consulted are the papers of Lamé and Liouville, and the treatises of Heine (*Kugelfunctionen*) and Halphen (*Fonctions elliptiques*, Vol. II.).\*

1. In working with ellipsoidal coordinates we shall use the following notation. We shall write  $au$ ,  $bu$ ,  $cu$  for the square roots of  $a^2 + \lambda$ ,  $b^2 + \lambda$ ,  $c^2 + \lambda$ , where  $\lambda$ ,  $u$  are connected by the relation

$$2u = \int_{\lambda}^{\infty} \{ (a^2 + \theta)(b^2 + \theta)(c^2 + \theta) \}^{-1} d\theta. \quad (1)$$

When  $u$  is real, positive, and small the values of  $au$ ,  $bu$ ,  $cu$  will be taken positive. In formulæ there will be complete symmetry as to  $a$ ,  $b$ ,  $c$ . For instance, to differentiate we have

$$\left. \begin{aligned} a'u &= -bu\,cu \\ b'u &= -cu\,au \\ c'u &= -au\,bu \end{aligned} \right\}. \quad (2)$$

\* The paragraphs that I recognize as due in substance to earlier writers are 2-7, 14, 20 and in part 10, 15.

The addition theorem is

$$a(u+v) = (au\,bv\,cv - av\,bu\,cu) / (a^2v - a^2u), \quad (3)$$

with two similar equations.

Let  $a, \beta, \gamma$  be the half periods, so that

$$aa = b\beta = c\gamma = 0, \quad a^2\beta = a^2 - b^2, \quad a^2\gamma = a^2 - c^2, \quad \&c. \quad (4)$$

The formulæ for the addition of half periods are of the types

$$\left. \begin{aligned} a(a+u) &= a'a/au \\ a(\beta+u) &= a\beta\,cu/bu \end{aligned} \right\}. \quad (5)$$

The three functions  $au, bu, cu$  are odd and for small values of  $u$  approach the limiting form  $u^{-1}$ .

We shall suppose  $a^2 - b^2$  and  $b^2 - c^2$  positive; thus  $\gamma$  may be taken real and positive,  $\alpha$  purely imaginary, and of the same sign as  $\iota$ ;  $\beta$  will be fixed by the equation

$$\alpha + \beta + \gamma = 0. \quad (6)$$

Then  $a\gamma, b\gamma, \iota ba, \iota ca, -a\beta, -\iota c\beta$  are all positive.

The relations between the Cartesian coordinates  $x, y, z$  and the ellipsoidal coordinates  $u, v, w$  are

$$\left. \begin{aligned} \frac{x^2}{a^2u} + \frac{y^2}{b^2u} + \frac{z^2}{c^2u} &= 1 \\ \frac{x^2}{a^2v} + \frac{y^2}{b^2v} + \frac{z^2}{c^2v} &= 1 \\ \frac{x^2}{a^2w} + \frac{y^2}{b^2w} + \frac{z^2}{c^2w} &= 1 \end{aligned} \right\}, \quad (7)$$

or

$$\left. \begin{aligned} x &= au\,av\,aw/a\beta\,a\gamma \\ y &= bu\,bv\,bw/b\gamma\,ba \\ z &= cu\,cv\,cw/ca\,c\beta \end{aligned} \right\}. \quad (8)$$

Of the three arguments,  $u$  is purely real, the real part of  $v$  is an odd multiple of  $\gamma$ , and the imaginary part of  $w$  an odd multiple of  $\alpha$ ; we shall suppose  $v$  to range between  $\beta \pm 2\alpha$  and  $w$  between  $\beta \pm 2\gamma$ , and then for any fixed value of  $u$  the corresponding ellipsoid will be covered twice, the points  $u, v, w$  and  $u, 2\beta - v, 2\beta - w$  coinciding. We have

$$a^2u > a^2\gamma > a^2v > a^2\beta > a^2w > 0. \quad (9)$$

Double integrals of the form

$$\iint \phi \frac{\partial \psi}{\partial \nu} dS$$

often occur,  $dS$  being an element of area on the surface of the ellipsoid  $u = \text{const.}$ , and  $d\nu$  an element of the outward normal to that surface. Since

$$\left. \begin{aligned} d\nu &= -(a^2u - a^2v)^{\frac{1}{2}} (a^2u - a^2w)^{\frac{1}{2}} du \\ dS &= -\iota (a^2u - a^2v)^{\frac{1}{2}} (a^2u - a^2w)^{\frac{1}{2}} (a^2v - a^2w) dv dw \end{aligned} \right\}, \quad (10)$$

this double integral becomes

$$\iint \phi \frac{\partial \psi}{\partial u} \iota (a^2v - a^2w) dv dw,$$

and we shall write  $d\omega$  for the positive quantity

$$-\iota (a^2v - a^2w) dv dw.$$

## 2. The transformation of Laplace's equation

$$\Delta \phi \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad (11)$$

$$\text{is} \quad (a^2v - a^2w) \frac{\partial^2 \phi}{\partial u^2} + (a^2w - a^2u) \frac{\partial^2 \phi}{\partial v^2} + (a^2u - a^2v) \frac{\partial^2 \phi}{\partial w^2} = 0, \quad (12)$$

and Lamé pointed out that, if this equation was satisfied by a value for  $\phi$  of the form  $fu fv fw$ , then the function  $f$  must be a solution of the equation

$$f''u = fu (Aa^2u + B), \quad (13)$$

$A, B$  being constants.

Lamé's functions are rational integral algebraic expressions in  $au, bu, cu$ , satisfying equations of the type (13). It is readily seen that such expressions will contain terms of odd degree only or else terms of even degree only, and that, if  $n$  is the degree of the highest term,  $A = n(n+1)$ . The general expression of the  $n$ -th degree that is to be considered may thus be written—

$$\begin{aligned} p_0 a^n u + p_1 a^{n-2} u + p_2 a^{n-4} u + \dots + bu (q_0 a^{n-1} u + q_1 a^{n-3} u + \dots) \\ + cu (r_0 a^{n-1} u + r_1 a^{n-3} u + \dots) \\ + bucu (s_0 a^{n-2} u + s_1 a^{n-4} u + \dots), \end{aligned}$$

and it contains  $2n+1$  terms. If it is substituted for  $fu$  in the equation (13), the terms of the degree  $n+2$  will disappear on account of the value

chosen for  $A$ , and there will be  $2n+1$  linear homogeneous equations that must be satisfied by the  $2n+1$  coefficients  $p, q, r, s$ . These equations are also linear in  $B$ , and thus the result of eliminating  $p, q, r, s$  is an equation for  $B$  of the degree of  $2n+1$ . A value satisfying this equation having been taken for  $B$ , the ratios of the coefficients  $p, q, r, s$  are determinate,\* and thus the function  $fu$  is found. A term in  $fu$  consists of a power of  $au$  multiplied by one or other, or both, or neither of the factors  $bu, cu$ , and a term of any one of these four forms gives terms of the same form only in  $f''u$ . Hence it appears that Lamé's functions are of four kinds, in each of which terms occur of one of the four forms only, and the equation for  $B$  accordingly breaks up into four factors.

3. Now let  $f'u, f_1u$  denote two of Lamé's functions. The products  $fuf_vfw$ , or  $\phi$ , and  $f_1uf_vf_1w$ , or  $\phi_1$ , satisfy the equation (11), and have no singularity except at infinity, since they are, as is readily proved, rational integral algebraic expressions in  $x, y, z$ . Hence

$$\int \left( \phi \frac{\partial \phi_1}{\partial u} - \phi_1 \frac{\partial \phi}{\partial u} \right) d\omega = 0, \quad (14)$$

by Green's equation, if the integral is taken over any ellipsoid  $u$ . The integral is doubled in value if we take  $\beta \pm 2a$  as the limits for  $v$  and  $\beta \pm 2\gamma$  for  $w$ , and these limits will always be understood when no others are stated or implied. Substituting for  $\phi, \phi_1$ , we have

$$(fuf_1'u - f_1uf'u) \int f_vfwf_1vf_1w d\omega = 0$$

for all real values of  $u$  between 0 and  $\gamma$ .

The first factor does not vanish unless  $f'u, f_1u$  are practically the same, their ratio being constant. Hence, if  $f, f_1$  are different Lamé functions

$$\int f_vfwf_1vf_1w d\omega = 0. \quad (15)$$

4. From (15) it follows that all the roots of the equation for  $B$  are real. For, if  $B, B_1$  were two conjugate complex roots, they would give conjugate complex values to the coefficients in the corresponding

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\* It is found later (§ 4) that the equation for  $B$  has no double root, and hence this statement is justified.

functions  $f, f_1$ . Hence  $fvf_1v$  and  $fwf_1w$  would be of constant sign over the surface, and the equation (15) could not therefore be true.

In somewhat the same way it may be proved that the equation for  $B$  cannot have equal roots. If a root  $B$  were repeated, then a quantity  $C$  and the coefficients in an expression  $gu$  of the same form as  $fu$  could be so chosen that the coefficient of  $\epsilon$  in

$$\frac{d^3}{du^3}(fu + \epsilon gu) - (Aa^3u + B + \epsilon C)(fu + \epsilon gu)$$

should vanish. That is, we should have

$$g''u = (Aa^3u + B)gu + Cfu. \quad (16)$$

Then  $gufvfw + fu gvfvgw + fu fvgv$  would satisfy Laplace's equation (11) and could be put in the place of  $\phi_1$  in equation (14), since it would be a rational integral algebraic function of  $x, y, z$ . The result would be

$$(g'u fu - f'u gu) \int f^2 v f^2 w dw = 0,$$

which is impossible, as  $gu$  is not a mere constant multiple of  $fu$ , and the subject of integration is of constant sign.

If  $C$  were zero, this argument would fail, since  $gu$  need not then be of the same form as  $fu$ , but only of the same degree, and then the expression put for  $\phi_1$  would not be rational in  $x, y, z$ . We should, however, have from the differential equation

$$g''u fu - gu f''u = 0,$$

and therefore  $g'u fu - gu f'u = h$ , a constant.

Thus 
$$2h fu gu = f^3 u \frac{d}{du} g^3 u - g^3 u \frac{d}{du} f^3 u$$

$$= au bu cu \times \text{an expression rational in } a^3 u.$$

This is impossible, unless  $h = 0$ , since  $fu gu$  is of even degree,  $2n$ , in  $au, bu, cu$ ; hence, again,

$$g'u fu - gu f'u = 0;$$

so that  $f, g$  are the same function practically.

Hence all the roots of the equation for  $B$  are real and distinct, and for any value of the order  $n$  there are  $2n + 1$  distinct Lamé functions with real coefficients.

5. The next matters to be discussed are superior and inferior

limits to the value of  $B$ , and the distribution of the values of  $u$  for which  $fu$  vanishes.

Now  $f^2u$  is a quantic in  $a^2u$ , and hence its derivative  $2fu f'u$  contains the factor  $au bu cu$ , and vanishes at  $\alpha, \beta, \gamma$ ; thus, for each of these values of  $u$ , either  $fu$  or  $f'u$  is zero. They cannot both vanish for any value of  $u$ ; otherwise, from the equation (13) and its derivatives, it would follow that all the successive derivatives of  $fu$  would vanish, which is impossible.

The limits between which  $B$  must lie are 0 and  $-n(n+1)a^2\gamma$ . For suppose the case otherwise; then the sign of  $f''u/fu$  is constant when  $u$  lies between  $\gamma$  and  $\gamma+\alpha$  or between  $\gamma+\alpha$  and  $\alpha$ . Take this sign to be positive; then the curve  $y = f(a+x)$  is convex to the axis of  $x$  for values of the abscissa between 0 and  $\gamma$ , and yet either  $y$  or  $dy/dx$  vanishes for each of these extreme values. The figure shows that this is absurd. If the sign of  $f''u/fu$  is taken negative, it will, in like manner, be found impossible to draw the curve  $y = f(\gamma+x)$  for values of  $x$  between 0 and  $-\alpha$ .

Since  $B$  lies between 0 and  $-n(n+1)a^2\gamma$ , it follows that  $f''u/fu$  is positive for all real values of  $u$ . Hence  $fu$  and  $f'u$  are constant in sign and increase in numerical value as  $u$  decreases from  $\gamma$  to 0. The same is true of  $fu$  and  $f'u$  as  $u$  decreases from  $-\alpha$  to 0. When  $u = 0$ ,  $fu$  and  $f'u$  become infinite.

Now  $fu$  contains as a factor a quantic in  $a^2u$ , and, since  $fu, f'u$  cannot vanish together, no root of this quantic can be repeated and none can be equal to 0 or  $a^2\beta$  or  $a^2\gamma$ ; also no root can be negative or greater than  $a^2\gamma$ . The coefficients in the polynomial are algebraic functions of  $a^2, b^2, c^2$ , and no variation in  $a^2, b^2, c^2$  can alter the number of roots of the polynomial that lie between 0 and  $a^2\beta$  or between  $a^2\beta$  and  $a^2\gamma$ ; for this would imply a transition stage in which two roots would be equal, or a root would be equal to 0 or  $a^2\beta$  or  $a^2\gamma$ . Hence the number of roots between 0 and  $a^2\beta$ , say, can be found by considering a special case: for instance, that in which  $b = c$ .

6. The elliptic functions then degenerate, and, writing  $\kappa$  for the value of  $a\beta$  or  $a\gamma$ , we have

$$\left. \begin{aligned} au &= \kappa \coth \kappa u \\ bu &= cu = \kappa \operatorname{cosech} \kappa u \end{aligned} \right\}. \quad (17)$$

Putting  $v = \gamma + \alpha\phi, \quad w = -\alpha + \psi,$

so that  $\phi, \psi$  are real, and using the formulæ (5), we have

$$\left. \begin{aligned} av &= a\gamma b\phi/c\phi = \kappa \\ bv/b\gamma &= a\phi/c\phi = \cos \kappa\phi \\ cv/b\gamma &= c'\gamma/b\gamma c\phi = -\epsilon \sin \kappa\phi \end{aligned} \right\}, \quad (18)$$

$$\text{and, similarly,} \quad \left. \begin{aligned} aw &= \kappa \tanh \kappa\psi \\ bw &= cv = \kappa \operatorname{sech} \kappa\psi \end{aligned} \right\}. \quad (19)$$

Lamé's equation (13) becomes, for  $fu$ ,

$$\frac{d^2 U}{du^2} = \{n(n+1)\kappa^2 \coth^2 \kappa u + B\} U; \quad (20)$$

$$\text{for } fv, \quad \frac{d^2 V}{d\phi^2} = -\{n(n+1)\kappa^2 + B\} V; \quad (21)$$

$$\text{for } fw, \quad \frac{d^2 W}{d\psi^2} = \{n(n+1)\kappa^2 \tanh^2 \kappa\psi + B\} W. \quad (22)$$

Now  $V$ , or  $fv$ , is a polynomial in  $\sin \kappa\phi$ ,  $\cos \kappa\phi$  of degree not higher than  $n$ , and, if it is to satisfy the equation (21), it must be a multiple of  $\sin m\kappa\phi$  ( $m = 1, 2, \dots, n$ ) or  $\cos m\kappa\phi$  ( $m = 0, 1, 2, \dots, n$ ), and the value of  $B$  will be  $m^2\kappa^2 - n(n+1)\kappa^2$ .

Thus when  $\mu$  is put for  $\tanh \kappa\psi$  the equation (22) becomes that which is satisfied by the zonal factor in a tesseral harmonic, and  $W$  is a constant multiple of

$$(\mu^2 - 1)^{1/2m} \frac{d^{m+n}}{d\mu^{m+n}} (\mu^2 - 1)^n.$$

Hence  $W$  vanishes for  $n-m$  real values of  $\psi$  of which one is zero when  $n-m$  is odd; thus the number of roots of the polynomial in  $a^2u$  that lie between 0 and  $a^2\beta$  is determined.

The real values of  $\kappa\phi$  for which  $V$  vanishes are, when  $V = \sin m\kappa\phi$ ,  $0, \pm\pi/m, \pm2\pi/m, \dots$ , and when  $V = \cos m\kappa\phi$ ,  $\pm\pi/2m, \pm3\pi/2m, \dots$ . To these correspond roots of the polynomial in  $a^2u$  between  $a^2\beta$  and  $a^2\gamma$ .

Thus a Lamé function of any order may be distinguished from others of the same order by the distribution of its zeros.\* Let  $C_m$ ,

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\* Klein, *Math. Ann.*, Vol. xviii., pp. 237-244.

$S_m$  denote those for which, in the limit when  $b = c$ ,  $V$  becomes  $\cos m\kappa\phi$ ,  $\sin m\kappa\phi$  respectively. Then the distribution is as follows:—

Zeros of	$n$ even				$n$ odd			
	$C_{2m-1}$	$C_{2m}$	$S_{2m-1}$	$S_{2m}$	$C_{2m-1}$	$C_{2m}$	$S_{2m-1}$	$S_{2m}$
At $a$ .....	1	0	1	0	0	1	0	1
From $a$ to $a+2\gamma^*$	$n-2m$	$n-2m$	$n-2m$	$n-2m$	$n-2m+1$	$n-2m-1$	$n-2m+1$	$n-2m-1$
At $a+\gamma$ .....	1	0	0	1	1	0	0	1
From $2a+\gamma$ to $\gamma^*$	$2m-2$	$2m$	$2m-2$	$2m-2$	$2m-2$	$2m$	$2m-2$	$2m-2$
At $\gamma$ .....	0	0	1	1	0	0	1	1

\* Exclusive of the half periods.

It follows that whenever  $fu$  vanishes  $a^2u$  is real. The classification might have been arrived at by considering the other extreme case,  $a = b$ , instead.

The process that we have used for finding the fourth row in the table amounts to this: that we write  $a^2 - b^2 \sin^2 \kappa\phi - c^2 \cos^2 \kappa\phi$  for  $a^2u$  in  $fu$ , divide out by as high a power as possible of  $b^2 - c^2$ , and then put  $b^2 = c^2$ . This reduces  $C_m$ ,  $S_m$  to multiples of  $\cos m\kappa\phi$ ,  $\sin m\kappa\phi$ ; so that their zeros are known.

7. In discussing expansions by means of Lamé's functions it seems necessary to consider another limiting case—namely, that in which  $a^2 - b^2$ ,  $b^2 - c^2$  are made infinitesimal in some given ratio. The ellipsoids of the confocal system thus become concentric spheres, and the hyperboloids confocal cones with vertex at the centre. The real argument  $u$  must be taken as infinitesimal, and, since in the limit it becomes  $r^{-1}$ , we are led to the following equations:—

$$x = r \frac{av \alpha w}{\alpha \beta \alpha \gamma}, \quad y = r \frac{bv \beta w}{\beta \gamma \beta \alpha}, \quad z = r \frac{cv \gamma w}{\gamma \alpha \gamma \beta}. \quad (23)$$

These only contain the ratios of  $a^2 - b^2$ ,  $b^2 - c^2$ ,  $a^2v$ ,  $a^2w$ , and so henceforth these quantities need not be supposed infinitesimal. Laplace's equation (11) becomes

$$(a^2v - a^2w) \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) - \frac{\partial^2 \phi}{\partial v^2} + \frac{\partial^2 \phi}{\partial w^2} = 0. \quad (24)$$

Any spherical surface harmonic of order  $n$  is a function of  $v, w$ , and the equation it satisfies is

$$\frac{\partial^2 \phi}{\partial v^2} - \frac{\partial^2 \phi}{\partial w^2} = n(n+1)(v^2 - w^2)\phi. \quad (25)$$

This is satisfied by the  $2n+1$  products  $f v f w$ ; so that these are surface harmonics. Let constant multiples of them be denoted by  $\theta_0, \theta_1, \theta_2, \dots, \theta_{2n}$ . The element of surface on the sphere is  $r^2 d\omega$ , and we proved above that

$$\int \theta_p \theta_q d\omega = 0 \quad (p \neq q). \quad (26)$$

In each of the functions  $\theta$  there is an arbitrary constant factor. Let this be so chosen that

$$\int \theta_p^2 d\omega = 8\pi/(2n+1) \quad (p = 0, 1, 2, \dots, 2n). \quad (27)$$

This would only be impossible if  $\int f^2 v f^2 w d\omega$  were zero, which cannot be, since the subject of integration is of constant sign. It follows that the harmonics  $\theta_0, \theta_1, \theta_2, \dots, \theta_{2n}$  are linearly independent; for, if  $\theta_{2n}$ , say, could be expressed in the form

$$a_0 \theta_0 + a_1 \theta_1 + \dots + a_{2n-1} \theta_{2n-1},$$

we should have, since

$$\int \theta_{2n} \theta_p d\omega = 0 \quad (p \neq 2n),$$

that

$$\int \theta_{2n}^2 d\omega = 0,$$

which is untrue.

8. A set of  $2n+1$  complete spherical surface harmonics,  $\sigma_0, \sigma_1, \dots, \sigma_{2n}$ , of order  $n$ , such that

$$\int \sigma_p \sigma_q d\omega = 0 \quad (p \neq q), \quad (28)$$

$$\int \sigma_p^2 d\omega = 8\pi/(2n+1), \quad (29)$$

has one or two important properties. Since the functions are linearly independent, any other complete harmonic of order  $n$  can be expressed as a linear function of them. Take, for instance,  $P_n(\cos PP')$ , where  $P$  is the current point,  $P'$  a fixed point on the sphere. Thus we have

$$P_n(\cos PP') = a_0 \sigma_0 + a_1 \sigma_1 + \dots,$$

where

$$8\pi a_p = (2n+1) \int P_n(\cos PP') \sigma_p d\omega,$$

as we find on multiplying both sides by  $\sigma_p$  and integrating. Now,

by a known theorem, since  $\sigma_p$  is a complete surface harmonic of order  $n$ , and the sphere is covered twice in the integration,

$$\int P_n(\cos PP') \sigma_p d\omega = 8\pi\sigma'_p/(2n+1),$$

where  $\sigma'_p$  denotes the value of  $\sigma_p$  at  $P'$ . Hence

$$\alpha_p = \sigma'_p,$$

and  $P_n(\cos PP') = \sigma_0\sigma'_0 + \sigma_1\sigma'_1 + \sigma_2\sigma'_2 + \dots$  to  $2n+1$  terms. (30)

Taking  $P'$  to coincide with  $P$ , we have

$$\sigma_0^2 + \sigma_1^2 + \sigma_2^2 + \dots + \sigma_{2n}^2 = 1. \quad (31)$$

We may call  $\sigma_0, \sigma_1, \dots, \sigma_{2n}$  a canonical set of surface harmonics of order  $n$ , and thus it follows from (31) that two canonical sets are connected by an orthogonal substitution. The converse also holds that a set derived from a canonical one by an orthogonal substitution is itself canonical.

Now the products  $\theta_0, \theta_1, \dots, \theta_{2n}$  form a canonical set, and thus it follows from (31) that no one of them can have a greater value than unity at any point of the surface.

9. We can now conveniently fix the constant factor in Lamé's function  $fu$ . The function will be taken positive for real values of  $u$  between 0 and  $\gamma$ . Then  $f^2vf^2v$  is negative or positive according as  $f\beta$  does or does not vanish. Let  $j$  stand for  $-1$  when  $f\beta = 0$ , and for  $+1$  when  $f\beta \neq 0$ ; then we shall suppose the constant factor\* so chosen that

$$\int f^2vf^2w d\omega = 8j\pi/(2n+1). \quad (32)$$

So then  $|f^2vf^2w|$  cannot exceed 1 at any point of the surface.

10. The chief problem to be solved is that of finding  $\phi$ , a function of  $x, y, z$ , without singularity within, or else without, a given ellipsoid, say  $u_0$ , which shall be a potential, that is, shall satisfy Laplace's equation  $\Delta\phi = 0$ , and which shall have assigned values on the surface of the ellipsoid  $u_0$ .

The first step is to express the assigned values by means of a series of the form  $\sum k f v f w$ . Let this value at a point  $(u_0, v, w)$  be denoted

\* Prof. G. H. Darwin (*Roy. Soc. Phil. Trans.*, Vol. cxvii.) has found another choice of this factor more convenient for purposes of computation. The factors could be compared by comparing the values of the integral in (32) in the two notations.

by  $F(v, w)$ ; then we may consider it as belonging to the corresponding point on the unit sphere, that is, to the point

$$\left( \frac{av}{a\beta} \frac{aw}{\alpha\gamma}, \frac{bv}{b\gamma} \frac{bw}{ba}, \frac{cv}{ca} \frac{cw}{c\beta} \right),$$

and expand it in a series of spherical surface harmonics

$$Y_0 + Y_1 + Y_2 + \dots + Y_n + \dots$$

The term  $Y_n$  can then be expressed linearly in terms of the harmonics  $\theta_0, \theta_1, \theta_2, \dots, \theta_{2n}$ , and thus we have the expansion desired.

For the present we shall assume that  $F(v, w)$  is an analytical function of the Cartesian coordinates over the whole surface of the ellipsoid; then the same will hold over the surface of the sphere, and the series  $\sum Y_n$  will be convergent by ratio; that is, the numerical value of  $Y_n$  will everywhere be less than  $pq^n$  where  $p, q$  are two positive quantities and  $q < 1$ . (See *Camb. Phil. Trans.*, Vol. xix., Part 2, pp. 205-7.)

$$\text{Let} \quad Y_n = k_0 \theta_0 + k_1 \theta_1 + \dots + k_{2n} \theta_{2n}.$$

$$\text{Then} \quad \int Y_n^2 d\omega = 8\pi (k_0^2 + k_1^2 + \dots + k_{2n}^2) / (2n+1).$$

$$\text{But} \quad \int d\omega = 8\pi \quad \text{and} \quad Y_n^2 < p^2 q^{2n};$$

$$\text{therefore} \quad k_0^2 + k_1^2 + \dots + k_{2n}^2 < (2n+1) p^2 q^{2n}.$$

Thus each of the coefficients  $k$  is numerically less than  $pq^n \sqrt{2n+1}$ , and, if we write  $\bar{Y}_n$  for  $\sum_{m=0}^{2n} |k_m \theta_m|$ , the series  $\bar{Y}_0 + \bar{Y}_1 + \dots$  is convergent by ratio, since  $\bar{Y}_n$  does not exceed  $pq^n (2n+1)^{\frac{1}{2}}$ .\* The series whose  $n$ -th term is  $\sum_{m=0}^{2n} |k_m|$  is also convergent by ratio.

Since  $fu fv fw$  is a potential, it is natural to suppose that the solution of our problem is given by the series  $\sum k fu fv fw / fu_0$ ; for this is a series of terms each of which is a potential without singularity except at infinity, and when  $u = u_0$  it takes the form  $\sum k fv fw$ , which is by hypothesis the expansion of  $F(v, w)$ . It is necessary, however, to prove (1) that the new series converges for all internal points, and that its sum is a potential; (2) that the limit of the sum when  $u$  approaches  $u_0$  is the sum of the series  $\sum k fv fw$ .

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\* In using a result of this kind we may drop such a factor as  $(2n+1)^{\frac{1}{2}}$  on condition of supposing  $q$  slightly increased, and possibly  $p$  increased also.

The expression not being a power series, the theorem of Abel's that is generally applied cannot be used to prove (2).

11. In dealing with series of this type we shall take the  $(n+1)$ -th term to mean the sum of the  $2n+1$  partial terms of order  $n$ . The products  $fvfw$  are all numerically  $\geq 1$ , and they may therefore be neglected in discussing absolute convergency. We need to prove that, if the series  $\sum |kf u_0|$  is convergent by ratio, the same is true for each of the series  $\sum |kfu|$ ,  $\sum |kf'u|$  for a continuous range of real values of  $u$ , including  $u_0$  and  $\gamma$ .

More generally, if  $\lambda u$  is a function of  $u$  satisfying one of Lamé's equations

$$\lambda''u = \{n(n+1)a^2u + B\}\lambda u,$$

and such that  $\lambda u_0$  or  $\lambda'u_0$  vanishes, where  $u_0$  is a fixed real quantity between 0 and  $2\gamma$  inclusive, and if either of the series  $\sum |k\lambda u_1|$ ,  $\sum |k\lambda'u_1|$  is convergent by ratio,  $u_1$  being a quantity between 0 and  $2\gamma$  exclusive, then the two series  $\sum |k\lambda u|$ ,  $\sum |k\lambda'u|$  converge by ratio for a continuous range of values of  $u$ , including both  $u_0$  and  $u_1$ .

Let  $\mu(\theta)$  denote  $\lambda u$  where  $u = u_0 + \theta(u_1 - u_0)$ , and take  $\theta$  in the first place to be between 0 and 1. Then

$$\mu''(\theta) = (u_1 - u_0)^2 \{n(n+1)a^2u + B\} \mu(\theta).$$

Thus  $\mu''(\theta)$ ,  $\mu(\theta)$  have the same sign, and the curve  $y = \mu(x)$  is always convex to the axis of  $x$ . We may, without loss of generality, take  $\mu(\theta)$  and  $k$  to be positive. As  $\theta$  increases  $\mu'(\theta)$  and  $\mu(\theta)$  increase.

Now first let  $\sum k\mu(1)$  be convergent by ratio. Since  $\mu(\theta) < \mu(1)$ , the series  $\sum k\mu(\theta)$  is also convergent by ratio.

$$\text{Also} \quad \mu(1) - \mu(\theta) = \int_{\theta}^1 \mu'(\theta) d\theta > (1-\theta)\mu'(\theta),$$

since  $\mu'(\theta)$  increases with  $\theta$ . The series  $\sum k\{\mu(1) - \mu(\theta)\}$  is convergent by ratio; so therefore is  $\sum k\mu'(\theta)$ .

$$\text{Also} \quad \mu''(\theta)/\mu(\theta) < (u_1 - u_0)^2 n(n+1)a^2,$$

and thus  $\sum k\mu''(\theta)$  also converges by ratio for values of  $\theta$  up to 1 inclusive. Integrating this from  $\theta$  to 1, we find that

$$\sum k\{\mu'(1) - \mu'(\theta)\}$$

is convergent by ratio; and so therefore is  $\sum k\mu'(1)$  since  $\sum k\mu'(\theta)$  has been proved to be.

Again, let  $\Sigma k\mu'(1)$  be supposed convergent by ratio. The same is true of  $\Sigma k\mu'(\theta)$ , which is less term by term. Let  $\theta'$  be a quantity between  $\theta$  and 1. Then

$$\begin{aligned}\mu'(\theta') - \mu'(\theta) &= (u_1 - u_0) \int_{u_0 + \theta(u_1 - u_0)}^{u_0 + \theta'(u_1 - u_0)} \{n(n+1)a^2u + B\} \lambda u \, du \\ &> n(n+1)(u_1 - u_0) \mu(\theta) \int_{u_0 + \theta(u_1 - u_0)}^{u_0 + \theta'(u_1 - u_0)} c^2 u \, du,\end{aligned}$$

since  $B > -n(n+1)a^2\gamma$  and  $\lambda u > \mu(\theta)$  throughout the range. Now  $\Sigma k\{\mu'(\theta') - \mu'(\theta)\}$  is convergent by ratio; so then is  $\Sigma k\mu(\theta)$ , and, since

$$\mu(1) - \mu(\theta) = \int_{\theta}^1 \mu'(\theta) \, d\theta,$$

so is  $\Sigma k\mu(1)$ .

12. The theorems are then proved for values of  $\theta$  between 0 and 1 inclusive. We are to show that they hold for a wider range of values. Take  $\theta > 1$ .

Let  $u_3, u_1, u_0$  be in order of magnitude, and take a quantity  $\eta$  that is greater than  $au$  for any value of  $u$  between  $u_3$  and  $u_0$ . Let  $\nu(\theta)$  stand for

$$\begin{aligned}\mu(1) \cosh \{(n+1)\eta(u_1 - u_0)(\theta - 1)\} \\ + \frac{\mu'(1)}{(n+1)\eta(u_1 - u_0)} \sinh \{(n+1)\eta(u_1 - u_0)(\theta - 1)\}.\end{aligned}$$

Then  $\nu''(\theta) = (n+1)^2 \eta^2 (u_1 - u_0)^2 \nu(\theta)$ ,

while  $\mu''(\theta) = [n(n+1)a^2u + B](u_1 - u_0)^2 \mu(\theta)$ ;

thus  $\mu(\theta)\nu''(\theta) - \mu''(\theta)\nu(\theta) = \mu(\theta)\nu(\theta) \times \text{a positive quantity,}$

so long as  $u$  is between  $u_2, u_3$ .

Also, when  $\theta > 1$ ,  $\nu(\theta)$  is positive and increases with  $\theta$ . By integration we have then

$$\begin{aligned}\{\mu(\theta)\nu'(\theta) - \mu'(\theta)\nu(\theta)\} - \{\mu(1)\nu'(1) - \mu'(1)\nu(1)\} \\ = \text{a positive quantity.}\end{aligned}$$

Now  $\nu(1) = \mu(1)$ ,  $\nu'(1) = \mu'(1)$ ;

hence  $\mu(\theta)\nu'(\theta) - \mu'(\theta)\nu(\theta)$  is positive, and  $\nu(\theta)/\mu(\theta)$  increases with  $\theta$ .

Thus  $\mu(\theta) < \nu(\theta)$ , since we are taking  $\theta < 1$ . Now  $\mu(1), \mu'(1)$  are both  $< pq^n$  where  $p, q$  are independent of  $n$  and of  $q < 1$ . If we take

$\theta > 1$ , but  $< 1 - (\log q)/\eta$  ( $u_1 \sim u_0$ ), the series  $\Sigma k\nu(\theta)$  will be convergent by ratio, and therefore also the series  $\Sigma k\mu(\theta)$ ,  $\Sigma k\mu'(\theta)$ . In the same way these series converge by ratio for negative values of  $\theta$  that are not too great. The proof does not apply to this case when  $u_0$  is 0 or  $2\gamma$ ; but, since  $\lambda u$  is then an even or odd function of  $u$  or  $u - 2\gamma$ , as the case may be, the theorem is otherwise evident.

13. In the present case  $u_0 = \gamma$  and  $\lambda u$  is  $fu$ ; the coefficient that has been called  $k$  in §§ 11, 12 will now be called  $k/fu_0$ ,  $u_0$  being put for  $u_1$ .

Now in Green's equation

$$\int \left( \phi \frac{\partial \phi_1}{\partial r} - \phi_1 \frac{\partial \phi}{\partial r} \right) dS = \iiint (\phi \Delta \phi_1 - \phi_1 \Delta \phi) dx dy dz \quad (33)$$

let

$$\phi = fu fv fw,$$

and let  $\phi_1$  be the reciprocal of the distance of  $P$  or  $(u, v, w)$  from an internal point  $(u', v', w')$ , the integration being over the surface of the ellipsoid  $u$  twice. We thus find

$$\int \left( fu \frac{\partial}{\partial u} \frac{1}{PP'} - f'u \frac{1}{PP'} \right) fv fw d\omega = 8\pi fu' fv' fw'. \quad (34)$$

Multiply this by  $k/fu$  and sum, changing the notation to agree with that used in § 10; thus

$$\Sigma k fu fv fw / fu_0 = \frac{1}{8\pi} \int \left( \Sigma k fv_0 fw_0 \frac{\partial}{\partial u_0} \frac{1}{PP_0} - \Sigma k \frac{f'u_0}{fu_0} fv_0 fw_0 \frac{1}{PP_0} \right) d\omega_0, \quad (35)$$

an expression which from its form represents a potential for the point  $P$ , having no singularity within the ellipsoid  $u_0$ .

We conclude that, if, in the series  $\Sigma k fu fv fw / fu_0$ ,  $|k|$  does not exceed  $pq^n$ , where  $p, q$  are positive,  $q < 1$ , and  $n$  is the order of Lamé's function  $fu$ , then the series is convergent when  $\gamma \geq u > u_0$ , and represents a potential without singularity in the space enclosed by the ellipsoid  $u_0$ .\*

The series is also convergent by ratio in a region outside the ellipsoid  $u_0$ . Take an ellipsoid  $u'_0$  in this region; in (34) write

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\* [The integration on the right-hand side of (35) gives a definite result, since the infinite series in the subject of integration are uniformly convergent over the surface, and their sums are therefore continuous.—December 16th, 1902.]

$u, v, w$  for  $u', v', w'$ , and  $u'_0$  for  $u$ , multiply by  $k/fu_0$ , and sum. Thus

$$\Sigma kfu fv fw/fu_0 = \frac{1}{8\pi} \int \left( \Sigma k \frac{fu'_0}{fu_0} fv'_0 fw'_0 \frac{\partial}{\partial u'_0} \frac{1}{PP'_0} - \Sigma k \frac{fu'_0}{fu_0} fv'_0 fw'_0 \frac{1}{PP'_0} \right) d\omega'_0,$$

which expression represents a potential without singularity within the ellipsoid  $u'_0$ .

Hence the series  $\Sigma kfu fv fw/fu_0$  is valid, and represents a potential without singularity, throughout the volume of an ellipsoid of the confocal system outside the ellipsoid  $u_0$ , and the surface  $u_0$  is included in its region of continuity. The value of this potential when  $u = u_0$  is then  $\Sigma kfv fw$ ; that is, it fulfils the assigned boundary conditions, and the problem is completely solved.

14. The value of  $k$  is easily found as a definite integral. Multiply the equation

$$F(v, w) = \Sigma kfv fw$$

by  $fv fw d\omega$  and integrate with the usual limits; thus

$$8\pi jk = (2n+1) \int F(v, w) fv fw d\omega. \quad (36)$$

The expansion is then

$$8\pi F(v', w') = \Sigma (2n+1) jfv' fw' \int F(v, w) fv fw d\omega. \quad (37)$$

This has been proved when  $F(v, w)$  is an analytical function of  $x, y, z$  all over the surface; the difficulty in extending the solution to other cases is that the series  $\Sigma (2n+1) jfv' fw' fv fw$  is not convergent.

15. With a view to this extension we shall next investigate Green's function for the ellipsoid; that is, the potential that vanishes over the surface  $u_0$  and has no singularity but that of  $1/PP_1$  within this surface,  $P$  being the current point and  $P_1$  a fixed internal point. To do this we must find the expansion of  $1/PP_1$ . Let  $(u_1, v_1, w_1)$  be the coordinates of  $P_1$ , and suppose the expansion

$$1/PP_1 = \Sigma Afv fw$$

to hold when  $P$  lies on the ellipsoid  $u_0$ . It follows that

$$8\pi A = (2n+1) j \int \frac{1}{P_0 P_1} fv_0 fw_0 d\omega_0. \quad (38)$$

$A$  is a function of  $u_0, u_1, v_1, w_1$ , and thus

$$8\pi \frac{\partial A}{\partial u_0} = (2n+1)j \int \frac{\partial}{\partial u_0} \frac{1}{P_0 P_1} f v_0 f w_0 d\omega_0. \quad (39)$$

We have then

$$\begin{aligned} 8\pi \left( \frac{\partial A}{\partial u_0} f u_0 - A f' u_0 \right) &= (2n+1)j \int \left( f u_0 \frac{\partial}{\partial u_0} \frac{1}{P_0 P_1} - f' u_0 \frac{1}{P_0 P_1} \right) f v_0 f w_0 d\omega_0 \\ &= (2n+1)j \cdot 8\pi f u_1 f v_1 f w_1 \quad [\text{by (34)}]. \end{aligned}$$

Hence 
$$A = j \cdot g u_0 f u_1 f v_1 f w_1, \quad (40)$$

where  $g u$  is a function of  $u$  such that

$$g' u f u - g u f' u = 2n+1, \quad (41)$$

and  $g0 = 0$ , since when  $P_0$  is at infinity  $A = 0$ . From (41), by differentiation

$$g'' u f u - g u f'' u = 0;$$

so that

$$g'' u / g u = f'' u / f u,$$

and  $g u$  is the second solution of Lamé's equation (13). Its value is  $(2n+1) f u \int_0^u \frac{du}{f^2 u}$ , and when  $u$  is small it is approximately a constant multiple of  $u^{n+1}$ . Thus when  $u < u_1$  we have

$$1/PP_1 = \Sigma j g u f v f w f u_1 f v_1 f w_1. \quad (42)$$

Similarly,  $\frac{\partial}{\partial u_0} \frac{1}{P_0 P_1}$  may be expanded, and we find from (39) that

$$\begin{aligned} \frac{\partial A}{\partial u_0} &\text{ is the coefficient of } f v_0 f w_0 \text{ in its expansion. Thus} \\ \frac{\partial}{\partial u} \frac{1}{P P_1} &= \Sigma j g' u f v f w f u_1 f v_1 f w_1. \end{aligned} \quad (43)$$

The higher derivatives and those with respect to  $u_1$  may be found similarly.

Now let  $\varpi(P, P_1)$  denote  $\Sigma j g u_0 f u f v f w f u_1 f v_1 f w_1 / f u_0$ . This series is valid so long as  $P, P_1$  are both within the ellipsoid  $u_0$ , and also when one lies on the surface and the other within. It is a potential without singularity for either point, and when one point lies on the surface its value is  $1/PP_1$ . Hence  $1/PP_1 - \varpi(P, P_1)$  is Green's function, say,  $G(P, P_1)$ , and, substituting it for  $\phi_1$  in (33), we have the value of a potential  $\phi$  at an internal point  $P_1$  given in terms

of its values on the surface by the equation

$$8\pi\phi(P_1) = \int \phi(P) \frac{\partial}{\partial u} G(P, P_1) d\omega, \quad (44)$$

which agrees with our former result, since

$$\frac{\partial}{\partial u} G(P, P_1) = \Sigma i \left( g'u - \frac{qu_0 f'u}{fu_0} \right) fvfwf u_1 f v_1 f w_1,$$

which becomes  $\Sigma(2n+1)jfvf w f u_1 f v_1 f w_1 / f u_0$  when  $u = u_0$ .

16. When  $u_1 = u_0$ ,  $G(P, P_1) = 0$ ,

and therefore  $\frac{\partial}{\partial u} G(P, P_1) = 0$ .

Our next step is to prove that the last result holds in the limit when  $u = u_0$  and  $u_1$  approaches the limit  $u_0$ . Let  $Q, Q_1$  denote the limiting positions of  $P, P_1$ , that is, the points  $(u_0, v, w)$   $(u_0, v_1, w_1)$ , and  $R_1$  the image of  $P_1$  in the tangent plane at  $Q_1$ . Then  $PP_1 < PR_1$  at every point of the surface except  $Q_1$ , and thus  $1/PP_1 - 1/PR_1 - G(P, P_1)$  is a potential without singularity throughout the region and positive throughout, since it is positive on the boundary; that is,

$$G(P, P_1) < 1/PP_1 - 1/PR_1.$$

Now let a sphere be described about  $Q$ , of radius  $r$  less than  $Q/Q_1$ ; let  $S$  be the region common to this sphere and the given ellipsoid,  $S_1$  the hemisphere cut off by the tangent plane at  $Q$  and including  $S$ ,  $S_2$  the rest of the sphere. Let  $\lambda$  be the potential which has no singularity within this sphere and is equal to  $-1$  over the curved surface of  $S_2$ ,  $+1$  on that of  $S_1$ . On the plane dividing  $S_1$  from  $S_2$ ,  $\lambda = 0$ , and, at  $Q$ ,  $\frac{\partial \lambda}{\partial \nu} = -\frac{3}{2r}$ ,  $d\nu$  being an element of the outward

normal. Let  $\kappa$  be the greatest value of  $G(P, P_1)$  on the spherical boundary of  $S$ . Then, on the whole boundary of  $S$ ,  $\kappa\lambda - G(P, P_1)$  is positive; this expression is also a potential without singularity within  $S$  and vanishing at  $Q$ . Hence it is positive throughout, and

must increase as  $P$  travels inwards from  $Q$ ; that is,  $\frac{\partial}{\partial \nu} \{ \kappa\lambda - G(P, P_1) \}$

is negative at  $Q$  or  $\left[ -\frac{\partial}{\partial \nu} G(P, P_1) \right]_{u=u_0} < 3\kappa/2r$ , where  $\kappa$  is less than the greatest value taken by  $1/PP_1 - 1/PR_1$  on the spherical boundary of  $S$ .

By taking  $u_1 - u_0$  small enough, we can make  $\kappa$  as small as we please, and thus the limit when  $u_1$  approaches  $u_0$  of  $\left[ \frac{\partial}{\partial u} G(P, P_1) \right]_{u=u_0}$  is zero, unless  $r = 0$ , that is, unless  $Q, Q_1$  coincide. This is the result desired.\* We may write it

$$\lim_{u \rightarrow u_0} \Sigma (2n+1) j f u f v f w f v_0 f w_0 / f u_0 = 0, \quad (45)$$

which holds unless  $v = v_0, w = w_0$ .

17. Now let  $\psi(u, v, w, u_0, v_0, w_0)$  stand for

$$\Sigma (2n+1) j f u f v f w f v_0 f w_0 / f u_0.$$

We have the following facts:—

- (a)  $\int \psi(u, v, w, u_0, v_0, w_0) d\omega_0 = 8\pi,$
- (b)  $\lim_{u \rightarrow u_0} \psi(u, v, w, u_0, v_0, w_0) = 0,$  unless  $v = v_0, w = w_0,$
- (c)  $\psi$  is always positive when  $u > u_0$  and  $\nabla \gamma$ ;

for  $G(P, P_1)$  can have no minimum within the ellipsoid, and cannot therefore be negative; it must then increase when  $P$  or  $P_1$  moves inwards from the surface.

Let  $\phi$  stand for  $\frac{1}{8\pi} \int F(v_0, w_0) \psi(u, v, w, u_0, v_0, w_0) d\omega_0$ ,  $F$  being one-valued on the surface, but otherwise as little restricted as possible. We have

$$\phi - F(v, w) = \frac{1}{8\pi} \int \{ F(v_0, w_0) - F(v, w) \} \psi(u, v, w, u_0, v_0, w_0) d\omega_0.$$

Suppose that when  $v \sim v_0$  and  $w \sim w_0$  do not exceed  $\epsilon$ , then  $F(v_0, w_0) \sim F(v, w)$  does not exceed  $\eta$ ; then the part of this integral contributed by values of  $v_0, w_0$  between  $v \pm \epsilon, w \pm \epsilon$  and by values between  $2\beta - v \pm \epsilon, 2\beta - w \pm \epsilon$  lies between

$$\pm \eta \frac{1}{8\pi} \int \psi(u, v, w, u_0, v_0, w_0) d\omega_0,$$

since  $\psi$  is always positive; that is, it lies between  $\pm \eta$ . The rest of

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\* The method here used applies to any ordinary kind of bounding surface unless the tangent plane meets the surface again. In such a case we only need to describe a sphere touching the surface externally and not meeting it in any other point, and then to invert from a point on the surface of this sphere.

the integral vanishes in the limit when  $u = u_0$ , since  $\psi = 0$  in the limit by (45). Hence, if  $F$  is continuous near  $v, w$ , so that  $\eta$  can be made as small as we please by diminishing  $\epsilon$ , we have

$$\lim_{u \rightarrow u_0} \phi = F(v, w).$$

This result is not affected by a discontinuity in  $F$  elsewhere on the surface. Hence the following theorem:—If  $F(v, w)$  is a one-valued function of position on the surface  $u_0$ , and is such that the integral  $\int F(v_0, w_0) \psi(u, v, w, u_0, v_0, w_0) d\omega_0$  has a definite value  $\phi(u, v, w)$  and may be differentiated twice by the ordinary rule, then  $\phi$  is a potential without singularity within the ellipsoid  $u_0$ , and when  $u$  approaches  $u_0$  the limiting value of  $\phi$  is  $F(v, w)$ , provided that  $F(v, w)$  is continuous in the neighbourhood of the point  $(v, w)$  on the surface  $u_0$ .

[It is clearly necessary that the boundary values  $F(v, w)$  should be integrable over the surface, and this condition is also sufficient to ensure the validity of the solution of the problem.

For let the whole surface be divided into small elements of which  $\sigma$  is any one, and let  $\epsilon$  be the difference between the greatest and least values of  $F$  at the points of  $\sigma$ , and  $\eta$  the difference between the greatest and least values of another function  $G(v, w)$  at the points of  $\sigma$ ; let  $M, N$  be the superior limits of  $\pm F, \pm G$  over the whole surface. Then the greatest and least values of the product  $FG$ , upon  $\sigma$ , differ by less than  $M\eta + N\epsilon$ .

Now, if  $F, G$  are integrable separately,  $\Sigma \epsilon \sigma$  and  $\Sigma \eta \sigma$  diminish indefinitely when all the elements  $\sigma$  do so, and therefore  $\Sigma (M\eta + N\epsilon) \sigma$  diminishes indefinitely, and the product  $FG$  is also integrable.\* If  $G$  is everywhere infinitesimal, the integral of  $FG$  will also be infinitesimal.

From these results it follows that when  $G$  is, as here, an analytical function of  $u, v, w, v_0, w_0$  and  $F$  is an integrable function of  $v_0, w_0$ , finite and one-valued everywhere, the function represented by  $\iint FG dv_0 dw_0$  has a definite meaning, and can be differentiated any number of times with respect to  $u, v, w$  by the ordinary rule, that is, by differentiating the factor  $G$  in the subject.—*Feb. 20th, 1903.*]

The expansion of  $\phi$  in Lamé's functions is convergent by ratio within the ellipsoid; but the question whether this expansion is valid on the surface is not affected; this question amounts to

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\* Compare du Bois Reymond, *Math. Ann.*, Vol. xx., pp. 122–24.

exactly the same as the corresponding one in the theory of spherical harmonics, and need not be treated here.

18. In order to prove that

$$\int \{F(v_0, w_0) - F(v, w)\} \psi(u, v, w, u_0, v_0, w_0) d\omega_0$$

was zero in the limit if  $(v, w)$  was outside the limits of the integration, we proved that the limit of  $\psi$  was zero when  $u$  approached  $u_0$ . A somewhat different proof may be given.

Let a sphere  $A$  be described touching the boundary at  $(u_0, v, w)$  and having no other point in common with it. Let a sphere  $B$  be described concentric with  $A$  and lying outside it, but not containing any of the points over which the integration extends. Let  $Q$  be any point on the sphere  $B$ , and let us write  $A(P)$  for the square of the tangent from  $P$  to  $A$ . Consider the potential without singularity within or on the boundary, whose value on the boundary is  $KA(P)/A(Q)$ , where  $K$  is the greatest numerical value of  $F(v_0, w_0) - F(v, w)$ . This potential at  $(u, v, w)$  is  $\frac{1}{8\pi}$  of

$$\int KA(P_0)/A(Q) \cdot \psi(u, v, w, u_0, v_0, w_0) d\omega_0;$$

every element in the integral is positive; and when  $P_0$  is outside the sphere  $B$  the element at  $P_0$  is numerically greater than the corresponding element in

$$\int \{F(v_0, w_0) - F(v, w)\} \psi(u, v, w, u_0, v_0, w_0) d\omega_0.$$

But, since  $KA(P)/A(Q)$  is analytical on the surface and  $(u_0, v, w)$  lies on the sphere  $A$ , the former integral tends to the limit zero when  $u$  approaches  $u_0$ ; the latter integral over the smaller range must then also vanish in the limit, since it is numerically less, element by element.

19. If a one-valued potential is to be constructed with given boundary values and given singularities within the region, it is possible, by differentiating or integrating Green's function, to form a potential vanishing along the boundary and having these singularities if they are such as a one-valued potential can have; adding to this a potential without singularities and with the given boundary values, we have the required result.

20. A potential having given values on the ellipsoid  $u_0$  and given singularities in the region outside may be found in like manner.

The typical term of the expansions is a multiple of  $gu fv fw$ . Since  $gu$  vanishes with  $u$ , we may use the results of §§ 11, 12, putting 0 for  $u_0$  and  $gu$  for  $\lambda u$ .

If the boundary values are denoted by  $F(v, w)$ , and there are no singularities, and the potential vanishes at infinity, its value is

$$\frac{1}{8\pi} \int F(v_0, w_0) \{ \Sigma j (2n+1) gu fv fw fv_0 fw_0 / gu_0 \} d\omega_0.$$

Green's function  $G(P, P_1)$  is  $1/PP_1 - \Sigma j fu_0 gu fv fw gu_1 fv_1 fw_1 / gu_0$ . The potential without singularities that has the value 1 at infinity and 0 at the bounding surface is  $1 - u/u_0$ .

21. Lamé's functions may also be used to solve the same problem for the region bounded by two confocal ellipsoids. Before discussing this I shall indicate a remarkable extension of which it is capable, and which may be illustrated by reference to an analogous problem in the theory of the logarithmic potential. Take a two-sheeted Riemann surface with two branch points. With the branch points as foci describe an ellipse in each sheet, and consider the boundary value problem (*Randwerthausgabe*) for the part of the surface bounded by these ellipses. Taking the axes of the ellipses as axes of coordinates, we may put

$$x = c \cosh u \cos v, \quad y = c \sinh u \sin v;$$

so that the curves  $u = \text{const.}$  are ellipses of the confocal system and the curves  $v = \text{const.}$  hyperbolas of the same system. The points  $(u, v)$  and  $(-u, -v)$  are the same in position, but belong to different sheets.

The functions corresponding to Lamé's are  $\cosh nu, \sinh nu,^*$   $\cos nv, \sin nv$ , and the problem may be solved by using expansions in multiples of  $\cosh nu \cos nv, \sinh nu \sin nv, \cosh nu \sin nv, \sinh nu \cos nv$ . In the simpler case when the potential is to be uniform over the area of an ellipse and there is only one sheet the products  $\cosh nu \sin nv$  and  $\sinh nu \cos nv$  do not come in, since they are not one-valued, but have opposite signs at  $(u, v), (-u, -v)$ ; but all four products are to be used in the case of the two-sheeted surface.

22. In dealing with space of three dimensions we shall speak of a change of colour instead of a change of sheet, and conceive of two

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\* Instead of  $2n+1$  of the order  $n$  there are only two; whether we take  $\cosh nu$  or  $\sinh nu$  to correspond to  $fu$ , we have  $e^{-nu}$  as corresponding to  $gu$ .

infinite spaces, red and blue, coextensive and connected by one or more doors which will be surfaces bounded by curves. In the present case there will be one such door—namely, the plane area that is bounded by the focal ellipse, and a point passing through this door in either direction changes its colour. The two points  $(u, v, w)$ ,  $(2\gamma - u, 2\gamma - v, w)$  are the same in position, but differ in colour; for we may pass from the first to the second by travelling along the intersection of the hyperboloids  $v, w$  to the point  $(2\gamma - u, v, w)$ , passing through the door at  $(\gamma, v, w)$ , and then along the intersection of the surfaces  $2\gamma - u, w$  to  $(2\gamma - u, 2\gamma - v, w)$ , passing round the edge of the door.

Let two ellipsoids of the system be described, one in each colour; then the problem is to find a potential for the space they enclose, having given boundary values and no singularity within the space except the branching: the potential will be an analytical function of  $x, y, z$  at all internal points except those on the focal ellipse, and even there it will be an analytical function of  $u, v, w$ , symmetrical as regards  $v, w$ .

23. Take the two bounding ellipsoids to be  $u_0, u_1$  where

$$0 < u_0 < \gamma < u_1 < 2\gamma.$$

The discussion will, for the most part, apply to the case of an ordinary shell where  $u_0, u_1$  are both between 0 and  $\gamma$ .

Let  $F_0, F_1$  denote the assigned values on the surfaces  $u_0, u_1$  respectively, and assume for the solution the series  $\Sigma (kfu + lgu)fvfw$ . Of the two potentials  $fu fv fw, gu fv fw$ , the former is unaffected by a change of colour, but the latter is not; so that the whole expression is two-valued in ordinary space, but one-valued in the space we are considering.

The values that must be given to  $k, l$  are easily found by substituting  $u_0, u_1$  for  $u$  and integrating over the two ellipsoids respectively. Thus

$$8\pi (kfu_0 + lgu_0) = (2n+1) \int \int F_0 fv fw d\omega,$$

$$8\pi (kfu_1 + lgu_1) = (2n+1) \int \int F_1 fv fw d\omega.$$

Let us first assume that  $F_0 = 0$  and  $F_1$  is an analytical function of  $x, y, z$  everywhere. Thus

$$kfu_0 + lgu_0 = 0,$$

and the series may be written  $\Sigma k \lambda u f v f w$ , where  $\lambda u$  stands for  $f u - f u_0 g u / g u_0$ , the solution of Lamé's equation which vanishes when  $u = u_0$ . Also the series  $\Sigma |k \lambda u_i|$  is convergent by ratio when the  $2n+1$  terms of the same order  $n$  are grouped together to make one term. The series  $\Sigma k \lambda u f v f w$ ,  $\Sigma k \lambda' u f v f w$ ,  $\Sigma k \lambda'' u f v f w$  are therefore all convergent by ratio (§§ 11, 12). Thus, integrating this last series twice between  $u$  and  $u_1$ , we find

$$\frac{\partial}{\partial u} \Sigma k \lambda u f v f w = \Sigma k \lambda' u f v f w,$$

$$\frac{\partial^2}{\partial u^2} \Sigma k \lambda u f v f w = \Sigma k \lambda'' u f v f w.$$

Again, the series  $\Sigma k \lambda u f'' v f w$  and  $\Sigma k \lambda u f v f'' w$  are also convergent by ratio, since  $f'' v / f v$  and  $f'' w / f w$  lie between  $\pm n(n+1)(a^2 - c^2)$ . Integrate the first twice between the limits  $v$ ,  $v_0$  where  $v_0$  is an arbitrary argument between  $\beta \pm 2\alpha$ . The result is

$$\Sigma k \lambda u f v \{f v - f v_0 - (v - v_0) f' v_0\},$$

which is therefore a series convergent by ratio. The series  $\Sigma k \lambda u f v f w$  and  $\Sigma k \lambda u f v_0 f w$  converge separately by ratio, and therefore the same is true of  $(v - v_0) \Sigma k \lambda u f' v_0 f w$  and of  $\Sigma k \lambda u f' v f w$ , since  $v_0$  is as arbitrary as  $v$ . Thus

$$\frac{\partial}{\partial v} \Sigma k \lambda u f v f w = \Sigma k \lambda u f' v f w$$

and

$$\frac{\partial^2}{\partial v^2} \Sigma k \lambda u f v f w = \Sigma k \lambda u f'' v f w.$$

Similarly for the derivatives with respect to  $w$ . Hence  $\Sigma k \lambda u f v f w$  satisfies the equation (12), that is, it is a potential. Its value when  $u = u_0$  is zero, and, since we may write it

$$\int_{u_0}^u \Sigma k \lambda' u f v f w du,$$

it varies continuously with  $u$  from  $u = u_0$  to  $u = u_1$  inclusive. The limit it approaches when  $u$  approaches  $u_1$  is therefore  $\Sigma k \lambda u_1 f v f w$  or  $F_1$ . It can be proved as in the former case that neither of the original bounding surfaces  $u_0$ ,  $u_1$  is a true boundary of the region within which this potential exists and is validly represented by the series, having no singularity except the branching.

Let  $\phi_1$  denote this potential; in like manner a potential  $\phi_0$  could

be found having the value zero on the surface  $u_1$  and  $F_0$ , if this is analytical, on the surface  $u_0$ , and with no singularity except the branching. Then  $\phi_1 + \phi_0$  solves the original problem in which boundary values  $F_0, F_1$ , other than zero, were given on both surfaces.

24. To complete the discussion we must find Green's function, whose only singularity, besides the branching, which will not be counted a singularity in future, is that of  $1/PP_2$  at the point  $P_2$  of a specified colour, and which vanishes over the whole boundary.

The expression for this will vary according to the relative magnitudes of  $u, u_0, u_1, u_2$ . Suppose first that  $0 < u_0 < u_2 < 2\gamma - u_1 < \gamma$ . Then the point  $P_2$  of the other colour is outside the region, and all we need to do is to find the coefficients  $k, l$ , so that  $\Sigma (kfu + lgu)fvfw$  shall have the value  $1/PP_2$  when  $u = u_0$  or  $u_1$ . This gives

$$kfu_0 + lgu_0 = jgu_0fu_2fv_2fw_2,$$

$$kfu_1 + lgu_1 = jgu_2fu_1fv_2fw_2,$$

and Green's function is  $\frac{1}{PP_2} - \Sigma (kfu + lgu)fvfw$  or

$$\begin{aligned} \frac{1}{PP_2} - \Sigma jfvfwfv_2fw_2\{fu_1gu(fu_0gu_2 - fu_2gu_0) \\ + gu_0fu(fu_2gu_1 - fu_1gu_2)\} \div (fu_0gu_1 - fu_1gu_0). \end{aligned} \quad (47)$$

When  $u_0 > 2\gamma - u_1$  and  $2\gamma - u_2$  lies between them the expression for Green's function is derived from (47) by putting  $2\gamma - u_1, 2\gamma - u_0, 2\gamma - u_2, 2\gamma - u$  for  $u_0, u_1, u_2, u$  respectively, and using the relations

$$f(2\gamma - u) = \pm fu,$$

$$g(2\gamma - u) = \mp (gu - e fu),$$

where  $e = (2n + 1) \int_0^{2\gamma} du/f^2u$ .

The value of  $e$  is finite and unambiguous even if  $f\gamma = 0$ .

25. When  $u_2$  is between  $u_1$  and  $2\gamma - u_1$ , the matter is not so simple; for  $1/PP_2$  has an infinity at each of the points  $P_2$ , and both lie within the region. We need to prove the existence of a potential with only one of these singularities. This can be done by means of the alternating method of combining volumes, adapted so as to avoid the usual infinite series of operations.

Suppose, to fix the ideas, that  $0 < u_0 \leq 2\gamma - u_1 < u_2 < \gamma < u_1$ .

Take  $u_4$  between  $u_3$  and  $2\gamma - u_3$ ,  $u_3$  between  $u_2$  and  $u_4$ . Then in  $(u_0, u_4)$ , that is, the region bounded by the ellipsoids  $u_0, u_4$ , the expression

$$\phi_1(P) \equiv \frac{1}{PP_2} - \Sigma jfvfwfv_2fv_3 \{ gu gu_2 fu_0 fu_4 - (fu gu_2 + fu_2 gu) gu_0 fu_4 + fu fu_2 gu_0 gu_4 \} \div (fu_0 gu_4 - fu_4 gu_0)$$

represents a potential vanishing on the boundary and having only the singularity of its first term; for this is a case in which (47) may be used.

The potential without singularity in the region  $(u_3, u_1)$  vanishing at the boundary  $u_1$  and having the same value as  $\phi_1(P)$  at the boundary  $u_3$  is

$$\phi_2(P) \equiv \Sigma jfvfwfv_2fv_3 \frac{fu gu_1 - fu_1 gu}{fu_3 gu_1 - fu_1 gu_3} \times \left[ gu_2 fu_3 - gu_3 gu_2 fu_0 fu_4 - \frac{(fu_3 gu_2 + fu_2 gu_3) gu_0 fu_4 + fu_2 fu_3 gu_0 gu_4}{fu_0 gu_4 - fu_4 gu_0} \right],$$

the first term in the bracket arising from the known expansion of  $1/PP_2$ , since  $u_3 > u_2$ .

On the surface  $u_4$  we have then, after reduction,

$$\phi_2(P_4) - \phi_1(P_4) = \Sigma jfv_4fw_4fv_2fv_3 \frac{(fu_4 gu_1 - fu_1 gu_4)(fu_0 gu_2 - fu_2 gu_0)(fu_3 gu_4 - fu_4 gu_3)}{(fu_3 gu_1 - fu_1 gu_3)(fu_0 gu_4 - fu_4 gu_0)}.$$

Let us apply the same process to a potential  $\chi_1(P)$ , having no singularity in the region  $(u_0, u_4)$ , vanishing on the surface  $u_0$ , and having on the surface  $u_4$  values represented by  $\Sigma \rho fvf w$ . Thus

$$\chi_1(P) = \Sigma \frac{fu_0 gu - fu gu_0}{fu_0 gu_4 - fu_4 gu_0} \rho fvf w.$$

The potential without singularity in  $(u_1, u_3)$  vanishing on  $u_1$  and equal to  $\chi_1(P)$  on  $u_3$  is

$$\chi_2(P) = \Sigma \frac{fu_0 gu_3 - fu_3 gu_0}{fu_0 gu_4 - fu_4 gu_0} \frac{fu gu_1 - fu_1 gu}{fu_3 gu_1 - fu_1 gu_3} \rho fvf w,$$

and on the surface  $u_4$  we have, after reduction,

$$\chi_2(P_4) - \chi_1(P_4) = \Sigma \frac{fu_0 gu_1 - fu_1 gu_0}{fu_0 gu_4 - fu_4 gu_0} \frac{fu_4 gu_3 - fu_3 gu_4}{fu_3 gu_1 - fu_1 gu_3} \rho fvf w.$$

This series is the same as that for  $\phi_2(P_4) - \phi_1(P_4)$  if

$$\rho = -jfv_2fv_3 (fu_0 gu_2 - fu_2 gu_0)(fu_4 gu_1 - fu_1 gu_4)/(fu_0 gu_1 - fu_1 gu_0).$$

Let us assume for the moment that the series  $\Sigma \rho f v f w$  is convergent when this value is taken for  $\rho$ , and consider the potentials  $\phi_1 - \chi_1$ ,  $\phi_2 - \chi_2$ . In the region  $(u_3, u_4)$  neither of these has any singularity, and on the boundaries of this region they have the same values. Hence throughout the region they are the same potential, say  $\omega$ . This potential therefore exists throughout  $(u_0, u_1)$ , and its only singularity is that of  $1/PP_2$  at the point  $(u_2, v_2, w_2)$ . There is no singularity at  $P_2$  in the other colour, and, as  $\phi_1, \chi_1$  vanish on  $u_0$  and  $\phi_2, \chi_2$  on  $u_1$ , the problem is solved, if we can prove the series  $\Sigma \rho f v f w$  convergent.

Now from its formation the expansion of  $\phi_2 - \phi_1$  is convergent in  $(u_3, u_4)$ , and this is the same as the series for  $\chi_2 - \chi_1$ . We therefore need to examine the fraction

$$\frac{fu_0gu_1-fu_1gu_0}{fu_0gu_4-fu_4gu_0} \frac{fu_3gu_4-fu_4gu_3}{fu_3gu_1-fu_1gu_3} (= \kappa, \text{ say}),$$

with the view of finding an inferior limit to its value. We have

$$1 - \kappa = \frac{fu_0gu_3-fu_3gu_0}{fu_0gu_4-fu_4gu_0} \frac{fu_4gu_1-fu_1gu_4}{fu_3gu_1-fu_1gu_3}.$$

Now, if  $\lambda u$  stands for  $fu_0gu - fu_1gu_0$ , we have seen that  $\lambda u, \lambda' u$  increase numerically with  $u - u_0$ , and thus  $\lambda u_3/\lambda u_4$  is positive but less than  $(u_3 - u_0)/(u_4 - u_0)$ , since  $u_0 < u_3 < u_4$ . Now  $\lambda u_3/\lambda u_4$  is the first factor in  $1 - \kappa$ ; in the same way the second factor is positive, but  $< (u_1 - u_4)/(u_1 - u_3)$ , since  $u_1 > u_4 > u_3$ . Thus  $1 - \kappa$  is positive and  $< \frac{u_3 - u_0}{u_4 - u_0} \frac{u_1 - u_4}{u_1 - u_3}$ ;  $\kappa < 1$  and  $> \frac{u_1 - u_0}{u_4 - u_0} \frac{u_4 - u_3}{u_1 - u_3}$ , a quantity independent of  $u$ . Now the series  $\Sigma |\kappa \rho|$  is convergent by ratio, and the same is therefore true for  $\Sigma |\rho|$  and  $\Sigma \rho f v f w$ . The solution  $\omega$  is accordingly valid.

26. The expression  $\phi_1 - \chi_1$  thus found for the potential is exactly the same as (47). This holds in the region  $(u_0, u_4)$ , and  $u_4$  may be as near as we please to  $2\gamma - u_2$ . Hence Green's function is given by the expression (47) for values of  $u$  between  $u_0$  and  $2\gamma - u_2$  when  $u_2$  is between  $u_0$  and  $\gamma$ .

If in (47) we write for  $1/PP_2$  the expansion  $\Sigma j g u_2 f u f v f w f v_2 f w_2$ , which holds when  $u > u_2$ , we arrive at another expression for Green's function, which agrees with  $\phi_2 - \chi_2$  and holds when  $u$  is between  $u_3$  and  $u_1$ , that is, between  $u_2$  and  $u_1$ , since  $u_3$  may be as near  $u_2$  as we

please. This expression is

$$\Sigma jfvfwfv_3fw_3(fu\ gu_1-fu_1\ gu)(fu_0\ gu_2-fu_2\ gu_0)/(fu_0\ gu_1-fu_1\ gu_0). \quad (48)$$

To pass to the case when  $u_2 > \gamma$  we need only write  $2\gamma-u_1$ ,  $2\gamma-u$ ,  $2\gamma-u_2$ ,  $2\gamma-u_0$  for  $u_0$ ,  $u$ ,  $u_2$ ,  $u_1$ .

27.\* There is a difficulty in the case when  $u_2 = \gamma$ , since we cannot take  $u_3$ ,  $u_4$  between  $u_2$  and  $2\gamma-u_2$ .

The expressions (47), (48) still represent potentials without singularity in the regions  $(u_0, \gamma)$ ,  $(\gamma, u_1)$  respectively, but, as these regions have no part in common, we need to prove that the two potentials are the same and to find what singularities they have on the surface  $u = \gamma$ .

Let  $\omega(P, P_2)$  denote the value of the function represented by (47), (48) so long as  $u_2 < \gamma$ . Describe two spheres  $A, B$  with a common centre  $O$  lying within the focal ellipse. Let  $\rho, \sigma$  be their radii and  $\sigma > \rho$ ; suppose also that  $B$  does not reach to any point on the bounding ellipsoids or the focal ellipse or to the point  $P_2$ . Then throughout  $B$  the potential  $\omega(P, P_2)$  can be expressed by an expansion in spherical harmonics, say  $\Sigma a_{n^2+m} \eta_{n^2+m} \left(\frac{r}{\rho}\right)^n$ , where  $r = OP$ ,  $m = 1, 2, \dots, 2n+1$  and  $\eta_{n^2+1}, \eta_{n^2+2}, \dots, \eta_{n^2+2n+1}$  are a canonical system of surface harmonics of order  $n$ , the quantities  $a$  being constant coefficients, depending on  $P_2$ . We shall examine the effect on  $a$  of gradually changing  $u_2$  into  $u'_2$  where  $u_2 < u'_2 < \gamma$ .

Now, throughout the whole region  $\omega(P, P_2) < 1/PP_2$ , for  $1/PP_2 - \omega(P, P_2)$  is a potential whose values on the boundary are positive, and whose only singularity within the region is a positive infinity at  $P'_2$ , the point at  $P_2$  in the other colour. Hence  $1/PP_2 - \omega(P, P_2)$  is positive throughout the region. It follows that over the surface  $B$  the value of  $\omega(P, P_2)$  is less than  $h$ , the reciprocal of the least distance from  $P_2$  in its different positions to  $B$ .

Let us take  $\gamma - u_2 < \delta$  and describe the ellipsoid  $u = \gamma - \delta$ . This will cut out from the surface of  $B$  a belt  $B'$ . Now we have

$$a_{n^2,m} = \left(\frac{\rho}{\sigma}\right)^n \frac{2n+1}{4\pi\sigma^3} \int_B \omega(P, P_2) \eta_{n^2,m} dS.$$

The contribution of  $B'$  to the integral here is a quantity of the same order of magnitude as the area of  $B'$ , that is, as  $\delta$ , since the subject of integration  $< h$  everywhere. Hence the change in value of this part of the integral is of the same order of magnitude as  $\delta$ , or a lower order.

Now consider the change in the contribution of  $B-B'$ . This depends on the change in  $\omega(P, P_2)$ . Let a sphere  $C$  of finite radius  $\rho'$  be described about  $P_2$ , so as not to reach any point of  $B$ , and let  $1/h'$  be the least distance between  $B$  and the various positions of  $C$ . Then  $\omega(P, P_2)$  is a potential for  $P_2$ , without singularity within  $C$ , and its values on  $C$  are less than  $h'$ . Hence its space derivative at  $P_2$  in any direction cannot exceed the finite quantity  $3h'/\rho'$ , so that the change in  $\omega(P, P_2)$  does not exceed this quantity multiplied by the length of the path; it is therefore of the same order of magnitude as the length of path, that is to say, as  $\delta$ . The same is therefore true of the integral in the expression for  $a_{n^2+m}$ , when taken over  $B-B'$ . Hence on the whole the change in  $a_{n^2+m}$  is of the same order of magnitude as  $\delta$ . Diminishing  $\delta$  without limit, we see that  $a_{n^2+m}$  approaches a definite limit when  $u_2$  approaches  $\gamma$ . Let this limit be  $\beta_{n^2+m}$ ; then  $\beta_{n^2+m} < \left(\frac{\rho}{\sigma}\right)^n (2n+1)h$ , since  $a_{n^2+m}$  is always less than

this quantity. Hence the series  $\sum \beta_{n^2+m} \gamma_{n^2+m} \left(\frac{r}{\rho}\right)^n$  is convergent by ratio throughout the sphere  $A$ , and represents a potential without singularity in that sphere. The sum of this series is the limit of the sum of  $\sum a_{n^2+m} \gamma_{n^2+m} \left(\frac{r}{\rho}\right)^n$ , since this latter is uniformly convergent up to the limit. Hence  $\lim_{u_2 \rightarrow \gamma} \omega(P, P_2)$  is a potential without singularity within  $A$ .

The above proof holds good even if  $P'_2$  lies within the sphere  $A$ , since it depends on the values of  $\omega(P, P_2)$  on the surface of  $B$ , and a superior limit of these is still  $h$ .

If  $P_2$  itself lies within  $A$ , then  $\omega(P, P'_2)$  has no singularity within  $A$ , and, as  $1/PP_2 - \omega(P, P_2) - \omega(P, P'_2)$  has no singularity within the whole region, it follows that  $\omega(P, P_2)$  has none but that of  $1/PP_2$  at  $P_2$ .

If  $P_2$  lies on the focal ellipse, it coincides with  $P'_2$ , and the difficulty of isolating the singularity does not arise. We have

$$\omega(P, P_2) = \omega(P, P'_2);$$

the function  $\omega(P, P_2)$  vanishes on the surface and has the singularity of  $1/2PP_2$  at  $P_2$ .

28. In all there are four expressions for Green's function in the region  $(u_0, u_1)$ , where  $u_1 > u_0$ , namely: (47), valid for the region common to  $(u_0, u_1)$  and  $(u_0, 2\gamma - u_2)$ ; (47) with  $u_0, u_1, u_2, u$  changed into  $2\gamma - u_1, 2\gamma - u_0, 2\gamma - u_2, 2\gamma - u$ —this is valid for the region common to  $(u_0, u_1)$  and  $(2\gamma - u_2, u_1)$ ; (48), valid for the region  $(u_2, u_1)$ ; (48) with  $u, u_2$  interchanged, valid for the region  $(u_0, u_2)$ . Of these the first, third, and fourth serve for an ordinary shell, in which  $u_0, u_1$  are both  $< \gamma$ .

By combination of volumes we may pass to the extreme case  $u_0 = 0, u_1 = 2\gamma$  in which the region includes the whole of the two-coloured space. Writing again  $e$  for  $(2n+1) \int_0^{2\gamma} du/f^2 u$ , we have the following expressions for the potential which vanishes at infinity in each colour and has no singularity but that of  $1/PP_2$  at  $(u_2, v_2, w_2)$  :—

$$\begin{aligned} & 1/PP_2 - \Sigma j e^{-1} gu gu_2 fv fw fv_2 fw_2 \quad (\text{when } u < 2\gamma - u_2), \\ & 1/PP_2 - \Sigma j e^{-1} (gu - e fu)(gu_2 - e fu_2) fv fw fv_2 fw_2 \quad (\text{when } u > 2\gamma - u_2), \\ & \Sigma j (fu - e^{-1} gu) gu_2 fv fw fv_2 fw_2 \quad (\text{when } u > u_2), \\ & \Sigma j (fu_2 - e^{-1} gu_2) gu fv fw fv_2 fw_2 \quad (\text{when } u < u_2). \end{aligned}$$

It is supposed of course that  $u, u_2$  lie between 0 and  $2\gamma$ .

29. The problem for the shell in one or two colours, when the boundary values are not analytical, may be discussed exactly as in the simpler case, except in one point. We have to prove that  $\frac{\partial}{\partial u} G(P, P_2)$  diminishes indefinitely when  $u = u_0$  or  $u_1$  and  $u_2$  approaches either  $u_0$  or  $u_1$ , but  $P_2$  does not coincide with  $P$  in the limit. The former proof fails when  $u = u_0$  and  $u_2$  approaches  $u_0$ , if  $2\gamma - u_0$  lies between  $u_0$  and  $u_1$ , but another is given by the combination method. Let us use the former notation (§§ 25, 26), but suppose  $0 < 2\gamma - u_1 \leq u_0 < u_2 < u_3 < u_4 < \gamma < u_1$ , which will not affect the former work, and write  $\omega^3, \omega^4, \dots$  for the greatest numerical value of  $\omega, \dots$  on the surfaces  $u_3, u_4$  respectively.

The potential which has no singularity in  $(u_0, u_4)$  vanishes when  $u = u_0$  and  $= 1$  when  $u = u_4$  is  $(u - u_0)/(u_4 - u_0)$ . The value of this when  $u = u_3$  is  $(u_3 - u_0)/(u_4 - u_0)$ ; hence, if a potential  $\theta$  has no singularity in  $(u_0, u_4)$ , has the value zero on  $u_0$  and values not exceeding  $M$  on  $u_4$ , its value on  $u_3$  cannot exceed  $M(u_3 - u_0)/(u_4 - u_0)$ , and  $\partial\theta/\partial u$  on  $u_0$  cannot exceed  $M/(u_4 - u_0)$ . Similarly in other cases. We wish to find a limit to  $\omega^4$ .

Now  $\omega = \phi_2 - \chi_2$   
 in  $(u_3, u_1)$ , and thus  $\omega^4 \nmid \phi_2^4 + \chi_2^4$ .

Also  $\phi_2^4 \nmid \phi_2^3 (u_1 - u_4)/(u_1 - u_3)$ , that is,  $\phi_1^3 (u_1 - u_4)/(u_1 - u_3)$ ,  
 and  $\chi_2^4 \nmid \chi_2^3 (u_1 - u_4)/(u_1 - u_3)$ , that is,  $\chi_1^3 (u_1 - u_4)/(u_1 - u_3)$ ,  
 $\chi_1^3 \nmid \chi_1^4 (u_3 - u_0)/(u_4 - u_0)$  or  $\omega^4 (u_3 - u_0)/(u_4 - u_0)$ ,

since  $\omega = -\chi_1$

on  $u_4$ . Thus  $\omega \nmid \phi_1^3 \frac{u_1 - u_4}{u_1 - u_3} + \omega^4 \frac{u_1 - u_4}{u_1 - u_3} \frac{u_3 - u_0}{u_4 - u_0}$ ,

and after reduction

$$\chi_1^4 \text{ or } \omega^4 \nmid \phi_1^3 \frac{u_1 - u_4}{u_1 - u_0} \frac{u_4 - u_0}{u_4 - u_3}.$$

The greatest value on  $u_0$  of  $\partial\chi_1/\partial u$  does not exceed  $\frac{1}{u_4 - u_0}$  of this, or  $\phi_1^3 (u_1 - u_4)/(u_1 - u_0)(u_4 - u_3)$ , which diminishes indefinitely when  $u_3$  approaches  $u_0$ , since  $\phi_1$  is Green's function for  $(u_0, u_4)$ . Since then, in  $(u_0, u_4)$ ,

$$\frac{\partial\omega}{\partial u} = \frac{\partial\phi_1}{\partial u} - \frac{\partial\chi_1}{\partial u},$$

and each of the terms on the right in this equation tends to the limit 0 when  $u = u_0$  and  $u_3$  approaches  $u_0$ , the limit of  $\partial\omega/\partial u$  is also 0, unless of course  $v_2 = v_0$ ,  $w_2 = w_0$ .

30. This method fails if  $u_0 = \gamma$ , but in this case Green's function may be put in a special form. Let  $G(P, P_2, u_0, u_1)$  denote Green's function for the points  $P, P_2$  in the region bounded by the ellipsoids  $u_0, u_1$ , and denote the point  $(2\gamma - u_2, v_2, w_2)$  by  $P_2''$ . Then, if  $P_2$  lies in the region  $(\gamma, u_1)$ ,  $P_2''$  does not, and hence

$$G(P, P_2, 2\gamma - u_1, u_1) - G(P, P_2'', 2\gamma - u_1, u_1)$$

has only one singularity in  $(\gamma, u_1)$ , namely, that at  $P_2$ . This expression vanishes when  $u = u_1, 2\gamma - u_1$ , or  $\gamma$ ; the last follows since its sign changes when  $u$  is changed into  $2\gamma - u$ . Hence the expression is  $G(P, P_2, \gamma, u_1)$ , and

$$\begin{aligned} \frac{\partial}{\partial u} G(P, P_2, \gamma, u_1) &= \frac{\partial}{\partial u} G(P, P_2, 2\gamma - u_1, u_1) - \frac{\partial}{\partial u} G(P, P_2'', 2\gamma - u_1, u_1) \\ &= 0, \end{aligned}$$

in limit when  $u = \gamma$  and  $u_2$  approaches  $\gamma$ , unless  $P, P_2$  coincide in the limit. Here the continuation of Green's function across the boundary is known, so that the usual difficulty does not arise.

31.\* Let us now discuss the problem of finding a potential without singularity within a given ellipsoid when the values at the boundary are given of its differential coefficient along the normal; that is, we are to find  $\phi$  so that, when  $u = u_0$ ,

$$\frac{\partial \phi}{\partial u} = F(v, w),$$

a given function, and that within the ellipsoid  $u_0$

$$\Delta \phi = 0.$$

It is necessary that the condition

$$\int F(v, w) d\omega = 0$$

should be satisfied.

There is no difficulty in proving, when the given function  $F$  is analytical everywhere on the surface, that the solution of the problem is

$$\phi = \frac{1}{8\pi} \Sigma (2n+1) j \frac{f'u}{f'u_0} f'v f'w \int F(v_0, w_0) f'v_0 f'w_0 d\omega_0.$$

It is not so easy to prove this when  $F$  is not everywhere analytical. The idea of the following proof is taken from C. Neumann (*Untersuchungen über das . . . Potential*, Leipzig, 1877).

Take  $d\nu$  to be an element of the outward normal to the ellipsoid  $u_0$ , and suppose an electrical distribution of surface density  $-F(v, w) \frac{\partial u}{\partial \nu}$  upon this surface. The potential of the distribution at an outside point  $(u, v, w)$  is

$$\frac{1}{2} \Sigma j g u f v f w f u_0 \int F(v_0, w_0) f v_0 f w_0 d\omega_0,$$

say  $\Sigma \lambda g u f v f w$ . This potential is continuous everywhere, even at the surface, if  $F(v, w)$  is everywhere finite. Let us denote it by  $\psi$  inside and by  $\psi'$  outside.

In the expansion of  $\psi'$  the term of order 0 is wanting, since by hypothesis

$$\int F(v_0, w_0) d\omega_0 = 0;$$

hence, since  $\psi'$  is continuous over the surface, Neumann's method of the arithmetic mean may be used to prove that the ellipsoid  $u_0$  can

\* §§ 31 to end have been added to the paper as originally written.—Feb. 20th, 1903.

be considered as a magnetic shell of such strength that its potential at an outside point will be  $\psi'$ . Let the potential of this shell be denoted by  $\chi$  inside and  $\chi'$  outside, so that  $\chi' = \psi'$ .

Since  $\psi, \psi'$  are the internal and external potentials of the same electric distribution, we have at the surface  $u_0$

$$\frac{\partial \psi'}{\partial \nu} - \frac{\partial \psi}{\partial \nu} = 4\pi F(v, w) \frac{\partial u}{\partial \nu}$$

or

$$\frac{\partial \psi'}{\partial u} - \frac{\partial \psi}{\partial u} = 4\pi F(v, w).$$

Similarly 
$$\frac{\partial \chi'}{\partial u} - \frac{\partial \chi}{\partial u} = 0,$$

since  $\chi, \chi'$  are the internal and external potentials of a magnetic shell. Since

$$\psi' - \chi' = 0$$

at all points outside, we have by subtraction

$$\frac{\partial}{\partial u} (\chi - \psi) = 4\pi F(v, w).$$

Hence the potential sought is  $\frac{1}{4\pi} (\chi - \psi)$ . It remains to expand this in terms of Lamé's functions. The expansion will be absolutely convergent, and valid, everywhere inside the ellipsoid  $u_0$ , since the potential has no singularity inside this surface.

32. We have 
$$\psi = \Sigma \lambda g u_0 f u f v f w / f u_0,$$

while 
$$\chi = \Sigma \nu f u f v f w,$$

where 
$$8\pi \nu f' u = (2n+1) j \int \frac{\partial \chi}{\partial u} f v f w d\omega,$$

taken upon the ellipsoid  $u$  ( $u > u_0$ ). But

$$8\pi \lambda g' u = (2n+1) j \int \frac{\partial \chi'}{\partial u} f v f w d\omega,$$

taken upon the ellipsoid  $u$  ( $u < u_0$ ), and the quantities  $\frac{\partial \chi}{\partial u}, \frac{\partial \chi'}{\partial u}$  approach the same limit at the surface. Hence, passing to the limit in each case, we have

$$\nu f' u_0 = \lambda g' u_0$$

and

$$\begin{aligned}\chi - \psi &= \Sigma \lambda f u f v f w \left( \frac{g' u_0}{f' u_0} - \frac{g u_0}{f u_0} \right) \\ &= \Sigma (2n+1) \lambda f u f v f w / f u_0 f' u_0,\end{aligned}$$

$$\frac{1}{4\pi} (\chi - \psi) = \frac{1}{8\pi} \Sigma (2n+1) j \frac{f u}{f' u_0} f v f w \int F(v_0, w_0) f v_0 f w_0 d\omega_0.$$

The theorem is therefore proved, and the solution holds good if the function  $F$  is integrable all over the surface and satisfies the condition

$$\int F(v, w) d\omega = 0.$$

Exactly the same method applies when the potential is to be found for external space. The expansion is

$$\frac{1}{8\pi} \Sigma (2n+1) j \frac{g u}{g' u_0} f v f w \int F(v_0, w_0) f v_0 f w_0 d\omega_0,$$

and the condition  $\int F(v, w) d\omega = 0$

is unnecessary.

33. Lastly, take the case of a region bounded by the two ellipsoids  $u_0, u_1$ , whether this is a shell in the ordinary sense of the word or a two-coloured region. Let  $F_0(v, w)$  and  $F_1(v, w)$  represent the assigned values of  $\frac{\partial \phi}{\partial u}$  on the two surfaces. Then it is necessary that

$$\int F_0(v, w) d\omega = \int F_1(v, w) d\omega.$$

If this condition is satisfied and the value of each side of the equation is  $8\pi A$ , we have

$$\phi = Au + \phi_0 + \phi_1,$$

where  $\phi_0, \phi_1$  are potentials without singularity within the region, and,

$$\text{when } u = u_0, \quad \frac{\partial \phi_0}{\partial u} = F_0(v, w) - A, \quad \frac{\partial \phi_1}{\partial u} = 0;$$

$$\text{when } u = u_1, \quad \frac{\partial \phi_0}{\partial u} = 0, \quad \frac{\partial \phi_1}{\partial u} = F_1(v, w) - A.$$

If these potentials  $\phi_0, \phi_1$  can be found, the problem can be solved; the same kind of process serves to find both.

Now the expression for  $\phi_0$ , which is certainly valid when  $F_0(v, w)$

is analytical, is

$$\frac{1}{8\pi} \Sigma (2n+1) j \frac{f'u g'u_1 - g'u f'u_1}{f'u_0 g'u_1 - g'u_0 f'u_1} f'v f'w \int \{F_0(v_0, w_0) - A\} f'v_0 f'w_0 d\omega_0,$$

or, say,

$$\Sigma k \frac{f'u g'u_1 - g'u f'u_1}{f'u_0 g'u_1 - g'u_0 f'u_1}.$$

We need to show that even when  $L_0$  is not assumed to be analytical, but only integrable, the limit of

$$\Sigma k \frac{f'u g'u_1 - g'u f'u_1}{f'u_0 g'u_1 - g'u_0 f'u_1}$$

when  $u$  approaches  $u_0$  is  $L_0(v, w) - A$ .

Suppose first that  $u_0 < \gamma$  and  $< u_1$ ; then the limit of  $u$  is  $u_0 + 0$ , and we know that  $\Sigma k f'u/f'u_0$  approaches the desired limit. Now

$$\begin{aligned} \frac{f'u g'u_1 - g'u f'u_1}{f'u_0 g'u_1 - g'u_0 f'u_1} - \frac{f'u}{f'u_0} &= \frac{f'u_1 (f'u g'u_0 - f'u_0 g'u)}{f'u_0 (f'u_0 g'u_1 - g'u_0 f'u_1)} \\ &= \frac{f'u g'u_0 - f'u_0 g'u}{f'u_0 g'u_0 - f'u_0 g'u_2} \left[ \frac{f'u_2 g'u_1 - f'u_1 g'u_2}{f'u_0 g'u_2 - g'u_0 f'u_1} - \frac{f'u_2}{f'u_0} \right], \end{aligned}$$

where  $u_2$  has any value. If now we take  $u_2 > u_0$ , but  $< u_1$  and  $< \gamma$ , and take  $u$  between  $u_0$  and  $u_2$ , each of the three fractions

$$\frac{f'u g'u_0 - f'u_0 g'u}{f'u_2 g'u_0 - f'u_0 g'u_2}, \quad \frac{f'u_2 g'u_1 - f'u_1 g'u_2}{f'u_0 g'u_2 - g'u_0 f'u_1}, \quad \frac{f'u_2}{f'u_0}$$

does not exceed  $pq^n$ , where  $p, q$  are certain positive constants and  $q < 1$ . Hence the series

$$\Sigma k \left[ \frac{f'u g'u_1 - g'u f'u_1}{f'u_0 g'u_1 - g'u_0 f'u_1} - \frac{f'u}{f'u_0} \right]$$

converges uniformly when  $u$  is between  $u_0$  and  $u_2$ . Its limit when  $u$  approaches  $u_0$  is therefore found by putting  $u = u_0$  and is zero, which was to be proved.

This method applies when  $u_0 - \gamma$  and  $u_0 - u_1$  have the same sign. When  $u_1 < u_0 < \gamma$  the series to be used for comparison is  $\Sigma k g'u/g'u_0$ , and when  $u_1 > u_0 > \gamma$  it is  $\Sigma k g'(2\gamma - u)/g'(2\gamma - u_0)$ . When  $u_0 = \gamma$  the method fails, but the gap may be filled up as follows.

34. In any region  $(u_0, u_1)$  bounded by the ellipsoids  $u_0, u_1$  we have constructed a potential  $\omega$  (§ 25), vanishing at the boundary and having no singularity except that of  $1/PP_2$  at the single point  $(u_2, v_2, w_2)$ .

When  $u_0 < u_1$ , and both differ from  $\gamma$ , we can construct a potential  $\psi$  without singularity within the region, and such that,

$$\text{when } u = u_0, \quad \frac{\partial \psi}{\partial u} = \frac{\partial \omega}{\partial u} - \frac{1}{2},$$

$$\text{when } u = u_1, \quad \frac{\partial \psi}{\partial u} = \frac{\partial \omega}{\partial u} + \frac{1}{2};$$

the terms  $\pm \frac{1}{2}$  are added on account of the condition to be satisfied by the values of  $\frac{\partial \psi}{\partial u}$  at the boundary. Then  $\omega - \psi$  is a potential with

the singularity at  $(u_2, v_2, w_2)$  and no other in the region, and its derivative with respect to  $u$  is constant over each of the bounding surfaces. Let this function be denoted by  $\Omega(P, P_2, u_0, u_1)$ . Let  $P'', L_2''$  denote the points  $(2\gamma - u, v, w)$ ,  $(2\gamma - u_2, v_2, w_2)$ . We then have, if  $u_1 > \gamma$ ,

$$\Omega(P, P_2'', 2\gamma - u_1, u_1) = \Omega(P'', P_2, 2\gamma - u_1, u_1).$$

$$\text{Thus} \quad \frac{\partial}{\partial u} \Omega(P, P_2'', 2\gamma - u_1, u_1) = \frac{\partial}{\partial u} \Omega(P'', P_2, 2\gamma - u_1, u_1).$$

But, when  $u = \gamma$ ,  $P$  and  $P''$  coincide, and

$$\frac{\partial}{\partial u} \Omega(P'', P_2, 2\gamma - u_1, u_1) = - \frac{\partial}{\partial u} \Omega(P, P_2, 2\gamma - u_1, u_1).$$

$$\text{Thus} \quad \frac{\partial}{\partial u} \{ \Omega(P, P_2, 2\gamma - u_1, u_1) + \Omega(P, P_2'', u_1, 2\gamma - u_1, u_1) \} = 0$$

when  $u = \gamma$ . It is therefore possible to form the function  $\Omega(P, P_2, u_0, u_1)$  even when  $u_0 = \gamma$ , and its value is in fact

$$\Omega(P, P_2, 2\gamma - u_1, u_1) + \Omega(P, P_2'', 2\gamma - u_1, u_1) + \frac{1}{2}u.$$

Its region of existence is certainly not bounded by the ellipsoid  $\gamma$  even when  $u_2 = \gamma$ .

Now consider the expression

$$\phi_0 = - \frac{1}{8\pi} \int \Omega(P, P_0, u_0, u_1) \{ F_0(v_0, w_0) - A \} d\omega_0,$$

in which  $P_2$  is replaced by a point  $P_0(u_0, v_0, w_0)$  on the surface  $u_0$ . As a function of the point  $P$  this is a potential without singularity within the region. If we take  $u = u_1$ , we have, since the ellipsoid  $u_1$  does not bound the region within which  $\Omega, \phi_0$  are analytical,

$$\frac{\partial \phi_0}{\partial u} = - \frac{1}{8\pi} \int \left( -\frac{1}{2} \right) \{ F(v_0, w_0) - A \} d\omega_0 = 0$$

for  $\frac{\partial \Omega}{\partial u} = -\frac{1}{2}$  at this surface, and by hypothesis

$$\int F'(v_0, w_0) d\omega_0 = 8\pi A.$$

If  $u_0 = \gamma$ , we have that  $\Omega - \frac{1}{2}u$ , and therefore also  $\phi_0$ , is unchanged by putting  $2\gamma - u$  for  $u$ ; so that the function  $\phi_0$  exists in both the regions  $(\gamma, u_1)$  and  $(2\gamma - u_1, \gamma)$ , and is continuous at the surface  $\gamma$ . On the other hand,  $\frac{\partial \phi_0}{\partial u}$  is an odd function of  $\gamma - u$  and is

discontinuous at this surface. There is no discontinuity in the part of the definite integral contributed by those regions of the surface which are at a finite distance from  $P$  or which are near it in position, but in the other colour. In considering the part of the surface which is near  $P$  we may write

$$\begin{aligned} \phi_0 = & -\frac{1}{8\pi} \int \left( \frac{1}{PP_0} + \frac{1}{P''P_0} \right) \{F_0(v_0, w_0) - A\} d\omega_0 \\ & - \frac{1}{8\pi} \int \left\{ \Omega - \frac{1}{PP_0} - \frac{1}{P''P_0} \right\} \{F_0(v_0, w_0) - A\} d\omega_0, \end{aligned}$$

and the discontinuity is wholly due to the first term, since  $\Omega - \frac{1}{PP_0} - \frac{1}{P''P_0}$  has no singularity at  $P_0$ . But the discontinuity in the first term of  $\frac{\partial \phi_0}{\partial u}$  is exactly  $2F_0(v, w) - 2A$ , that is,

$$\left( \frac{\partial \phi_0}{\partial u} \right)_{r \rightarrow 0} - \left( \frac{\partial \phi_0}{\partial u} \right)_{r \rightarrow 0} = 2F_0(v, w) - 2A,$$

and, since  $\frac{\partial \phi_0}{\partial u}$  is an odd function of  $\gamma - u$ , its two limiting values when  $u = \gamma \pm 0$  must be  $\pm F(v, w) \mp A$ .

The expression  $\pm \frac{1}{8\pi} \int \Omega(P, P_0, u_0, u_1) \{F'(v_0, w_0) - A\} d\omega_0$  gives

then the desired value of  $\phi_0$  when  $u_0 = \gamma$ , and there is no difficulty in proving that this agrees in form with the general solution. The upper or lower sign is to be taken according as  $u_1 < \text{or} > \gamma$ .