ON THE UNIFORM APPROACH OF A CONTINUOUS FUNCTION TO ITS LIMIT

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1. In a recent paper I introduced the term "uniform divergence at a point," and proved incidentally that various theorems involving uniformity of approach of a function to its limit still held whether or no that limit was a bounded function. In the paper in question I was concerned more particularly with the distribution of particular points, and the behaviour of the limiting function in the neighbourhood of those points. In a large class of theorems we are concerned with the behaviour of the limiting function throughout an interval, and the question naturally forces itself on our notice how we are to characterise uniformity of approach throughout an interval, when the sense of the words is the generalised one in question. In the present note I give the formulation in the case in which the generating functions are continuous (but not necessarily bounded). As an illustration of the use of this formulation, I prove various theorems leading up to the following generalisation of Weierstrass' theorem that any continuous (bounded) function can be expressed as the sum of a uniformly converging series of polynomials:—

THEOREM.—*An unbounded continuous function is expressible as the sum of a uniformly converging and diverging series of polynomials or rational fractions according as in the extension of the definition of continuity to unbounded functions, the two infinities* $+\infty$ *and* $-\infty$ *are regarded as distinct or not.*

The paper concludes with a formulation of the property of uniform continuity throughout an interval in the case of unbounded continuous functions.

Uniform Convergence and Divergence at a Point.

2. The definition of uniform divergence at a point given in my paper on the subject published lately in the *Proceedings of the London Mathematical Society* (p. 36) was as follows:—

The series of functions $f_1, f_2, \ldots$ is said to diverge uniformly at a point

P where it has no finite limit, if, given any quantity A, an interval d_P can be described, having P as internal point, so that for all points x within the interval d_P,

$$f_n(x) > A,$$

for all values of $n > m_P$, where m_P is an integer, independent of x, which can always be determined.

It is also said to diverge uniformly at P, if in this condition we alter the inequality to

$$f_n(x) < A.$$

In this definition the values $+\infty$ and $-\infty$ are distinguished, in accordance with the extended view of continuity adopted, where a function which is not finite is still regarded as continuous, if it is infinite with determinate sign at a point P, and is the only limit of values in the neighbourhood.

It is to be remarked that, just as in extending the idea of continuity to non-finite functions it is not necessary to distinguish the two infinities, so it is not necessary to do so in defining uniform divergence. The first point of view is equivalent to regarding the axis of y, where y is the dependent variable, as a segment with two end-points, the points $+\infty$ and $-\infty$. The second point of view is that of regarding the axis of y as a loop, without any end-point. The definition of uniform divergence when the two infinities are identified will only differ from the above in the two inequalities, which are replaced by the single inequality

$$|f_n(x)| > A.$$

In either case the limiting function will have a point of continuity at such a point of uniform divergence, *whether or no* $f_1, f_2, \ldots$ *are continuous at the point.* This indicates that the definition is open to objection except when $f_1, f_2, \ldots$ are continuous functions. We shall, however, confine our attention to this latter case, which is by far the most important one in practice.

3. The definition so given, though analogous to the recognised definition of uniform convergence at a point, labours under the disadvantage that the inequality employed is different in form according as there is convergence, or divergence, at the point considered. Moreover, the analogous definition of uniform convergence itself presents certain disadvantages, to obviate which it was shewn in the paper referred to that it might be replaced by another in the case when the functions $f_1, f_2, \ldots$ were *continuous.* This new definition had the advantage of being the

same in form whether the series converged, or diverged, at the point considered. It depended on the definition of the peak and chasm functions, uniform convergence, or divergence, taking place at any point where these are equal, and at such points only.

I propose, first of all, to transform the original definition of uniform convergence and divergence at a point in such a way that, without using the peak and chasm functions, its form is the same whether the series converges, or diverges, at the point considered. The new definition is as follows:—

Let $f_1, f_2, \ldots$ be a series of continuous functions which converges, or diverges to a definite limit $F(x)$ at every point of an interval. The series is said to approach uniformly to its limit at a point P of this interval if, corresponding to any segment on the y-axis containing the point*

$$y = F(P)$$

as internal point,† we can find an interval d_P containing the point P of the x-axis, and determine an integer m_P, such that the points

$$y = F(x) \quad \textit{and} \quad y = f_n(x)$$

lie for all values of $n \geqslant m_P$ inside the given segment, provided x lies inside the interval d_P.

It is evident that if this is the case, there is uniform convergence, or uniform divergence, at the point P, according as the point $y = F(P)$ is, or is not, at infinity. We have, in fact, in the former case, only to choose the segment of length $2e$ with the point $y = F(P)$ as middle point, and in the latter case to choose as segment all the part of the y-axis beyond the point $y = A$, on one side or the other, if the two infinities are distinguished, while, if the two infinities are identified, we have only to choose as segment all the part of the y-axis at a distance greater than A from the origin.

To shew conversely that when the series is uniformly convergent, or divergent, at the point P, this property holds, we proceed as follows. First, let P be such that the corresponding point of the y-axis,

$$p = F(P),$$

* In the generalised sense, $+\infty$ and $-\infty$ being distinguished, or identified.

† In the exceptional case when the point $y = F(P)$ is an end-point of the range of y (whether this range is finite, or infinite with a finite end-point, or the whole straight line with $+\infty$ distinguished from $-\infty$), $y = F(P)$ is to be included as an "internal point" in any segment having it as end-point. A similar remark applies to the range of x if this has one, or two, end-points.

is not at infinity. Let the distance of p from the nearest end-point of the given segment be $3e$. Then, since the series is uniformly convergent at P, we can find an interval d_P containing P as internal point, and determine an integer m_P, such that

$$| F(x)-f_n(x) | \leqslant e,$$

provided only the point x lies in the interval d_P, and $n \geqslant m_P$.

Also, since F is known to be continuous at P, we can so choose the interval d_P, that

$$| F(P)-F(x) | \leqslant e.$$

From these two inequalities it follows that

$$| F(P)-f_n(x) | \leqslant 2e,$$

or, in other words, the point $y = f_n(x)$

lies in the same segment as the point $p = F(P)$.

Secondly, let the point p be at infinity, and suppose first that the two infinities are identified. Then the point p is, as before, internal to the given segment.

Let that one of the two end-points of the segment which is nearest to the origin be denoted by

$$y = A.$$

Then, since there is uniform divergence at P, we can determine an interval d_P containing P as internal point, and an integer m_P, such that

$$|f_n(x)| > A,$$

provided the point x lies in the interval d_P and $n \geqslant m_P$. Thus the point

$$y = f_n(x)$$

lies in the segment containing the point p.

Further, since F is known to be continuous, we can secure that the interval d_P is such that

$$|F(x)| > A,$$

so that the point $y = F(x)$

also lies in the segment. We have therefore the same relation as when the point p was not at infinity.

The argument when the two infinities are distinguished is precisely analogous. For definiteness take

$$p = +\infty.$$

Then the given segment consists of all the points

$$y \geqslant A,$$

including the point p, which, though an end-point, is now to be regarded as tantamount to an internal point.

Then, since there is uniform divergence at P, we can determine an interval d_P, containing P as internal point, and an integer m_P, such that

$$f_n(x) > A,$$

provided the point x lies in the interval d_P and $n \geqslant m_P$. Thus the point

$$y = f_n(x)$$

lies in the given segment.

Further, since F is continuous, we can secure that this interval d_P is such that

$$F(x) > A,$$

so that the point

$$y = F(x)$$

also lies in the given segment.

4. *If a series of functions converges, or diverges, uniformly at every point of an interval, or of a set of points, it is said to converge uniformly throughout the interval, or set.*

It now follows that *the necessary and sufficient condition that a series of functions $f_1(x)$, $f_2(x)$, ... should approach uniformly to a limiting function throughout a closed interval, or set, is that however we divide up the range of the dependent variable $y = F(x)$ into a finite number of segments, we can find a corresponding division of the range of the independent variable x into a finite number of intervals, and a fixed integer m, such that, if the points x and x' belong to the same interval, n being any integer $\geqslant m$, the points*

$$y = F(x), \qquad y' = f_n(x')$$

lie in the same segment of the y-axis, or in the same two adjacent segments.

First this condition is necessary. For, taking any particular division of the y-axis, each point P of the range on the axis of x determines a segment on the axis of y, viz., that part, or that pair of adjacent parts, inside which the corresponding point $y = F(P) = p$ lies. This segment, provided the given series is uniformly convergent or divergent at P, determines, as explained in the preceding section, an interval d_P containing the point P, and an integer m_P. By the Heine-Borel theorem a finite number of these intervals suffice to cover every point of the range of x, provided that range is a closed interval or set. Let m be the greatest of the corresponding integers m_P, and let the intervals be $d'_1, d'_2, \ldots, d'_i$. Then, provided $n \geqslant m$, and that the points x and x' both belong to one of these intervals, say d'_r, the points

$$y = F(x), \qquad y' = f_n(x'),$$

both lie in the same segment, or the same pair of adjacent segments, determined by the interval d'_r. These intervals $d'_1, d'_2, \ldots$ however, overlap; if we now replace d'_2 by the part of it not internal to d'_1, and each succeeding interval in turn by the part of it not internal to the preceding intervals, we get a finite number of non-overlapping intervals $d_1, d_2, \ldots, d_r$, each of which determines a segment, or a pair of adjacent segments, on the y-axis, inside which the points

$$y = F(x), \qquad y' = f_n(x')$$

both lie, whenever x and x' both lie in the corresponding interval d_1 and $n \geqslant m$. Thus the given condition is necessary.

It is, moreover, sufficient, for, supposing it true, however we divide the y-axis, then given any point P of the x-axis, this determines a point

$$p = F(P)$$

of the y-axis. Taking any segment containing p, let us make any convenient division of the y-axis in which that part which contains p, as well as the adjacent part, or parts, lie inside that segment. By hypothesis this division determines a finite number of non-overlapping intervals containing all the points x, and determines also an integer m. If P belongs to only one interval d, then, corresponding to the chosen segment of the y-axis, we have found a d and an m, such that, if x is any point belonging to d, and $n \geqslant m$, the points

$$y = F(x), \qquad y = f_n(x),$$

lie in the same part, or pair of parts, as p and $y = f_n(P)$, so that they lie inside the chosen segment of the y-axis. If P belongs to two adjacent intervals, then, taking together these two intervals, we get a d and, as before, an m. Thus in either case the criterion for uniform convergence at P, given in § 3, is satisfied. Thus every point P is a point of uniform convergence, so that the condition is not only necessary but sufficient.

5. THEOREM.—*Let $f_{i,1}, f_{i,2}, \ldots, f_{i,n}, \ldots$ be a series of continuous functions which approaches uniformly throughout an interval S to a limiting function f_i, for each integral value of i. Also, let $f_1, f_2, \ldots, f_i, \ldots$ approach to a limiting function F. Then we can find a series of the continuous functions $f_{i,n}$ which approach throughout the interval S to the limiting function F.*

Let $R_1, R_2, \ldots$ be a countable set of points dense everywhere on the

y-axis $e.g.$, the rational points. The first i points $R_1, R_2, \ldots, R_i$, determine uniquely a division of the y-axis into a finite number of segments i or $i+1$, in number according as we identify or distinguish $+\infty$ and $-\infty$. The characteristic of the i-th division, performed in this way, is that if R_j and R_k are the end-points of the same segment, there is no point R_n inside that segment, whose index $n \leqslant i$, *a fortiori*, whose index $n < j$ or $< k$. At the n-th and at all subsequent divisions such a point R_n will be itself the end-point of two adjacent segments, whose other end-points may at first be R_j and R_k, but will, if not from the first, certainly from and after some subsequent stage, always lie inside the segments (R_j, R_n) and (R_n, R_k) respectively.

Hence it follows that if a series of segments, one from each successive division, is given, say (R_1, R'_1), (R_2, R'_2), ..., in such a way that points $P_1, P_2, \ldots$, one from each segment, have only one limiting point P, then the same will be true of $R_1, R_2, \ldots$ and of $R'_1, R'_2, \ldots$; and therefore of any other set of points $Q_1, Q_2, \ldots$ lying in the same segments, or, indeed, by similar reasoning, in either of the adjacent segments at each stage.

This being premised, let us determine, corresponding to the i-th division, the integer m_i, such that, whatever x may be, the points

$$y = f_i(x), \qquad y = f_{i,n}(x),$$

always lie in the same segment, or in adjacent segments, provided $n \geqslant m_i$. This we can do, since the functions $f_{i,n}$ converge, or diverge, uniformly throughout the interval S. Then, since, by hypothesis, the points

$$P_i = y = f_i(x),$$

for fixed x, have the single limiting point

$$P = F(x),$$

it follows, from what has been pointed out, that the points

$$Q_i = f_{i, m_i}(x)$$

have the same single limiting point. Thus the series of continuous functions

$$f_{1, m_1}, f_{2, m_2}, f_{3, m_3}, \ldots$$

has at each point x the limit $F(x)$, which proves the theorem.

COR.—*If we know further that the series $f_1, f_2, \ldots$ approaches uniformly at the point P to its limit $F(P)$, then the series*

$$f_{1, m_1}, f_{2, m_2}, \ldots$$

approaches uniformly at P to the same limit.

We shall suppose, for convenience of wording, that P is not one of the points R_1, R_2, ... The argument is, however, the same when this is not the case, only that the point p, or $y = F(P)$, determines then two adjacent segments, instead of a single segment.

Take any segment d containing the point p of the y-axis. Then we can determine the integer i so that that segment in which p lies at the i-th division by means of the points R_1, R_2, ..., R_i, together with the adjacent segment or segments, all lie inside the given segment d. Now, since the series $f_1, f_2, \ldots$ converges, or diverges, uniformly at P, we can, by § 3, find an interval d_P containing the point P of the x-axis, and determine an integer m_P greater than i, so that, if x is any point of d_P and k any integer $\geqslant m_P$, the points

$$y = F(x) \quad \text{and} \quad y = f_k(x)$$

lie in that segment of the i-th division in which the point p lies. Now the points

$$y = f_k(x) \quad \text{and} \quad y = f_{k, m_k}(x)$$

lie in the same segment at the k-th division, and therefore, since $k > i$, in the same segment at the i-th division. Thus the points

$$y = F(x) \quad \text{and} \quad y = f_{k, m_k}(x),$$

both lie in the given segment d; for the former lies in the same segment as p at the i-th division, and the latter in the same, or, if $y = f_k(x)$ is an end-point of this segment, in one of the adjacent segments, which, by our choice of i, all lie in the given segment d.

Thus the criterion for uniform convergence, or divergence, at P, as given in § 3, is satisfied, which proves the theorem.

6. We now proceed to the extension of Weierstrass' theorem.

LEMMA.—*If $F(x)$ is a continuous function which is always positive (or always negative) but not necessarily finite, then F is the limit of a series of bounded positive continuous functions approaching its limit uniformly throughout the interval considered.*

For, if n be any positive integer, the points x at which

$$F(x) \leqslant n,$$

form a closed set, including no infinity of $F(x)$. Thus the infinities (F being positive) are internal to the black intervals of this set, and, since the infinities form a closed set, to a finite number of those black intervals. In each of the intervals so determined put

$$f_n(x) = n,$$

and at the remaining points $f_n(x) = F(x)$.

Then, since at the end-points of each of the intervals in question

$$F(x) = n,$$

and $F(x)$ is finite and continuous outside the intervals, $f_n(x)$ is a finite and continuous function.

Now, if $m < n$, the closed set

$$F(x) \leqslant n,$$

contains the closed set $F(x) \leqslant m$;

and therefore the black intervals of the former set lie inside those of the latter set. Thus, throughout the intervals in which, by definition,

$$f_m(x) = m,$$

we have for all values of $n > m$,

$$f_n(x) \geqslant m,$$

and, throughout the intervals complementary to the intervals just mentioned, we have

$$f_n(x) = f_m(x) = F(x).$$

This shews that at every infinity of $F(x)$, the series $f_1(x), f_2(x), \ldots$ diverges uniformly to $F(x)$, while at every other point it converges uniformly to $F(x)$, which proves the theorem.

THEOREM.—*Any function which, without being always finite, is continuous when $+\infty$ is distinguished from $-\infty$, is expressible as the limit of a series of polynomials, which approaches its limit uniformly for every value of x for which the function is defined.*

Let $F(x)$ be the function, and A any positive finite number. Then we define two new functions $U(x)$ and $V(x)$, as follows:—

$U(x) = F(x)+A$, wherever $F(x)$ is positive, and elsewhere $U(x) = A$.

$V(x) = F(x)-A$, wherever $F(x)$ is negative, and elsewhere $V(x) = -A$.

Then, at every point $F(x) = U(x)+V(x)$.

But, by the preceding Lemma, $U(x)$ is the limit of a series of bounded continuous functions $u_1(x), u_2(x), \ldots$ approaching its limit uniformly.

By the known theorem of Weierstrass, $u_i(x)$ is the limit of a uniformly convergent series of polynomials. Since this is true for each value of i, we can apply the corollary to the theorem of § 5, and state that $U(x)$ itself is the limit of a suitably chosen series of the polynomials, say

$$u_1(x),\ u_2(x),\ \ldots,$$

approaching its limit uniformly.

Similarly $V(x)$ is the limit of a series of polynomials

$$v_1(x),\ v_2(x),\ \ldots,$$

approaching its limit uniformly.

Since $U(x)$ and $V(x)$ have no common infinities, their sum $F(x)$ is the limit of the sum of corresponding polynomials, say

$$f_i(x) = u_i(x) + v_i(x),$$

and the series $f_1(x), f_2(x), \ldots$ approaches uniformly to its limit $F(x)$, which proves the theorem.

7. Before proceeding to the second case we shall prove the following theorem :—

THEOREM.—*If $f_1, f_2, \ldots$ is a series of functions of x which approaches uniformly a limiting function $F(x)$, each function being continuous (but not necessarily finite) at any point P, and $g(x)$ any other function continuous also at P, then the series $g[f_1(x)], g[f_2(x)], \ldots$ approaches uniformly $g[F(x)]$ as limit at the point P.*

For, taking three axes corresponding to variables x, y, and z, and taking the point P of the x-axis, let us choose any segment on the z-axis containing the point

$$z = g(p),$$

where

$$p = F(P),$$

we can, since g is continuous, find a segment d_p of the y-axis, containing the point

$$y = p = F(P),$$

such that, whatever point y be taken in this interval d_p, the corresponding point

$$z = g(y)$$

of the z-axis lies inside the chosen segment.

But, since the series of continuous functions $f_1, f_2, \ldots$ approaches uniformly $F(x)$ as limit, we can, corresponding to the segment d_p of the y-axis, find an interval d_P of the x-axis, containing the point P, and determine an integer m_P, such that, for all points x inside the interval d_P, and for all values of $n \geqslant m_P$, the points

$$y = f_n(x)$$

lie in the interval d_p of the y-axis, and therefore the points

$$z = g(y) = g[f_n(x)]$$

lie inside the chosen interval. But this is the condition for uniform convergence, or divergence, of the series $g[f_1(x)]$, $g[f_2(x)]$, ... at the point P of the x-axis to the limit $g[F(x)]$.

One of the most important applications of the preceding theorem consists in the process of inverting a given series. In other words, if the series

$$f_1(x),\ f_2(x),\ \ldots$$

converges, or diverges, uniformly at a point P, so does the series

$$\frac{1}{f_1(x)},\ \frac{1}{f_2(x)},\ \ldots.$$

This process was not allowable in dealing with uniformly convergent, but not divergent series, a point where the series had the limit zero leading to a point of divergence of the inverted series.

Moreover, it is only allowable if we adopt the definition of continuity and divergence which depends on the two infinities being identified.

8. Weierstrass' theorem requires modification in the case when the two infinities are identified. We have, in fact, the following theorem:—

THEOREM.—*A function which is continuous if, and only if, the two infinities are identified, cannot* be expressed as the limit of a series of bounded continuous functions, which converges, or diverges, uniformly, and this, whether or no the two infinities are identified in defining uniform divergence.*

For, if P be a point of uniform divergence, we can assign an interval d_P and an integer m_P, such that, for all values of $n \geqslant m_P$, and all points x of the interval d_P,

$$|f_n(x)| > A;$$

thus $f_n(x)$ never vanishes in the interval d_P, and therefore, being a continuous bounded function is throughout the interval d_P of one sign. Thus there is either one series of continually increasing integers n such that $f_n(x)$

* Except in the trivial case when it is always infinite and indeterminate as to sign, *e.g.*,

$$f_n(x) = (-)^n n \operatorname{cosec} x.$$

is always positive in the interval d_P, and another series always negative, or else any such series of integers always determine the same sign. In the latter case the infinity at P will have the same sign, and will be a point of continuity without identifying the two infinities. In the former case, however, at each point x of the interval d_P the one series of functions $f_n(x)$ will give rise to a limit which is positive, and the other to a limit which is negative. These two limits must, however, coincide, and are therefore both infinite at every point of the interval d_P. In this trivial case the limiting function is indeterminately infinite at every point of a closed interval, since, by the above, the end-points of an interval throughout which the function was indeterminately infinite could not be points of uniform divergence without the function being indeterminately infinite at these points also. Apart from this trivial case, the theorem is therefore true.

9. On the other hand, a function which is continuous if, and only if, the two infinities are identified, may be expressed as the limit of a uniformly converging, and diverging, series of rational functions.

To prove this we remark first, as in proving the Lemma, that the infinities of the function F lie in a finite number of the black intervals of the closed set of points at which

$$|F| \leqslant A.$$

Let these be (B_1, C_1), (B_2, C_2), (B_3, C_3), ..., (B_n, C_n), and let the whole interval considered be (B, C). Then in each of these partial intervals

$$F \neq 0,$$

at each of their end-points $F = A$ or $-A$,

while in the remaining partial intervals F is finite and continuous.

We now define n functions $f_1(x), f_2(x), \ldots, f_n(x)$, as follows:—

$$\begin{aligned} f_i(x) &= F(x) \text{ in the interval } (B_i, C_i) \\ &= F(B_i) \text{ in the interval } (B, B_i) \\ &= F(C_i) \text{ in the interval } (C_i, C). \end{aligned}$$

Then each of these functions $f_i(x)$ $(i = 1, 2, \ldots, n)$, is continuous and numerically never less than A; their reciprocals are therefore finite and continuous, so that, by Weierstrass' theorem, we may express each of these reciprocals as the limit of a polynomial,

$$\frac{1}{f_i(x)} = \operatorname*{Lt}_{r=\infty} P_{i,r}(x),$$

which converges uniformly throughout the interval (B, C) to its limit. Therefore (the two infinities being now identified)

$$f_i(x) = \frac{1}{\underset{r=\infty}{\mathrm{Lt}}\, P_{i,r}(x)} = \underset{r=\infty}{\mathrm{Lt}}\, \frac{1}{P_{i,r}(x)},$$

the convergence, or divergence, of the rational function to its limit being uniform (§ 7).

Now, by their definition, no two of the functions $f_i(x)$ have an infinity at the same point; therefore their sum is, like each of them, continuous throughout the whole interval (B, C); in each of the intervals (B_i, C_i) it differs from $F(x)$ only by a constant, say K_i, and in each of the remaining intervals it is constant, the value in the interval (C_{i-1}, B_i) being, since the function is continuous,

$$F(B_i)+K_i = F(C_{i-1})+K_{i-1}, \text{ or say } K_i'.$$

Thus, if we define another function $f_{n+1}(x)$ in the following manner:—

$$\begin{aligned} f_{n+1}(x) &= F(x)-K_1' \text{ in the first interval } (B, B_1) \\ &= -K_1 \text{ in the second interval } (B_1, C_1) \\ &= F(x)-K_2' \text{ in the third interval } (C_1, B_2) \\ &= -K_2 \text{ in the next interval } (B_2, C_2), \end{aligned}$$

and so on, this function will be continuous throughout the whole interval (B, C), and will be finite, since $F(x)$ is finite and continuous in each of the intervals in which $f_{n+1}(x)$ is not constant. Hence, by Weierstrass' theorem, we may write

$$f_{n+1}(x) = \underset{r=\infty}{\mathrm{Lt}}\, P_{n+1,r}(x),$$

the convergence of the polynomial to its limit being uniform.

The sum of $f_{n+1}(x)$ to the sum of the n functions $f_i(x)$, will then be $F(x)$ at every point, thus

$$F(x) = \underset{r=\infty}{\mathrm{Lt}}\, \frac{1}{P_{1,r}(x)} + \underset{r=\infty}{\mathrm{Lt}}\, \frac{1}{P_{2,r}(x)} + \ldots + \underset{r=\infty}{\mathrm{Lt}}\, \frac{1}{P_{n,r}(x)} + \underset{r=\infty}{\mathrm{Lt}}\, P_{n+1,r}(x).$$

Since, in this sum of limits, no two of the limits are infinite at the same point, the sum of the limits is the limit of the sum. Also, since each rational function converges, or diverges, uniformly throughout the whole interval (B, C), the same is true of the sum. Thus, finally,

$$F(x) = \underset{r=\infty}{\mathrm{Lt}}\, \left(\frac{1}{P_{1,r}(x)} + \frac{1}{P_{2,r}(x)} + \ldots + \frac{1}{P_{n,r}(x)} + P_{n+1,r}(x)\right),$$

the convergence, or divergence, being uniform, which proves the theorem.

10. We conclude by pointing out that the formulation of uniform approach to a limit throughout an interval given in § 4, gives us a corresponding formulation of the property of uniform continuity applicable to any continuous non-finite function. For continuity at a point P is neither more nor less than uniform convergence, or divergence, of $f(x+h)$ to $f(x)$ at the point P. The continuous variable h which approaches in any manner the limit 0, takes the place now of the discontinuous variable n approaching its limit $+\infty$.

Thus, if $f(x)$ is continuous at every point x of a finite closed interval, we can, corresponding to any given division of the y-axis into a finite number of segments, find a value of h, for which and all smaller values, the points

$$y = f(x) \quad \textit{and} \quad y = f(x+h),$$

lie in the same segment, or in the same pair of adjacent segments of the y-axis, this segment, or pair of segments, being determined only by the particular point x chosen.

We can, if we please, further modify the wording so as to permit of the point $x = \infty$ entering as an internal or end-point into the closed interval of the x-axis in which the function is continuous.