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XXXII. *The Logical Spectrum.*

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IN my work on the Algebra of Logic I investigated the meaning of Boole's analysis by the aid of diagrams. In this paper I propose to make a further use of the same method.

When the class of things considered (that is, the universe) is subdivided by not more than three qualities, a modified use of Euler's circles is sufficient. Let a square represent the universe of objects considered, and let a circle separate those having a particular quality. Diagram 1 represents the two classes formed by the presence and absence of the mark a; and diagram 2 represents each of the four classes ab, ab', $a'b$,

Diagram 1.

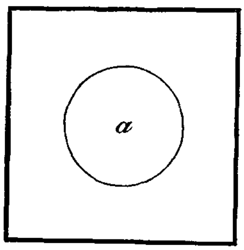
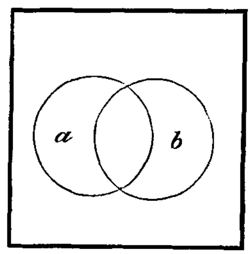


Diagram 2.



$a'b'$ formed by the presence or absence of each of the two marks a and b. Here the dash is used to denote *non*, as a' for non-a. It is a contraction for the complementary expression $1-a$. The diagram 3 is quite general, because it represents each of the eight classes abc, abc', $ab'c$, $ab'c'$, $a'bc$, $a'bc'$, $a'b'c$, $a'b'c'$ formed by the three marks a, b, and c. But if we

Diagram 3.

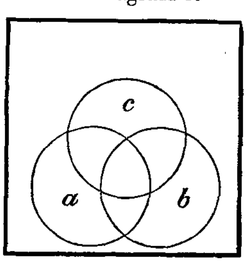
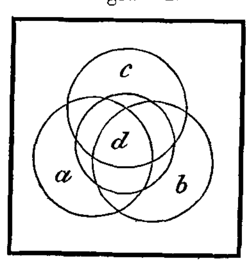


Diagram 4.



attempt by means of four circles to represent each one of the sixteen possible classes formed by four marks a, b, c, d, we shall find that it is impossible. For example, in diagram 4

* Communicated by the Author.

two possible classes are not represented ; hence the diagram is not general.

Another method, which I propose to call the *logical spectrum*, is capable of representing quite generally the universe subdivided by any number of marks. Let the universe be represented by a rectangular strip, as in diagrams 5 to 8.

Diagram 5.

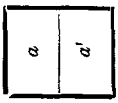


Diagram 6.

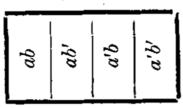


Diagram 7.

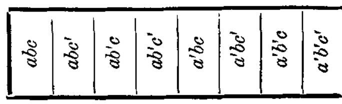
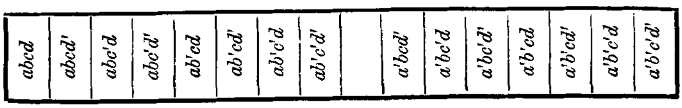


Diagram 8.



By subdividing into two we represent the possible classes formed by one mark a ; by subdividing each of these parts again we represent the four possible classes formed by two marks a and b ; and by subdividing each of these four parts again we represent the eight possible classes formed by the three marks a, b, c ; and so on. This method allows all the a part to be contiguous ; but the b part is broken up into two portions, the c part into four portions, the d part into eight portions. However, the regularity of the spectrum enables us easily to find all the portions belonging to any one mark.

I shall apply this method to verify the solution of the logical equations

$$\mathrm{U}\{ax+by=c\},$$

$$\mathrm{U}\{dx-ey=f\}.$$

Here U is the symbol for the whole of the objects considered; it corresponds to the strip of paper in the diagram. The letters a, b, c, d, e, f represent known marks. There are two unknown marks denoted by x and y, which are such that the part of U having the marks a and x, together with the part having the marks b and y, is identical with the part having the mark c ; and the part having the marks d and x, excepting the part having the marks e and y, is identical with the part having the mark f. It is required to select the part of U which has the mark x, and also the part which has the mark y.

X 2

By the ordinary process of solution of simultaneous equations we get

$$x=\frac{ce+bf}{ae+bd} \text{ and } y=\frac{cd-af}{ae+bd}.$$

Now, according to Boole's method,

$$\frac{ce+bf}{ae+bd}=A_1abcdef+A_2abcdef'+A_3abcde'f+A_4abcde'f'$$
$$+ \quad + \quad + \quad +$$
$$+A_{61}a'b'c'd'ef+A_{62}a'b'c'd'ef'+A_{63}a'b'c'd'e'f$$
$$+A_{64}a'b'c'd'e'f';$$

where the coefficients $A_1, A_2, \ldots A_{64}$ are numerical. The coefficient for any term is found by supposing that term coextensive with the universe, and substituting 1 or 0, as the case may be, for each of the letters in $\frac{ce+bf}{ae+bd}$. If U *abcdef* is identical with the universe, then *a, b, c, d, e, f* are each 1, and

$$A_1=\frac{1+1}{1+1}=1.$$

If U $abcdef'=$ U, then *f* is 0 and all the others 1, and

$$A_2=\tfrac{1}{2}.$$

According to Boole, these coefficients are susceptible of one or other of four interpretations: 1 means *all*, 0 means *none*, $\frac{0}{0}$ means *none* or *a portion* or *all*, and every other coefficient shows that the term is impossible.

When I sought to determine the coefficients for *x* and *y* by the above method, I found that many coefficients assumed the indeterminate form; and I found, on verifying the solution by means of the logical spectrum, that some of those terms could not be really indeterminate. Eventually I discovered a simpler method of finding the coefficients; and the solution obtained by it may be verified by the spectrum. It consists in substituting the special values of *a, b, c, d, e, f* (due to supposing a particular term identical with the universe) in the original equations, and then solving for *x* and *y*. The values so found are the coefficients for the terms.

For the first term *abcdef* we get

$$x+y=1, \quad x-y=1;$$

from which

$$x=1 \text{ and } y=0;$$

hence

$$A_1=1 \text{ and } B_1=0.$$

For the second term $abcdef'$ we get

$$x+y=1, \quad x-y=0;$$

hence y is complementary to x and also identical with x, which is impossible unless the term $abcdef'$ is null.

For the fifth term $abcd'ef$ we get

$$x+y=1, \quad -y=1;$$

but y cannot be negative unless $abcd'ef$ is null.

For the seventh term,

$$x+y=1, \quad 0=1;$$

but the latter equation is impossible unless the term vanish.

For the eighth term,

$$x+y=1, \quad 0=0;$$

hence x and y are indeterminate, but complementary to one another. In the case of the 58th term, we get x and y both indeterminate but identical with one another. These are interpretations which are not conceived of by Boole.

For the nineteenth term,

$$x=1, \quad x=1;$$

hence A_{19} is 1 and B_{19} is indeterminate, that is may have any value between 0 and 1. In a similar manner, by solving each of the sixty-four sets of equations the several coefficients are found. In the last case only are both coefficients indeterminate and independent of one another.

I have exhibited the solution obtained by this analysis in the accompanying spectrum (diagram 9). Each null term is shaded; each term which is wholly included is white; each term which is totally excluded is black; and each term which is indeterminate is partly black and partly white. The two complementary indeterminates have complementary parts white, and the two identical indeterminates have the same parts white.

The solution is verified by finding whether the ax together with the by is identical with the c, and whether the dx, excepting the ey, is identical with the f; and by testing whether any other solution not comprehended as a particular case of that obtained would satisfy the conditions.

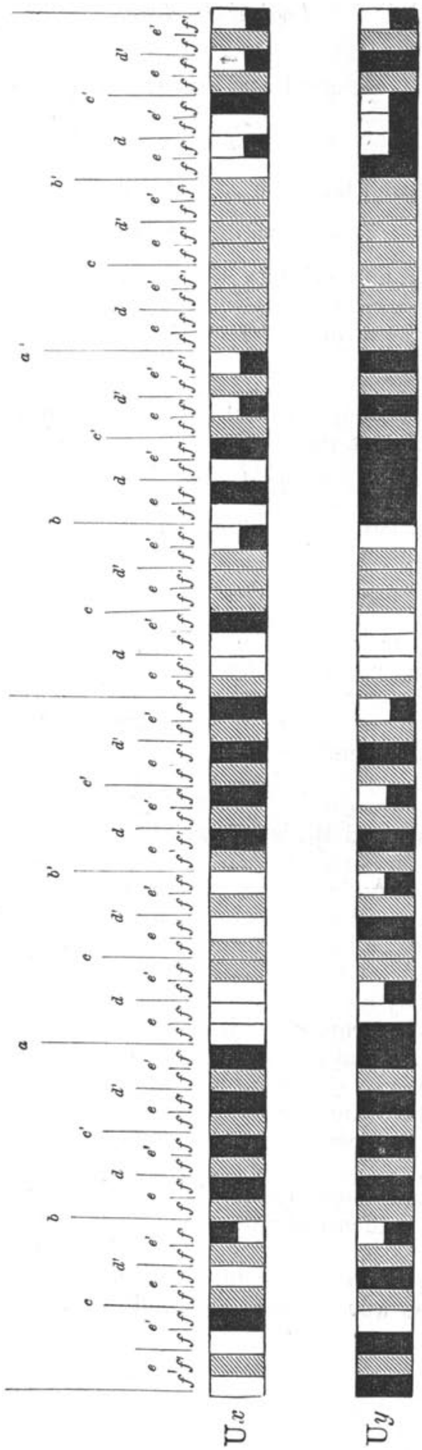


Diagram 9.