



LIX. The bakerian lecture.—On the theory of the astronomical refractions

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sensitive character of his preparation is lost, and the value of less sensitive modes is reduced in a still greater proportion.

Daguerre informs us, that under very favourable circumstances a drawing may be obtained by exposing his plates in the camera during from three to five minutes. If then, by correcting his focus, he were enabled to use a lens of equal power to the one by which the inclosed drawings were produced, he would be enabled to make the necessary impression in from ten to twelve seconds.

During the discussion which took place at the Institute, after M. Arago had publicly announced the process of Daguerreotype, it was allowed to be a great desideratum that the art might be applied to taking portraits from life. The use of large lenses, which the correction of the focus enables us to adopt, would, I should imagine, render such an application of the art practicable; and the value of each use to which this important invention is applied, must also be increased by a knowledge of the means of obtaining the best possible effect in the least possible time.

I am, Gentlemen, your obedient servant,
JOHN T. TOWSON.

LIX. *The Bakerian Lecture.—On the Theory of the Astronomical Refractions.* By JAMES IVORY, K.H., M.A., F.R.S. L. & E., *Instit. Reg. Sc. Paris, Corresp. et Reg. Sc. Götting. Corresp.*

[Continued from p. 109.]

Atmosphere of Air mixed with aqueous Vapour.

CONTINUING to represent the pressure and temperature at the earth's surface by p' and τ' , and the like quantities at the height z by p and τ , the symbols (ρ') , (ρ) may be used to denote the respective densities in the case of air mixed with aqueous vapour. When the pressure and density vary, all the gases, and mixtures of gases and vapours, are found to follow the same laws of dilatation and compression; and hence the same equations that express the equilibrium of an atmosphere of dry air, will hold equally in one of moist air. In the present case these equations will therefore be,

$$p = \int \frac{-dz \cdot (g)}{\left(1 + \frac{z}{a}\right)^2},$$

$$\frac{p}{p'} = \frac{1 + \beta \tau}{1 + \beta \tau'} \cdot \frac{(\rho)^*}{(\rho')} :$$

* This equation is equivalent to the one in p. 18 of M. Biot's dissertation, on which that author lays so much stress.

and, if we put

$$\sigma = \frac{z}{1 + \frac{z}{a}}, \quad \frac{1 + \beta \tau}{1 + \beta \tau'} = 1 - q, \quad \frac{(\rho)}{(\rho')} = c^{-u},$$

the same equations will be thus written,

$$p = (\rho') f - d \sigma c^{-u},$$

$$p = p' (1 - q) c^{-u}.$$

The three quantities σ , q , u are severally equal to zero at the surface of the earth: so that, by the same procedure as before, we shall obtain these formulas,

$$q = f u - (f - f') \frac{u^2}{2} + \&c.$$

$$\sigma = \frac{p'}{(\rho')} \cdot \left\{ u - f \cdot \frac{d \cdot c^{-u} R_2}{c^{-u} d u} - f' \cdot \frac{d d \cdot c^{-u} R_2}{c^{-u} d u^2} - \&c. \right.$$

But it is to be observed that, in these expressions, the coefficients $f, f', \&c.$, are not exactly the same as in an atmosphere of dry air: for the quantities mentioned, although they have determinate values in the same quiescent atmosphere, depend upon the manner in which the temperature q , or the height z , varies relatively to the density, or to u .

If we suppose that the height z is not very great, so that the powers of q may be neglected, we shall obtain from the foregoing equations,

$$z = \frac{p'}{(g')} \cdot \frac{1 + f}{f} \cdot q:$$

and hence

$$\frac{1 + f}{f} = \frac{1 + \beta \tau'}{\beta} \cdot \frac{(\rho')}{p'} \cdot \frac{z}{\tau' + \tau}.$$

In order to ascertain how far this value is different from the like value in the case of dry air, we must resolve the complex density (ρ') into its elements. The hygrometer will discover the tension of the vapour at the earth's surface; and if ϕ' denote this tension in inches of mercury, and g' be the density of dry air under the pressure p' and at the temperature τ' , the following equation is proved in all the late treatises on Natural Philosophy.

$$(g') = g' \left(1 - \frac{3}{8} \cdot \frac{\phi'}{p'} \right):$$

by means of which we obtain

$$\frac{1 + f}{f} = \frac{1 - \frac{3}{8} \cdot \frac{\phi'}{p'}}{\beta L} \times \frac{z}{\tau' - \tau},$$

$$L = \frac{p'}{\rho' (1 + \beta \tau')}.$$

Now the small additional factor in the value of $\frac{1+f}{f}$ is not taken into account in the measurement of heights by the barometer, no distinction being usually made between dry air and moist air. In order to form some estimate of its effect, we may instance the mean atmosphere of our climate, the temperature of which is 50° Fahrenheit; the greatest possible tension of vapour in such an atmosphere is .36 of an inch of mercury; at a medium, if we make $\phi' = .18$, and $p' = 30$ inches, we shall have,

$$1 - \frac{3}{8} \cdot \frac{\phi'}{p'} = 1 - \frac{1}{444}.$$

It thus appears that in our climate, when the mean portion of aqueous vapour is mixed with the air, the value of $\frac{1+f}{f}$ is less than it would be if the air were perfectly dry by its $\frac{1}{444}$ -th part, a quantity too minute to be perceptible in most experiments. A small part only of the refractions depend upon f , about a twelfth part of the whole at the horizon; so that, neglecting the minute variations which f undergoes by the greater or less portions of aqueous vapour mixed with the air, the effect of which on the refractions is insensible, we may assume that it has the same value in all atmospheres. The same thing applies with greater force to the other coefficients $f', f'', \&c.$, which having themselves hardly any influence on the refractions, their minute changes in different atmospheres may be wholly disregarded.

If we substitute for (ρ') its equivalent $\rho' \left(1 - \frac{3}{8} \cdot \frac{\phi'}{p'}\right)$ in the foregoing value of σ , we shall obtain the following equation, which is sufficient for the problem of the refractions in an atmosphere of moist air:

$$\sigma = \frac{1}{1 - \frac{3}{8} \cdot \frac{\phi'}{p'}} \cdot \frac{p'}{\rho'} \cdot \left\{ u - f \cdot \frac{d \cdot c^{-u} R_2}{c^{-u} d u} - f' \cdot \frac{d \cdot d \cdot c^{-u} R_4}{c^{-u} d u^2} - \&c. \right\} \dots\dots\dots (10.)$$

In which expression the coefficients $f, f', \&c.$, may be considered the same in all atmospheres, the quantity u varying from zero at the earth's surface to be infinitely great at the top of the atmosphere.

8. In the foregoing analysis, every formula has been strictly deduced from the equations of equilibrium: no quantities have been introduced except such as really exist in nature, and might be determined experimentally, if we had the means of exploring the phenomena of the atmosphere with the requisite accuracy. It may not be improper to notice here an obvious consequence of the equation

$$p = \rho' \int -d\sigma c^{-u},$$

which holds in an atmosphere of dry air; namely, that the integral

$$\int -d\sigma c^{-u},$$

being extended from the surface of the earth to the top of the atmosphere, is the analytical expression of $\frac{p'}{\rho'}$, or of the height of the homogeneous atmosphere, that is, of a column of air equiponderant to the whole atmosphere, and every part of which has the same density and the same weight which it would have at the surface of the earth. This height varies only with the temperature, and is thus determined:

$$\frac{p'}{\rho'} = \frac{p'}{\rho' (1 + \beta \tau')} \cdot (1 + \beta \tau') = \frac{p'}{D} (1 + \beta \tau') = L (1 + \beta \tau').$$

In like manner, in an atmosphere of air mixed with aqueous vapour, the same integral is equal to $\frac{p'}{(\rho')}$: and we have

$$\frac{p'}{(\rho')} = \frac{p'}{\rho'} \cdot \frac{1}{1 - \frac{3}{8} \cdot \frac{\phi'}{p'}} = \frac{L (1 + \beta \tau')}{1 - \frac{3}{8} \cdot \frac{\phi'}{p'}}.$$

Thus the analytical theory agrees in every respect with the real properties of the atmosphere, as far as these have been ascertained; and we now proceed to show that the same theory represents the astronomical refractions with a fidelity that can be deemed imperfect only in so far as the constants $f, f', &c.$, which can only be determined by experiment, are liable to the charge of inaccuracy.

9. The apparent zenith-distance of a star being represented by θ , and the refraction by $\delta \theta$, the following formulas have already been obtained (§ 2. equations (2.) and (3.)).

$$d \cdot \delta \theta = \frac{dy}{\sqrt{r^2 - y^2}},$$

$$y = a \sin \theta \times \sqrt{\frac{1 + 2 \phi (g')}{1 + 2 \phi (\rho)}},$$

the quantity $\delta \theta$ being supposed to increase from the surface

of the earth to the top of the atmosphere. For the sake of perspicuity, we shall, in the first place, confine our attention to an atmosphere of dry air, in which case it is known by experiment that the refractive power $\phi(g)$ is proportional to the density ρ ; so that

$$\phi(g) = K \times \rho,$$

K being a constant. Adverting to the mode of expression before used, we have

$$\rho = \rho' c^{-u};$$

and hence

$$\begin{aligned}\phi(g) &= K \times \rho = K \rho' \cdot c^{-u}, \\ y &= a \sin \theta \times \sqrt{\frac{1+2K\rho'}{1+2K\rho'c^{-u}}};\end{aligned}$$

and by introducing new symbols in order to abridge expressions,

$$\begin{aligned}\alpha &= \frac{K\rho'}{1+2K\rho'}, \\ \omega &= 1 - c^{-u}, \\ y &= \frac{a \sin \theta}{\sqrt{1-2\alpha\omega}}.\end{aligned}$$

Let this value of y be substituted in the differential of the refraction; then

$$\begin{aligned}r^2 &= (a+z)^2 = a^2 \left(1 + \frac{z}{a}\right)^2 = \frac{a^2}{\left(1 - \frac{\sigma}{a}\right)^2}, \\ d \cdot \delta \theta &= \sin \theta \times \frac{\alpha}{1-2\alpha\omega} \times \frac{d\omega}{\sqrt{\frac{1-2\alpha\omega}{\left(1 - \frac{\sigma}{a}\right)^2} - \sin^2 \theta}}.\end{aligned}$$

In further transforming this expression, it is to be observed that α is a very small fraction less than .0003; and if the atmosphere extend fifty miles above the earth's surface, $\frac{z}{a}$ or $\frac{\sigma}{a}$ when greatest will not exceed .012. If we now put

$$\frac{\sigma}{a} = \frac{s}{a} + \alpha\omega,$$

we shall have

$$\frac{1-2\alpha\omega}{\left(1 - \frac{\sigma}{a}\right)^2} = \frac{(1-\alpha\omega)^2 - \alpha^2\omega^2}{\left(1-\alpha\omega - \frac{s}{a}\right)^2} = 1 + 2\frac{s}{a} + 3\frac{s^2}{a^2},$$

the quantities rejected being plainly of no account relatively to those retained. Further, because ω is always less than 1,

$\frac{\alpha}{1-2\alpha\omega}$ is contained between α and $\alpha(1+2\alpha)$; and it may be taken equal to α , or to the mean value $\alpha(1+\alpha)$. Thus we have

$$d \cdot \delta \theta = \sin \theta \times \frac{\alpha(1+\alpha) du c^{-u}}{\sqrt{\cos^2 \theta + 2 \frac{s}{a} + 3 \frac{s^2}{a^2}}}.$$

Again, the formula (9.) gives

$$\sigma = s + a \cdot \alpha \omega = \frac{p'}{g'} \cdot \left\{ u - f' \cdot \frac{d \cdot c^{-u} R_2}{c^{-u} du} - f' \cdot \frac{dd \cdot c^{-u} R_4}{c^{-u} \cdot du^2} - \&c. \right\}.$$

Now, $\frac{p'}{g'} = \frac{p'}{g'(1+\beta\tau')} \cdot (1+\beta\tau') = L(1+\beta\tau')$:
and if we make

$$\begin{aligned} s \cdot \frac{g'}{p'} &= \frac{s}{L(1+\beta\tau')} = x, \\ \frac{p'}{g'} \cdot \frac{1}{a} &= \frac{L(1+\beta\tau')}{a} = i, \\ \frac{a \cdot p' \cdot \alpha}{p'} &= \frac{\alpha}{i} = \lambda, \end{aligned}$$

we shall have

$$\frac{s}{a} = ix$$

$$x = u - \lambda(1-c^{-u}) - f \cdot \frac{d \cdot c^{-u} R_2}{c^{-u} du} - f' \cdot \frac{dd \cdot c^{-u} R_4}{c^{-u} du^2} - \&c.$$

Let $\Psi(u)$ stand for all the terms in this value of x except the first, so that

$$x = u - \Psi(u):$$

from this we deduce by Lagrange's theorem,

$$c^{-u} = c^{-x} - c^{-x}(\Psi)(x) - \frac{1}{2} \cdot \frac{dd \cdot c^{-x} \Psi^2(x)}{dx} - \&c.$$

consequently,

$$du c^{-u} = dx c^{-x} + \frac{d \cdot c^{-x} \Psi(x)}{dx} dx + \frac{1}{2} \cdot \frac{dd \cdot c^{-x} \Psi^2(x)}{dx^2} dx + \&c.$$

By means of the values that have been found, the differential of the refraction can be expressed in terms of one variable x . In making the substitutions, the smallest term of the radical quantity is to be neglected in all the terms of $du c^{-u}$, except

the first and greatest; and the denominator of that term is to be expanded. Thus we obtain

$$d \cdot \delta \theta = \sin \theta \cdot \alpha (1 + \alpha) \cdot \left\{ \int \frac{dx}{\sqrt{\cos^2 \theta + 2ix}} \cdot \left(c^{-x} + \frac{d \cdot c^{-x} \Psi(x)}{dx} \right) + \frac{1}{2} \int \frac{dx}{\sqrt{\cos^2 \theta + 2ix}} \cdot \frac{dd \cdot c^{-x} \Psi^2(x)}{dx} - \frac{3}{2} \int \frac{dx \cdot c^{-x} \cdot i^2 x^2}{(\cos^2 \theta + 2ix)^{\frac{3}{2}}} \right\}$$

In order to estimate the relative magnitude of the several parts of this formula we must find the numerical values of the quantities α and i . If η stands for the refraction at 45° of altitude, determined very exactly from many astronomical observations, we shall have

$$\alpha = \eta (1 - 2i + 2\eta),$$

as will readily appear from the formula according to Cassini's method given in § 1. MM. Biot and Arago have ascertained the value of α with great exactness in a different way, by means of experiments on the gases with the prism. In some of the best attempts to determine α , the refractions at 45° of altitude, being reduced to the barometer 29.6 and to the temperature 50° Fahr., are as follows:

Dr. Brinkley	57.42
De Lambre	57.58
Bessel, Tab. Reg.	57.55
Experiments of MM. Biot and Arago	57.65

Mean 57.55

It appears that Bessel's determination has the best claim to be preferred: but as it differs very little from De Lambre's result, which is adopted in the paper of 1823, the same value will be retained in the calculations which follow. According to De Lambre, the value of α is $60''.616^*$ at the temperature 0° centigrade, and the barometric pressure $0^m.76$: wherefore, when the temperature is 50° Fahrenheit, and the pressure 30 inches ($= 0^m.762$), we shall have

$$\alpha = 60.616 \times \frac{762}{760 \times 1.0018} \times \frac{1}{1 + \frac{18}{480}} = 58''.47:$$

* *Tableaux Chronomiques, publiées par le Bureau des Longitudes de France.*

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and in parts of the radius,

$$\alpha = \cdot 0002835.$$

It has been found that $L = 4347\cdot 8$ fathoms at 0° centigrade or 32° Fahr.: wherefore, if we make $a =$ mean radius of the earth $= 3481280$ fathoms, we shall have at the temperature of our climate, or 50° Fahrenheit,

$$i = \frac{L (1 + \beta \tau')}{a} = \frac{4347\cdot 8 \left(1 + \frac{18}{483}\right)}{3481280} = \cdot 0012958;$$

and hence

$$\lambda = \frac{\alpha}{i} = \cdot 21878.$$

We can now inquire into the values of the last two terms of the foregoing formula for the refraction, both of which are very small. With respect to the first of them, we have

$$\Psi(x) = \lambda (1 - c^{-x}) + f \cdot \frac{d \cdot c^{-x} R_2}{c^{-x} dx} + f' \cdot \frac{d d \cdot c^{-x} R_4}{c^{-x} d x^2} - \&c.:$$

and, by performing the differential operations,

$$\Psi(x) = \lambda (1 - c^{-x}) + f (R_1 + R_2) + f' (R_2 + 3 R_3 + R_4);$$

and, by substituting the values of the functions,

$$h = 2f - \lambda = \cdot 22566$$

$$\Psi(x) = -h (1 - c^{-x}) + f x + 4 f' \left(1 - x + \frac{3 x^2}{8} - \frac{x^3}{24} - c^{-x}\right).$$

It might not be very objectionable to neglect the term multiplied by f' , for the same reasons that the terms which follow it are neglected, that is, both on account of the nature of the functions and because the coefficients are small: but, in order to leave no room for scruples respecting accuracy, the square of the entire expression set down, may be thus represented:

$$\Psi^2(x) = G - 8 h f' \cdot G' + 8 f f' \cdot G'' + 16 f'^2 \cdot G'''.$$

The integral in the term under consideration is greatest when the radical quantity in the denominator is least, that is, when $\cos \theta = 0$: and if the integration be performed between the limits $x = 0, x = \infty$, we shall obtain a result greater than if the integral were extended only to the top of the atmosphere. Now we have,

$$G = h^2 (1 - 2 c^{-x} + c^{-2x}) + 2 h f \cdot x c^{-x} - 2 h f \cdot x + f^2 \cdot x^2:$$

and, by operating on the terms separately, the part of the integral depending on G , will be as follows:

$$\int_0^\infty \frac{dx}{\sqrt{2ix}} \cdot \frac{d d \cdot c^{-x} G}{d x^2} =$$

$$\frac{\sqrt{\pi}}{\sqrt{2i}} \times \left(h^2 (1 - 4\sqrt{2} + 3\sqrt{3}) - 3hf(\sqrt{2}) - 1 \right) + \frac{3}{4}f^2 \\ = \frac{\sqrt{\pi}}{\sqrt{2i}} \times \cdot 00216.$$

The other parts depending on G' , G'' , G''' are complicated; but they are troublesome more on account of the number of terms they contain than from any difficulty in the integrations. The following results have been obtained:

$$8hf' \times \int_0^\infty \frac{dx}{\sqrt{2ix}} \cdot \frac{dd \cdot c^{-x} G'}{dx^2} = -f' \times \frac{\sqrt{\pi}}{\sqrt{2i}} \times \cdot 01759,$$

$$8ff' \times \int_0^\infty \frac{dx}{\sqrt{2ix}} \cdot \frac{dd \cdot c^{-x} G''}{dx^2} = -f' \times \frac{\sqrt{\pi}}{\sqrt{2i}} \times \cdot 02043,$$

$$16f'^2 \int_0^\infty \frac{dx}{\sqrt{2ix}} \cdot \frac{dd \cdot c^{-x} G'''}{dx^2} = +f'^2 \times \frac{\sqrt{\pi}}{\sqrt{2i}} \times \cdot 00855.$$

Collecting all the parts, the term sought is found, viz.

$$\frac{\alpha(1+\alpha)}{2} \cdot \int_0^\infty \frac{dx}{\sqrt{2ix}} \cdot \frac{dd \cdot c^{-x} \Psi^2(x)}{dx^2} = \\ \frac{\alpha(1+\alpha)\sqrt{\pi}}{\sqrt{2i}} \times (\cdot 00108 - f' \times \cdot 00142 + f'^2 \times \cdot 00427).$$

To this must be added the other term, which, being integrated in the same circumstances, gives,

$$-\frac{3}{2} \cdot \int_0^\infty \frac{dx c^{-x}}{\sqrt{2ix}} \times \frac{ix}{2} = -\frac{3}{8} \cdot \frac{\sqrt{\pi}}{\sqrt{2i}} = -\frac{\sqrt{\pi}}{\sqrt{2i}} \times \cdot 00049.$$

It thus appears that the two small terms of the expression of the refraction are, together, equal to

$$\frac{\alpha(1+\alpha)\sqrt{\pi}}{\sqrt{2i}} \cdot (\cdot 00059 - f' \times \cdot 00142 + f'^2 \times \cdot 00427):$$

and as
$$\frac{\alpha(1+\alpha)\sqrt{\pi}}{\sqrt{2i}} = 2036'' \cdot 5,$$

the greatest amount of both is about $1''$.

The whole refraction will therefore be thus expressed:

$$d\delta\theta = \sin\theta \times \alpha(1+\alpha) \cdot \int \frac{dx}{\sqrt{\cos^2\theta + 2ix}} \cdot \left(c^{-x} + \frac{d \cdot c^{-x} \Psi(x)}{dx} \right),$$

with the assurance that the error cannot exceed $1''$. If we substitute what $\Psi(x)$ stands for, we shall have

$$d \cdot \delta \theta = \sin \theta \times \alpha (1 + \alpha) \times \int \frac{dx}{\sqrt{\cos^2 \theta + 2ix}} \times \\ (c^{-x} + \lambda \cdot \frac{d \cdot (-x - c^{-2x})}{dx} + f \cdot \frac{d \cdot d \cdot c^{-x} R_2}{d x^2} + f' \cdot \frac{d^3 \cdot c^{-x} R_4}{d x^3}) + \&c.$$

This expression being regular, it may be continued to any number of terms, and it has the advantage of being linear with respect to the coefficients. Adverting to what x stands for, it will appear that $L \times x$ is nearly equal to s , or to z , that is, to the elevation in the atmosphere; so that, if we suppose the greatest height of the atmosphere is $10 \times L$, or about fifty miles, the greatest value of x will be 10; and all the integrals in the foregoing expression must be taken between the limits zero and 10. But the quantity c^{-x} is so small when x has increased to 8 or 10, that the results are not sensibly different whether the integrals be extended to those limits or be continued to infinity. By substituting the values of the functions, the expression of $\delta \theta$ will take this form :

$$\delta \theta = \sin \theta \times \alpha (1 + \alpha) \times \left\{ \int \frac{dx c^{-x}}{\sqrt{\cos^2 \theta + 2ix}} \right. \\ \left. + \lambda \int \frac{dx (2c^{-2x} - c^{-x})}{\sqrt{\cos^2 \theta + 2ix}} \right. \\ \left. - f \int \frac{dx}{\sqrt{\cos^2 \theta + 2ix}} \cdot (4c^{-2x} - 3c^{-x} + x c^{-x}) \right. \\ \left. + f' \int \frac{dx}{\sqrt{\cos^2 \theta + 2ix}} \cdot (8c^{-2x} - 8c^{-x} + 7x c^{-x} \right. \\ \left. - 2x^2 c^{-x} + \frac{x^3 c^{-x}}{6}) \right. \\ \left. - f'' \int \frac{dx}{\sqrt{\cos^2 \theta + 2ix}} \cdot (16c^{-x} - 16c^{2-x} + 16x c^{-x} \right. \\ \left. - \frac{15}{2} x^2 c^{-x} \right. \\ \left. + \frac{11}{6} x c^{-x} - \frac{5}{24} x^4 c^{-x} + \frac{x^5 c^{-x}}{120}) \right\}.$$

In order to illustrate the rapidity with which the terms decrease, it may be proper to find the limit of $\delta \theta$, by making $\cos^2 \theta = 0$, and integrating between the limits $x = 0$, $x = \infty$; which limit is not sensibly different from the refraction at the horizon. Now it will be found that, in the circumstances mentioned,

$$\delta \theta = \frac{\alpha (1 + \alpha) \sqrt{\pi}}{\sqrt{2i}} \times \left\{ 1 + \lambda (\sqrt{2} - 1) \right.$$

$$\begin{aligned}
 & -f \left(2\sqrt{2} - \frac{5}{2} \right) \\
 & + f' \left(4\sqrt{2} - \frac{91}{16} \right) \\
 & - f'' \left(8\sqrt{2} - \frac{2895}{256} \right) \\
 & - \&c. :
 \end{aligned}$$

or, in seconds,

$$\delta\theta = 2072'' \cdot 46 - f'' \times 62 \cdot 4 - f'' \times 10'' \cdot 2 - \&c.$$

From this calculation it appears that the term multiplied by f'' and all the subsequent terms are too small to be sensible; and as f' is much less than f , even the term multiplied by f' can hardly exceed a few seconds at low altitudes. There is great probability that the horizontal refraction is very near $34' 30''$, and does not exceed this quantity.

To prepare the foregoing expression of $\delta\theta$ for integration, put

$$m = 10, \frac{\sqrt{2im}}{\cos\theta} = \tan\phi, e = \tan\frac{\phi}{2};$$

then

$$\cos^2\theta = \frac{(1-e^2)^2}{4e^3} \times 2im,$$

$$\sqrt{\cos^2\theta + 2im} = \frac{\sqrt{5i}}{e} \cdot \sqrt{(1-e^2)^2 + 4e^2} \cdot \frac{x}{m} = \frac{\sqrt{5i}}{e} \cdot \Delta;$$

and we shall have

$$\delta\theta = \sin\theta \times \frac{\alpha(1+\alpha)}{\sqrt{5i}} \times \left\{ \begin{aligned}
 & \int_0^m \frac{e dx}{\Delta} \cdot c^{-x} \\
 & + \lambda \int_0^m \frac{e dx}{\Delta} \cdot (2c^{-2x} - c^{-x}) \\
 & - f \int_0^m \frac{e dx}{\Delta} \cdot (4c^{-2x} - 3c^{-x} + xc^{-x}) \\
 & + f' \int_0^m \frac{e dx}{\Delta} \cdot (8c^{-2x} - 8c^{-x} + 7xc^{-x} - 2x^2c^{-x} \\
 & \quad + \frac{x^3}{6} c^{-x}).
 \end{aligned} \right\} \quad (C.)$$

For the sake of abridging, the several integrals in succession may be represented by Q_0, Q_1, Q_2, Q_3 ; so that the value of $\delta\theta$ will be thus written:

$$\delta\theta \sin\theta \times \frac{\alpha(1+\alpha)}{\sqrt{5i}} \cdot (Q_0 + \lambda Q_1 - f Q_2 + f' Q_3).$$

[To be continued.]