ON THE DISTRIBUTION OF THE SET OF POINTS $(\lambda_n\theta)$

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1. In a paper on "Some Problems of Diophantine Approximation", recently published,* Messrs. G. H. Hardy and J. E. Littlewood have investigated in detail the distribution of the set of points $(\lambda_n\theta)$ [$(\lambda_n\theta)$ denotes the fractional part of $\lambda_n\theta$] in the particular case in which

$$\lambda_n = a^n,$$

where a is an integer greater than 1, and n takes the values $1, 2, \ldots, \rightarrow \infty$. The authors point out that this is equivalent to studying the distribution of the digits in the expression of θ as a decimal in the scale of a, and it is this view of the problem which is the more interesting in this particular case, and on which they concentrate their attention. In the following paper some of these results are extended, by means of an adaptation of Messrs. Hardy and Littlewood's arguments, to the set of points resulting from any sequence λ_n which satisfies the inequalities

$$(1) \qquad \lambda_n/\lambda_{n-1} \geqslant \beta^{n^{-1+\zeta}} \quad (n \geqslant n_0),$$

where ζ is any number > 0, and $\beta > 1$. It is easily verified that these inequalities imply that

$$(2) \qquad \lambda_n > \exp(Hn^{\zeta}) \quad (n \geqslant n_1),$$

where H is a constant. Of course the relation (2) does not imply the relations (1); but, to give some idea of the range of applicability of the results that we shall obtain, we may say, speaking roughly, that *our theorems apply to sufficiently regular sequences which increase faster than*

$$\exp(n^{\zeta})$$

for some positive value of ζ.

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* *Acta Mathematica*, Vol. 37, pp. 155-190.

In the particular case discussed by Messrs. Hardy and Littlewood, there are, as we have mentioned, two ways of regarding the results which are both of great interest. In the more general case, though the discussion proceeds by the construction of a quasi-decimal expression for θ and the determination of the distribution of the integers employed (corresponding to the digits in a decimal), this aspect of the problem is no longer fundamental, and the results must be exhibited from the other point of view.

Messrs. Hardy and Littlewood prove the following theorems:—

THEOREM 1·48.*—*It is almost always true that, when a number θ is expressed in any scale of notation, the number of occurrences of any digit, or any combination of digits, is asymptotically equal to the average number that might be expected.*

THEOREM 1·481.*—*It is almost always true that the deviation from the average in the first n places of decimals, is not of order exceeding* $\sqrt{n \log (n)}$.

THEOREM 1·482.*—*It is almost always true that the deviation, in both directions, is sometimes of order exceeding* $\sqrt{n}$.

THEOREM 1·483.*—*The number* Δ_n, *of the first n numbers* $(a^\nu\theta)$ *which fall inside an interval of length* δ *included in the interval* (0, 1) *is almost always asymptotically equal to* δn.

In these theorems, a statement is said to be almost always true when it is true for all θ's between 0 and 1, with the exception of a set of measure zero.

Theorem 1·483 is, of course, practically the same as Theorem 1·48, regarded from the other point of view. Messrs. Hardy and Littlewood suggest that it may be possible to give corresponding forms to Theorems 1·481, 1·482. It is, however, very doubtful if this can be done.

In the case of the more general sequences considered in this paper, I shall prove theorems which are to a certain extent analogous to Theorem 1·481, and strictly analogous to Theorem 1·48. They differ from strict analogues of Theorem 1·481 in that there seems to be no way of proving—at any rate by Messrs. Hardy and Littlewood's methods—that the error term is of an order so small as $\sqrt{n \log (n)}$. I have not succeeded in proving any analogue of Theorem 1·482 for these general sequences.

* These theorems are so numbered in the paper referred to, p. 190.

2. We shall start by considering sequences for which

$$\lambda_n/\lambda_{n-1} \geqslant \beta^n \quad (n \geqslant n_0),$$

where β is a constant > 1; we can then extend the theorems so obtained to less restricted sequences by another argument. There will be no loss of generality in supposing that $n_0 = 2$, and $\lambda_1 > 1$.

Let G be any integer conveniently chosen. G will eventually be made large, but for the present it is to be regarded as fixed. Let

$$n = n_1 + i,$$

and consider the sequence $(\lambda_i\theta)$ which is formed by omitting the first n_1 terms of the old sequence $(\lambda_n\theta)$. If n_1 be so chosen that

$$\beta^{n_1} \geqslant G,$$

i.e., if
$$n_1 \geqslant \log(G)/\log(\beta),$$

the new sequence will satisfy the inequalities

$$\lambda_i/\lambda_{i-1} > G\beta^i,$$

for all values of
$$i \geqslant 2.$$

Let a be the greatest integer contained in $\sqrt{G}$. Then the λ's in our new sequence $(\lambda_i\theta)$ satisfy the inequalities

$$\lambda_1 < a\lambda_1 < \lambda_2 < a\lambda_2 \ldots < \lambda_i < a\lambda_i < \lambda_{i+1} < \ldots.$$

By means of this increasing sequence of numbers, we can build up a quasi-decimal expression for θ which will enable us to make use of Messrs. Hardy and Littlewood's arguments, and so to obtain the desired result. The possibility of the required expansion is established by the Lemma of the next section.

3. Let $[x]$ denote *the integer next less than* x, so that when x is not an integer,

$$x > [x] > x-1,$$

and when x is an integer,

$$[x] = x-1.$$

Let us also write

$$[x]+1 = x+(x)_f; \tag{3}$$

$(x)_f$ will satisfy the inequalities

$$1 > (x)_f \geqslant 0.$$

Let K denote the set of expressions of the form

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} \left\{ \phi_i + \frac{\chi_i}{a} + x_i \right\}, \tag{4}$$

where ϕ_i takes any one of the values 0, 1, 2, ..., $[\lambda_i/a\lambda_{i-1}]$,* χ_i any one of the values 0, 1, 2, ..., $a-1$, and x_i is that number satisfying

$$1 > x_i \geqslant 0,$$

which is such that $\lambda_i \sum_{s=1}^{i-1} \frac{1}{\lambda_s} \left\{ \phi_s + \frac{\chi_s}{a} + x_s \right\} + x_i$

is an integer. Referring to the equation (3) we see that

$$x_i = \left(\lambda_i \sum_{s=1}^{i-1} \frac{1}{\lambda_s} \left\{ \phi_s + \frac{\chi_s}{a} + x_s \right\} \right)_f. \tag{5}$$

For given values of $\phi_1 \dots \phi_{i-1}$, $\chi_1 \dots \chi_{i-1}$, the values of $x_1 \dots x_i$ are uniquely determined and can be calculated in order, beginning with x_1, by means of the equation (5). A change in the values of ϕ_i and χ_i will therefore not affect the values of $x_1 \dots x_i$, though, in general, the values of x_{i+1}, x_{i+2}, ... will all be changed. It may be observed that $x_1 = 0$.

Any expression of the form (4), in which all the ϕ's and χ's after a certain stage are zero, will be spoken of as a terminating expression, though, in general, of course, such an expression will contain the non-terminating convergent series

$$\sum_{i=1}^{\infty} x_i/\lambda_i.$$

It is easily seen that the series (4) will in all cases converge, and so will always represent a definite positive number.

Let ${}_0x_i$ be the value assumed by x_i when all the ϕ's and all the χ's have their greatest permissible values. Let Λ denote the sum of the convergent series

$$1 + \frac{(\lambda_1)_f}{\lambda_1} + \sum_{i=2}^{\infty} \frac{1}{\lambda_i} \left\{ \left(\frac{\lambda_i}{a\lambda_{i-1}} \right)_f + {}_0x_i \right\}. \tag{6}$$

Then $$\Lambda \geqslant 1.$$

We shall prove the following Lemma :—

LEMMA I.—*Every member of K represents a definite number between* 0 *and* Λ *inclusive.*

Conversely every number between 0 *and* Λ *inclusive is represented by at least one member of K.*

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* $\lambda_i/a\lambda_{i-1}$, when $i = 1$, is always to be taken to represent λ_1.

It is an obvious consequence of Lemma I that K is a set of measure Λ.

We have already pointed out that any member of K denotes a definite number θ. Consider the change produced in any member k of K by adding 1 to any ϕ or χ which has not its maximum value, to χ_s for example. We have

$$k < \sum_{i=1}^{s} \frac{1}{\lambda_i} \left\{ \phi_i + \frac{\chi_i}{a} + x_i \right\} + \sum_{i=s+1}^{\infty} \frac{1}{\lambda_i} \left\{ \phi_i + \frac{\chi_i}{a} + 1 \right\},$$

and $$k' > \sum_{i=1}^{s} \frac{1}{\lambda_i} \left\{ \phi_i + \frac{\chi_i}{a} + x_i \right\} + \frac{1}{a\lambda_s} + \sum_{i=s+1}^{\infty} \frac{1}{\lambda_i} \left\{ \phi_i + \frac{\chi_i}{a} \right\},$$

where the ϕ's, χ's, and x's are the same in both expressions. Therefore

$$k' - k > 1/a\lambda_s - \sum_{i=s+1}^{\infty} (1/\lambda_i).$$

But it is easily seen that

$$\sum_{i=s+1}^{\infty} (1/\lambda_i) < (1/\lambda_{s+1}) \sum_{r=0}^{\infty} \beta^{-r(s+1)} = \beta^{s+1}/\lambda_{s+1} \{\beta^{s+1} - 1\}.$$

Now $$\beta^{s+1}/\{\beta^{s+1}-1\} < \beta/\{\beta - 1\},$$

and $$\lambda_{s+1} > \sqrt{G}\, a\lambda_s,$$

so that, provided G be greater than a certain constant depending only on β,

$$k' - k > 0.$$

Therefore any member of K is increased by increasing any one of its constituent integers. The first part of the Lemma is an immediate consequence of this, for it follows that every member of K is less than the number obtained by giving their maximum values to *all* the ϕ's and χ's, and it is easily verified that the number so formed is Λ as defined by (6).

To prove the converse, suppose that θ is any number $< \Lambda$. Since the sum of the series (4) is increased by increasing any of its constituent integers, and since the sum is Λ when all these integers have their maximum values, it must be possible to determine the first q ϕ's and the first q χ's in such a way that

$$\sum_{i=1}^{q} \frac{1}{\lambda_i} \left\{ \phi_i + \frac{\chi_i}{a} + x_i \right\} + \sum_{i=q+1}^{\infty} \frac{x_i}{\lambda_i} \leqslant \theta < \sum_{i=1}^{q} \frac{1}{\lambda_i} \left\{ \phi_i + \frac{\chi_i}{a} + x_i \right\} + \sum_{i=q+1}^{\infty} \frac{1}{\lambda_i} \left\{ \left[\frac{\lambda_i}{a\lambda_{i-1}}\right] + 1 - \frac{1}{a} + x_i' \right\}.$$

The first q terms on each side agree exactly, so that the first $q+1$ x's are the same on each side; there is, however, no reason to suppose that the later values of x_i will be the same on both sides. This fact is indicated by writing x_i' for x_i in the later terms on the right. This choice of the ϕ's and χ's may be effected by giving to each ϕ and to each χ in turn the greatest possible value that leaves the expression

$$\sum_{i=1}^{q} \frac{1}{\lambda_i} \left\{ \phi_i + \frac{\chi_i}{a} + x_i \right\} + \sum_{i=q+1}^{\infty} \frac{x_i}{\lambda_i}$$

not greater than θ at each stage of the process, which can be continued indefinitely if no case of equality ever occurs. If a case of equality does turn up, we have a terminating expression of the form (4) for θ. If no such case turns up, we construct thus a definite convergent series which is such that the difference between the sum of its first q terms and θ tends to 0 as $q \to \infty$. It therefore represents the number θ, and the truth of the Lemma is established.

4. The set of terminating expressions included in K is enumerable, and therefore is a set of measure zero. We shall in future consider only the set K' of non-terminating expressions included in K, which consequently is a set of measure Λ. It should also perhaps be pointed out that we have not proved that more than one member of the set K' cannot represent the same number θ. It can, in fact, be shown that more than one member of K' can represent the same θ (at least for some values of θ) except when all the members $\lambda_i/a\lambda_{i-1}$ are integers. Consequently the relation between the set K' and the set of numbers θ between 0 and Λ, which cannot have a terminating expression, is a many-one, not a one-one relation. This fact, however, need not concern us further, for the following reason. For we proceed to prove theorems about almost all members of K', and the fact that the relation may be many-one does not prevent us from deducing the corresponding theorems about almost all the θ's.

5. All members of K', whose first ν ϕ's and first ν χ's have assigned values, can be enclosed in an interval whose length is certainly not greater than I_ν, where

$$I_\nu = \sum_{i=\nu+1}^{\infty} \frac{1}{\lambda_i} \left\{ \left[\frac{\lambda_i}{a\lambda_{i-1}}\right] + 2 - \frac{1}{a} \right\}.$$

On using the equation (3), we find that

$$I_\nu = \frac{1}{a\lambda_\nu} + \sum_{i=\nu+1}^{\infty} \frac{1}{\lambda_i} \left\{ 1 + \left(\frac{\lambda_i}{a\lambda_{i-1}}\right)_f \right\}. \tag{7}$$

Among the terminating expressions (in number a^ν),

$$\sum_{i=1}^{\nu} \frac{1}{\lambda_i} \left\{ \phi_i + \frac{\chi_i}{a} + x_i \right\} + \sum_{i=\nu+1}^{\infty} \frac{x_i}{\lambda_i},$$

in which the ϕ's are assigned and the χ's are not, there will be $p(\nu, m)$ in which χ_i takes the particular value b for exactly m values of i, where

$$p(\nu, m) = \frac{\nu!}{m! \, \{\nu - m\}!} \{a-1\}^{\nu-m}.* \tag{8}$$

All such members of K' can be enclosed in intervals of total length $< p(\nu, m) I_\nu$. Consequently *all* members of K' whose first ν χ's contain exactly m b's can be enclosed in intervals whose total length L_ν satisfies the inequality

$$L_\nu \leqslant \prod_{i=1}^{\nu} \{1 + [\lambda_i / a\lambda_{i-1}]\} I_\nu p(\nu, m). \tag{9}$$

On using the equations (3) and (7), this becomes

$$L_\nu \leqslant \left[\prod_{i=1}^{\nu} \left\{ 1 + \frac{a\lambda_{i-1}}{\lambda_i} \left(\frac{\lambda_i}{a\lambda_{i-1}} \right)_f \right\} \right]$$
$$\times \left[1 + a\lambda_\nu \sum_{i=\nu+1}^{\infty} \frac{1}{\lambda_i} \left\{ 1 + \left(\frac{\lambda_i}{a\lambda_{i-1}} \right)_f \right\} \right] a^{-\nu} p(\nu, m).$$

Now we have assumed that

$$\lambda_i / \lambda_{i-1} > G\beta^i;$$

and therefore

$$L_\nu < \left\{ 1 + 2\beta^{-\nu} \sum_{r=0}^{\infty} \beta^{-r(\nu+1)} \right\} \prod_{r=1}^{\infty} \{1 + \beta^{-r}\} a^{-\nu} p(\nu, m), \tag{10}$$
$$< M a^{-\nu} p(\nu, m),$$

where M depends only on β.

6. We can now treat the set K' in exactly the same way as Messrs. Hardy and Littlewood treat the set of irrational numbers expressed as decimals in the scale of a. We shall need, however, to revise the Lemmas on which the discussion is based, since it will be necessary to let $a \to \infty$.

Let $$\mu(\nu) = m - \nu / a,$$

so that $\mu(\nu)$ is the excess of the number of b's above the average. We

* *Loc. cit.*, 1·441.

proceed to prove that—

LEMMA II.—*If $f(\nu)$ is any function of ν that $\to \infty$ with ν, and satisfies the inequalities*

$$1 < f(\nu) < \nu/a - 1,$$

then, for all values of ν,

$$a^{-\nu}\left\{\sum_{|\mu(\nu)| \geqslant f(\nu)} p(\nu, m)\right\} < M\sqrt{\{a\nu\}} \exp\left\{-\tfrac{1}{4}a[f(\nu)]^2/\nu\right\},$$

*where M is a constant independent of a, ν, and $f(\nu)$.**

Let us consider values of $\mu(\nu)$, such that

$$\mu(\nu) \geqslant f(\nu),$$

where $f(\nu)$ is positive, so that

$$m \geqslant \nu/a + f(\nu).$$

Then
$$\begin{aligned} p(\nu, m+1)/p(\nu, m) &= \{\nu - m\}/\{a-1\}\{m+1\} \\ &\leqslant \{\nu\{a-1\} - af(\nu)\}/\{a-1\}\{\nu + af(\nu)\} \\ &< 1. \end{aligned}$$

It follows that, for all values of $\mu(\nu) \geqslant f(\nu)$,

$$\begin{aligned} p(\nu, m) &< p[\nu, \{\nu/a + f(\nu)\}] \\ &< \frac{\nu!}{\{\nu/a + f(\nu)\}!\,\{\nu - \nu/a - f(\nu)\}!}\{a-1\}^{\nu - \nu/a - f(\nu)}. \end{aligned}$$

Now it follows from Stirling's theorem that, if $s \geqslant 1$,

$$K \exp\{(s+\tfrac{1}{2})\log(s) - s\} > s! > k \exp\{(s+\tfrac{1}{2})\log(s) - s\},$$

where K and k are independent of s. Provided, therefore, that

$$\nu \geqslant 1, \quad \nu/a + f(\nu) \geqslant 1, \quad \nu - \nu/a - f(\nu) \geqslant 1,$$

we have
$$a^{-\nu} p(\nu, m) < M\nu^{-\frac{1}{2}} \exp\{F[\nu, a, f(\nu)]\},$$

where M is independent of ν, a, and $f(\nu)$, and $F[\nu, a, f(\nu)]$ is a function which can easily be reduced to the form

$$\begin{aligned} F[\nu, a, f(\nu)] = -\{\nu/a\}\{&(1+a\zeta)\log(1+a\zeta) \\ &+ (a-1-a\zeta)\log[1 - a\zeta/(a-1)]\} \\ &-\tfrac{1}{2}\log\{(1/a+\zeta)(1-1/a-\zeta)\}, \end{aligned}$$

* This Lemma is merely an adaptation of Lemma 1·444, *loc. cit.*, p. 185.

where $$\zeta = f(\nu)/\nu.$$

The coefficient of $-\nu/a$ in this expression may be written in the form

$$A = (1+x)\log(1+x) + (a-1-x)\log[1-x/(a-1)],$$

where $$x = a\zeta.$$

On differentiating A twice with respect to x, we find that

$$d^2A/dx^2 = a/\{(a-1-x)(1+x)\},$$

while A and dA/dx both vanish when $x = 0$. Therefore, provided

$$x < 1,$$

i.e., provided $$f(\nu)/\nu < 1/a,$$

we have $$d^2A/dx^2 > \tfrac{1}{2},$$

and $$A = \tfrac{1}{2}a^2\zeta^2 [d^2A/dx^2]_{x=\theta a\zeta} \quad (0 < \theta < 1),$$

so that $$A > \tfrac{1}{4}a^2\zeta^2.$$

Hence it is easily deduced that

$$a^{-\nu}p(\nu, m) < M\{a/\nu\}^{\frac{1}{2}} \exp\{-\tfrac{1}{4}a[f(\nu)]^2/\nu\},$$

where M is a constant independent of a, ν, and $f(\nu)$, if the necessary provisions are complied with. Similar arguments can be applied to the terms for which $$\mu(\nu) \leqslant -f(\nu),$$

provided $$\nu \geqslant 1, \quad \nu/a - f(\nu) \geqslant 1, \quad \nu - \nu/a + f(\nu) \geqslant 1.$$

All these conditions are covered by the conditions of the Lemma. Since the total number of terms $\leqslant \nu$, the truth of the Lemma has been established.

7. Lemma II leads at once to the following theorem :*—

THEOREM I.—*Any member of the set K', which is such that*

$$|\mu(\nu)| \nless \{4\tau\nu \log(\nu)/a\}^{\frac{1}{2}},$$

for all values of $\nu \geqslant n$, where τ is any number $> \frac{3}{2}$, can be enclosed in a

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* *Loc. cit.*, Theorem 1·45.

set of intervals whose total length is less than

$$Ma^{\frac{1}{2}}n^{\frac{3}{2}-\tau},$$

where M is a constant independent of a and n.

We omit the proof of this theorem, for it is a mere repetition of the proof of the corresponding theorem in Messrs. Hardy and Littlewood's paper.

From this theorem, it follows at once that the numbers θ between 0 and Λ, *or between* 0 *and* 1, which are such that their expressions in the form (4) do not satisfy the inequality

$$|\mu(\nu)| < \{4\tau\nu \log(\nu)/a\}^{\frac{1}{2}},$$

for *all* values of $\nu \geqslant n$, form a set of measure less than

$$Ma^{\frac{1}{2}}n^{\frac{3}{2}-\tau} \quad (\tau > \tfrac{3}{2}).$$

8. We have now to transform this result into a theorem about the distribution of the points $(\lambda_i\theta)$ over the interval (0, 1).

We have $$(\lambda_\nu\theta) = \frac{\chi_\nu}{a} + \lambda_\nu \sum_{i=\nu+1}^{\infty} \frac{1}{\lambda_i} \left\{ \phi_i + \frac{\chi_i}{a} + x_i \right\},$$

so that $$0 < (\lambda_\nu\theta) - \chi_\nu/a < \lambda_\nu I_\nu < \gamma/a,$$

where γ is a constant depending only on β.

Now let δ be the length of any interval included in the interval (0, 1), and let a_1, a_2 be the numbers corresponding to its end-points. For any given value of a, we can choose integers p_1, p_2, q_1, q_2, such that

$$p_1/a \geqslant a_1, \qquad \{q_1+\gamma\}/a < a_1,$$

$$\{p_2+\gamma\}/a < a_2, \qquad q_2/a \geqslant a_2,$$

where p_1, q_2 are the least possible integers, and p_2, q_1 are the greatest possible integers satisfying their respective inequalities. It follows that

$$a\delta = p_2 - p_1 + O(1) = q_2 - q_1 + O(1).$$

Now, if $$p_1 \leqslant \chi_i \leqslant p_2,$$

the point $(\lambda_i\theta)$ must fall inside the interval δ, while if it is not true that

$$q_1 < \chi_i < q_2,$$

the point $(\lambda_i \theta)$ cannot lie in the interval δ. But if θ does not belong to any one of a set of intervals whose total length is less than

$$M\{q_2-q_1-1\}\, a^{\frac{1}{2}} n^{\frac{3}{2}-\tau},$$

and if B_ν be the number of times than any integer b, $(q_2 > b > q_1)$, occurs among the first ν χ's $(\nu \geqslant n)$, then B_ν satisfies the relation

$$B_\nu = \nu/a + \epsilon\{4\tau\nu \log(\nu)/a\}^{\frac{1}{2}} \quad (|\epsilon| < 1). \tag{11}$$

Let Δ_ν denote the number of the first ν members $(\lambda_i \theta)$ that fall inside the interval δ. Then it follows from (11) that, *if θ does not belong to any one of a set of intervals whose total length is less than*

$$M a^{\frac{3}{2}} n^{\frac{3}{2}-\tau} \quad (\tau > \tfrac{3}{2}),$$

then

$$\Delta_\nu = \{a\delta + O(1)\}\{\nu/a + \epsilon[4\tau\nu \log(\nu)/a]^{\frac{1}{2}}\} \quad (|\epsilon| < 1). \tag{12}$$

We have so far been considering the sequence obtained by omitting n_1 terms from the original sequence, where

$$n_1 = O\{\log(a)\}. \tag{13}$$

Taking these terms into account, we see that we have established the truth of the following statement:—

If $\{\lambda_n\}$ be any sequence of positive numbers such that for all values of n,

$$\lambda_n/\lambda_{n-1} > \beta^n \quad (\beta > 1),$$

then, if θ be any number between 0 *and* 1 *which does not belong to a set of intervals whose total length is less than*

$$M a^{\frac{3}{2}} n^{\frac{3}{2}-\tau},$$

it follows that

$$\Delta_\nu = \{a\delta + O(1)\}\{\nu/a + \epsilon[4\tau\nu \log(\nu)/a]^{\frac{1}{2}}\} + O\{a^{\frac{1}{2}} \log(a)\} \quad (|\epsilon| < 1), \tag{14}$$

for any value of $\nu \geqslant n$, and for any value of $n > H \log(a)$. M, H, and the constants implied by the O's, depend solely upon β.

We must now choose the relative order of n and a in such a way that the error term in the relation (14) is made as small as possible. The two most important terms are of orders n/a and $[an \log(n)]^{\frac{1}{2}}$, when $\nu = n$; to make these of the same order (we shall thus achieve the best result), we must take

$$a = g\{n/\log(n)\}^{\frac{1}{3}},$$

where g is a constant. This being so, let N be any large integer, and let

$$n = \nu = N,$$

$$a = [\{N/\log(N)\}^{\frac{1}{3}}].$$

The necessary conditions will all be satisfied if N is greater than a certain number n_0 depending only on β. Thus it appears that if θ does not belong to a set of intervals whose total length is less than

$$MN^{2-\tau},$$

then

$$\Delta_N = N + \epsilon[\tau N^{\frac{2}{3}}\{\log(N)\}^{\frac{1}{3}}] \quad (|\epsilon| < \eta), \tag{15}$$

for all values of $N \geqslant n_0$, where M, η, n_0 depend only upon β.

We may now repeat the arguments used to establish Theorem I,* and deduce therefrom the following theorem :—

THEOREM II.—*The numbers θ between* 0 *and* 1 *for which it is not true that*

$$\Delta_n = \delta n + \epsilon\tau n^{\frac{2}{3}}[\log(n)]^{\frac{1}{3}} \quad (|\epsilon| < \eta), \tag{16}$$

for all values of $n \geqslant N \geqslant n_0$, can be enclosed in a set of intervals whose total length is less than

$$MN^{3-\tau} \quad (\tau > 3).$$

M, η, and n_0 depend only upon β.

Theorem III, that follows, is an obvious deduction from Theorem II.

THEOREM III.—*If $\{\lambda_n\}$ is any sequence of positive numbers satisfying*

$$\lambda_n/\lambda_{n-1} \geqslant \beta^n \quad (\beta > 1),$$

for all values of $n \geqslant n_0$, if δ is the length of any interval included in the interval (0, 1), *if θ is any number between* 0 *and* 1, *and if Δ_n denotes the number of the first n numbers* $(\lambda_\nu \theta)$ *that fall inside the interval δ, then for almost all values of θ,*

$$\Delta_n - \delta n = O\{n^{\frac{2}{3}}[\log(n)]^{\frac{1}{3}}\}. \tag{17}$$

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* *Loc. cit.*, p. 186.

9. We now proceed to extend this result to less heavily restricted sequences. Let $\{\lambda_n\}$ be any sequence of positive numbers satisfying

$$\lambda_n/\lambda_{n-1} \geqslant \beta > 1,$$

for all values of n. [If necessary, a finite number of terms may be omitted from the beginning.] We split this sequence up into an infinite number of subsequences by placing in the first subsequence all terms whose suffices can be expressed in the form

$$\tfrac{1}{2}s(s+1) \quad (s \text{ integral, and } \geqslant 1);$$

in the second, all those whose suffices can be expressed in the form

$$1+\tfrac{1}{2}s(s+1) \quad (s \text{ integral, and } \geqslant 1);$$

and in the r-th ($r \geqslant 2$), all those whose suffices can be expressed in the form

$$r-1+\tfrac{1}{2}(s+r-2)(s+r-1) \quad (s \text{ integral, and } \geqslant 1).$$

Let n be any integer, and N any integer satisfying

$$\tfrac{1}{2}n(n+1) \leqslant N < \tfrac{1}{2}(n+1)(n+2),$$

and let us consider the first N terms of the given sequence. Of these N terms, n terms will be regarded as belonging to the first subsequence, and, in general, $n-r+1$ terms to the r-th subsequence, while a certain number will be left unattached. The system of division is perhaps made clearer by the following diagram showing the terms in the first few subsequences.

Number of Subsequence	Number of Terms taken.	Suffix Formula.	Suffices.
1	n	$\frac{1}{2}s(s+1)$	1, 3, 6, 10, ...
2	$n-1$	$1+\frac{1}{2}s(s+1)$	2, 4, 7, 11, ...
3	$n-2$	$2+\frac{1}{2}(s+1)(s+2)$	5, 8, 12, ...
4	$n-3$	$3+\frac{1}{2}(s+2)(s+3)$	9, 13, ...
⋮	⋮	⋮	⋮ ⋮ ⋮

Suppose we make use of all the subsequences which contain $[\sqrt{n}]$, or more, terms in the above scheme. The total number of such subsequences will be

$$n-[\sqrt{n}]+1,$$

and the total number of terms accounted for by them will be

$$\tfrac{1}{2}\{n+[\sqrt{n}]\}\{n-[\sqrt{n}]+1\},$$

or

$$\tfrac{1}{2}n^2+O(n).$$

There will therefore be at most $O(n)$ terms left unattached among the total number N.

If $\{\lambda_s\}$ denote any one of these subsequences, the terms of $\{\lambda_s\}$ satisfy

$$\lambda_s/\lambda_{s-1} \geqslant \beta^s \quad (\beta > 1),$$

for all values of s, *the value of* β *being the same for all the subsequences.* It follows from this that, if n, and so N, be chosen sufficiently large (*i.e.*, if $n \geqslant n_0$, where n_0 depends only upon β), Theorem II can be applied to each subsequence. The longest set of intervals of which exception must be made is the set belonging to the last subsequence, and its total length is less than

$$Mn^{\frac{1}{2}(3-\tau)};$$

the total number of such excepted sets is less than n. Consequently if θ does not belong to any one of a set of intervals whose total length is less than

$$Mn^{\frac{1}{2}(5-\tau)},$$

i.e.,

$$MN^{\frac{1}{4}(5-\tau)},$$

$$(18) \quad \Delta_N = \delta\left\{\sum_{r=0}^{n-[\sqrt{n}]}(n-r)\right\}+O(n)+\epsilon\tau\left\{\sum_{r=0}^{n-[\sqrt{n}]}(n-r)^{\frac{2}{3}}[\log(n-r)]^{\frac{1}{3}}\right\} \quad (|\epsilon| < \eta),$$

$$= \delta N+O(n)+\epsilon\tau n^{\frac{5}{3}}[\log(n)]^{\frac{1}{3}} \quad (|\epsilon| < \eta),$$

$$= \delta N+\epsilon\tau N^{\frac{5}{6}}[\log(N)]^{\frac{1}{3}} \quad (|\epsilon| < \eta).$$

Without further comment, we may enunciate the following extensions of Theorems II and III:—

THEOREM IV.—*If* $\{\lambda_n\}$ *is any sequence of positive numbers satisfying*

$$\lambda_n/\lambda_{n-1} \geqslant \beta > 1,$$

for all values of $n \geqslant n_0$, *then the numbers* θ *for which it is not true that*

$$\Delta_n = \delta n+\epsilon\tau n^{\frac{5}{6}}[\log(n)]^{\frac{1}{3}} \quad (|\epsilon| < \eta),$$

for all values of $n \geqslant N \geqslant n_1$, *can be enclosed in a set of intervals whose* *total length is less than*

$$MN^{\frac{1}{4}(9-\tau)} \quad (\tau > 9).$$

where M, η, and n_1 depend only upon β.

THEOREM V.—*If $\{\lambda_n\}$ is any sequence of positive numbers satisfying*

$$\lambda_n/\lambda_{n-1} \geqslant \beta > 1,$$

and the other conditions of Theorem III are unaltered, then for almost all values of θ between 0 *and* 1,

$$\Delta_n - \delta n = O\{n^{\frac{2}{3}}[\log(n)]^{\frac{1}{3}}\}. \tag{19}$$

10. In the preceding section, a particular subdivision of the given sequence has been worked out in detail in order to show how Theorems II and III may be extended to the less restricted sequences contemplated in Theorems IV and V. The use of a more powerful subdivision and the application of identical arguments enables us to extend Theorems IV and V (or Theorems II and III) to sequences which satisfy the inequalities

$$\lambda_n/\lambda_{n-1} \geqslant \beta^{n^{-1+\zeta}} \quad (n \geqslant n_0,\ \zeta > 0,\ \beta > 1).$$

Choose an integer k, so that

$$k\zeta > 1,$$

and let n, N be large integers such that

$$\frac{n(n+1)\ldots(n+k-1)}{k!} \leqslant N < \frac{(n+1)(n+2)\ldots(n+k)}{k!}.$$

Then we divide up the original subsequence so that the first subsequence contains terms whose suffices are of the form

$$s(s+1)\ldots(s+k-1)/k! \quad (s \text{ integral, and } \geqslant 1),$$

and take the first n terms of this subsequence, and so on as in the last section. The r-th subsequence will have suffices of the form

$$a_r + (s+\beta_r)(s+\beta_r+1)\ldots(s+\beta_r+k-1)/k!,$$

where a_r, β_r are integers depending on r. If we take all the subsequences which have more than $[\sqrt{n}]$ terms, it may be proved that the number of terms taken will be

$$N + O(N^{1-\mu}),$$

where μ is a positive number depending on k. If λ_s and λ_{s-1} are consecutive terms of the r-th subsequence

$$\lambda_s/\lambda_{s-1} \geqslant \beta^{f(s)},$$

where $f(s) = \dfrac{(s+\beta_r)\ldots(s+\beta_r+k-2)}{(k-1)!}\left[\dfrac{k!}{a_r k!+(s+\beta_r)\ldots(s+\beta_r+k-1)}\right]^{1-\zeta}$

It may be verified that, whatever be the value of r,

$$f(s) \geqslant (s+\beta_r)^{\frac{1}{2}(k\zeta-1)} \geqslant 1,$$

provided that at most a finite number of terms be omitted from the subsequence. We can therefore apply Theorems IV and V to each subsequence in turn, obtaining finally the following theorem :*—

THEOREM VI.—*If* $\{\lambda_n\}$ *be any sequence of positive numbers satisfying*

$$\lambda_n/\lambda_{n-1} \geqslant \beta^{n^{-1+\zeta}} \quad (\zeta > 0,\ \beta > 1),$$

and the other conditions of Theorem III are unaltered, then for almost all values of θ *between* 0 *and* 1,

$$\Delta_n - \delta n = O\{n^{1-\mu}\},$$

where μ *is some positive number depending on* ζ.

11. We shall conclude the paper by considering the extension of the foregoing results to the m-dimensional set of points whose m coordinates are

$$(\lambda_n\theta_1),\ (\lambda_n\theta_2),\ \ldots,\ (\lambda_n\theta_m).$$

The method† by which this extension is to be made will be sufficiently explained by considering the extension of the foregoing theorems to the 2-dimensional set of points

$$(\lambda_n\theta_1), \quad (\lambda_n\theta_2).$$

Let θ_1, θ_2 be expressed in the forms (§ 3),

$$\theta_1 = \sum_{i=1}^{\infty} \frac{1}{\lambda_i}\left\{\phi_i^{(1)} + \frac{\chi_i^{(1)}}{a} + x_i^{(1)}\right\},$$

$$\theta_2 = \sum_{i=1}^{\infty} \frac{1}{\lambda_i}\left\{\phi_i^{(2)} + \frac{\chi_i^{(2)}}{a} + x_i^{(2)}\right\}.$$

* We could also have proved the same result by a continued application of the particular subdivision $k = 2$. This is clearly equivalent to confining ourselves to values of $k = 2^a$, where a is an integer.

† This is the method used by Messrs. Hardy and Littlewood to establish Theorem 1·491, of which the following theorem is the analogue.

We suppose that $$\lambda_i/\lambda_{i-1} \geqslant G\beta^i \quad (i \geqslant 2),$$

and that $$a = [\sqrt{G}].$$

The point of coordinates (θ_1, θ_2) will be any point in a square of side Λ ($\geqslant 1$). We correlate this point (θ_1, θ_2) with the point

$$\psi_{1,2} = \sum_{i=1}^{\infty} \frac{1}{\lambda_i^2} \left\{ \phi_i^{(1)} \left\{ \left[\frac{\lambda_i}{a\lambda_{i-1}}\right] + 1 \right\} + \phi_i^{(2)} + \frac{\chi_i^{(1)} a + \chi_i^{(2)}}{a^2} \right\}.*$$

It is easily verified that $\psi_{1,2}$ may be any point along a line of length Λ' ($\geqslant 1$). Moreover, a set of intervals on the line of $\psi_{1,2}$, of total length L, may be shown to correspond to a set of areas in the (θ_1, θ_2) plane of total area ML, where M is independent of the particular intervals and areas concerned.

We now apply Theorem I to show that if $B_\nu^{1,2}$ be the number of times that the particular combination $b^{(1)}a + b^{(2)}$ occurs in first ν terms of the expression for $\psi_{1,2}$, then

$$B_\nu^{1,2} = \nu/a^2 + \epsilon\,\{4\tau\nu \log(\nu)/a^2\}^{\frac{1}{2}} \quad (|\epsilon| < 1),$$

for all $(\psi_{1,2})$'s which do not belong to a set of intervals of total length $Ma\nu^{\frac{3}{2}-\tau}$.

From this it follows at once that, if $\Delta_\nu^{1,2}$ be the number of the first ν points whose coordinates are $(\lambda_n\theta_1)$, $(\lambda_n\theta_2)$ that are included in any rectangle (included in the unit square) whose sides are of lengths δ_1, δ_2, then

$$\Delta_\nu^{1,2} = \{a\delta_1 + O(1)\}\{a\delta_2 + O(1)\}\{\nu/a^2 + \epsilon\,[4\tau\nu \log(\nu)/a^2]^{\frac{1}{2}}\} \quad (|\epsilon| < 1),$$

unless $\{\theta_1, \theta_2\}$ belongs to a set of points of plane measure less than

$$Ma^3\nu^{\frac{3}{2}-\tau}.$$

From this point on, all the arguments can be repeated as before, and the restriction

$$\lambda_i/\lambda_{i-1} \geqslant G\beta^i$$

lightened in the manner of § 10. It is, moreover, clear that the arguments can be extended equally well to the case of the m-dimensional set of points

* There is no need to insert the term x_i, as we are not considering $(\lambda_i^2\psi_{1,2})$.

$\{\theta_1, \theta_2, \ldots, \theta_m\}$. We may, therefore, without further preface, enunciate our final theorem.

THEOREM VII.—*If* $\{\lambda_n\}$ *be any sequence of positive numbers satisfying*

$$\lambda_n/\lambda_{n-1} \geqslant \beta^{n^{-1+\zeta}} \quad (\zeta > 0,\ \beta > 1),$$

and if Δ_n *be the number of the first* n *points*

$$(\lambda_\nu\theta_1),\ (\lambda_\nu\theta_2),\ \ldots,\ (\lambda_\nu\theta_m),$$

which lie inside an m*-dimensional "rectangle" of "area"* δ, *then for almost all the points* $\{\theta_1, \theta_2, \ldots, \theta_m\}$,

$$\Delta_n - \delta n = O\{n^{1-\mu}\},$$

where μ *is some positive number depending on* ζ.