ordinary Algebra, differing from the latter in the substitution of three arbitrary quantities z, i, and u for the quantities 0, 1, and ∞.

Mr. Tucker read a Note, A Theorem in Conics, by the Rev. T. C. Simmons, M.A.

The following presents were received:—

"Proceedings of the Royal Society," Vol. XXXIX., No. 241.

"Proceedings of the Cambridge Philosophical Society," Vol. V., Part 5, Mich. 1885.

"Mathematical Questions, with their Solutions, from the 'Educational Times,'" Vol. XLIV.

"Educational Times," April, 1886.

"Proceedings of the Canadian Institute," Third Series, Vol. III., Fasc. No. 3; Toronto, 1886.

"Jahrbuch über die Fortschritte der Mathematik," XV., 2, Jahrgang 1883.

"Bulletin des Sciences Mathématiques," T. X., March and April, 1886.

"Bulletin de la Société Mathématique de France," T. XIV., No. 1.

"Beiblätter zu den Annalen der Physik und Chemie," B. X., St. 3, 1886.

"Catalogue de la Bibliothèque de l'Ecole Polytechnique," 8vo; Paris, 1881.

"Atti della R. Accademia dei Lincei—Rendiconti," Vol. II., F. 4, 5, 6, Feb., March, 1886.

"Atti del R. Istituto Veneto," T. II., Ser. V., Disp. 3 to 10; T. III., Ser. VI., Disp. 1 to 9; 1883–85.

"Memorie del R. Istituto Veneto," Vol. XXII., Parts I. and II.; di Scienze, Lettere, ed Arti, 1884–85.

On some Results connected with the Theory of Reciprocants.

By C. LEUDESDORF, M.A.

[*Read April 8th*, 1886.]

1. Let x and y be two variables connected by any relation, and let $y_1, y_2, \dots$ denote the successive differential coefficients of y with respect to x, and $x_1, x_2, \dots$ those of x with respect to y. Then

$$\begin{aligned}
x_1 &= 1 && \div\, y_1,\\
x_2 &= -y_2 && \div\, y_1^3,\\
x_3 &= -y_1y_3+3y_2^2 && \div\, y_1^5,\\
x_4 &= -y_1^2y_4+10y_1y_2y_3-15y_2^3 && \div\, y_1^7,
\end{aligned}$$

and so on. If the numerators on the right be denoted by Y_1, Y_2, ..., we notice the following properties of Y_n :—

It is homogeneous, and every term of it is of degree $n-1$.

It is isobaric, and every term of it is of weight $2(n-1)$ (if the weight of y_r be taken to be r).

It only involves y_n once, in the term $-y_1^{n-2}y_n$.

The denominator corresponding to it is y_1^{2n-1}.

2. Let $R(y_1, y_2, ...)$ be a reciprocant every term of which is of degree i and weight w, and which is equal to $qy_1^{\lambda}R(x_1, x_2, ...)$, where $q = \pm 1$. Then evidently q is $+1$ or -1 according as the number of factors in that term of R which contains the highest power of y_1 is (neglecting the power of y_1) even or odd. For

$$y_1 = +\frac{x_1}{x_1^2},$$

but
$$y_2 = -\frac{x_2}{x_1^3}, \quad y_3 = -\frac{x_3}{x_1^4}+...,$$

and in general
$$y_r = -\frac{x_r}{x_1^{r+1}}+....$$

If R does not contain y_1, then evidently $q = (-1)^i$.

Again, λ must be equal to the sum of the degree (i) and the weight (w) of R. For, if we write ky for y (where k is a constant), y_r^m becomes $k^m y_r^m$, but x_r^m becomes $k^{-mr}x_r^m$; thus every term of $R(y_1, y_2, ...)$ will be multiplied by k^i and every term of $R(x_1, x_2, ...)$ will be multiplied by k^{-w}; therefore

$$k^i R(y_1, y_2, ...) = qk^{\lambda}y_1^{\lambda}k^{-w}R(x_1, x_2, ...),$$

so that $k^i = k^{\lambda-w}$, which shows that $\lambda = w+i$. More generally, it is seen that any homogeneous and isobaric function $F(y_1, y_2, ...)$ of degree i and weight w will, if transformed by means of the formulæ reciprocal to those in § 1, become a homogeneous and isobaric function $y_1^{w+i}\Phi(x_1, x_2, ...)$ of x_1, x_2, ...

3. Let now F be *any* function of y_1, y_2, ...; it can of course be expressed in terms of x_1, x_2, ...; let it become $x_1^{-\lambda}\Phi(x_1, x_2, ...)$ when so expressed. If we write $x-y\theta$ (where θ is an infinitesimal) in place of x, and leave y unaltered, the change in Φ will be

$$-\theta\frac{d}{dx_1}(\Phi x_1^{-\lambda});$$

for the change makes x_1 into $x_1-\theta$, and x_2, x_3, ... are unaffected by it. Let us examine the effect of the change upon F.

We have $\delta y = 0$, $\delta x = -y\theta$, and therefore, writing $\omega = \delta y - y_1\,\delta x = yy_1\theta$,

$$\delta y_1 = \omega' - yy_2\theta = y_1^2\theta,$$

$$\delta y_2 = \omega'' - yy_3\theta = 3y_1y_2\theta,$$

$$\delta y_3 = \omega''' - yy_4\theta = (4y_1y_3+3y_2^2)\,\theta,$$

and so on. Therefore

$$\delta F = \frac{dF}{dy_1}\delta y_1 + \frac{dF}{dy_2}\delta y_2 + \frac{dF}{dy_3}\delta y_3 + \dots$$

$$= \left[y_1^2\frac{d}{dy_1} + 3y_1y_2\frac{d}{dy_2} + (4y_1y_3+3y_2^2)\frac{d}{dy_3} + \dots\right]F\theta$$

$$= \left[-y_1^2\frac{d}{dy_1} + y_1\left(2y_1\frac{d}{dy_1} + 3y_2\frac{d}{dy_2} + \dots\right) + V\right]F\theta,$$

where V is the well-known operator

$$3y_2^2\frac{d}{dy_3} + 10y_2y_3\frac{d}{dy_4} + \dots,$$

the coefficient of $\dfrac{d}{dy_m}$ in V being

$$\frac{d^m(yy_1)}{dx^m} - yy_{m+1} - (m+1)\,y_1y_m,$$

that is, $$c_2^m y_2y_{m-1} + c_3^m y_3y_{m-2} + \dots + c_{m-1}^m y_{m-1}y_2,$$

c_r^m denoting the number of combinations of m things taken r together. Equating the changes in F and Φ, there results

$$y_1^2\frac{dF}{dy_1} - y_1\left(2y_1\frac{d}{dy_1} + 3y_2\frac{d}{dy_2} + \dots\right)F - VF = \frac{d(\Phi x_1^{-\lambda})}{dx_1} \quad\text{......(1).}$$

In the case where F is homogeneous and isobaric, this last equation can be simplified. For, if F be of degree i and weight w, then, by Euler's theorem and the isobaric theorem,

$$\left(y_1\frac{d}{dy_1} + y_2\frac{d}{dy_2} + \dots\right)F = iF,$$

$$\left(y_1\frac{d}{dy_1} + 2y_2\frac{d}{dy_2} + \dots\right)F = wF,$$

so that the equation reduces in this case to

$$y_1^2 \frac{dF}{dy_1} - (w+i)\, y_1 F - VF = \frac{d}{dx_1} (\Phi x_1^{-\lambda}),$$

or, since $\lambda = w+i$ (§ 2), to

$$y_1^2 \frac{dF}{dy_1} - VF = y_1^{(w+i)} \frac{d\Phi}{dx_1} \quad \text{......................}(2).$$

If F is a *pure* function, *i.e.*, one which does not contain y_1, then

$$VF = -y_1^{w+i} \frac{d\Phi}{dx_1} \text{............................}(3),$$

a result in which is included the well-known proposition that if F is a *pure reciprocant* $\qquad VF = 0.$

4. The following preliminary proposition will be required in § 5.

Let $F(y_2, y_3, \ldots)$ be a rational homogeneous isobaric function, of degree i and weight w, of the differential coefficients of y with respect to x (*excluding the first*). If $y_1^{-(w+i)} F$ is such that it is unchanged by the substitution of $x - y\theta$ for x (where θ is an infinitesimal), y remaining unaltered, then F must be a pure reciprocant.

To prove this, let the substitutions

$$(1)\ \left.\begin{aligned} x &= X - Y \\ y &= Y \end{aligned}\right\}, \qquad (2)\ \left.\begin{aligned} X &= X' \\ Y &= X' + Y' \end{aligned}\right\}, \qquad (3)\ \left.\begin{aligned} X' &= \xi - \eta \\ Y' &= \eta \end{aligned}\right\}$$

be made successively in $y_1^{-(w+i)} F$. Since, by hypothesis, this function is unchanged when $x - y\theta$, y are written for x, y respectively, it follows that any number of such infinitesimal changes made successively in x will have no effect on the function, and therefore that we may write $x - y$ for x and y for y without altering it. Therefore

$$y_1^{-(w+i)} F(y_2, y_3, \ldots) = Y_1^{-(w+i)} F(Y_2, Y_3, \ldots) \text{ ....}(4),$$

where Y_r denotes $\dfrac{d^r Y}{dX^r}$.

Now let the second substitution be made in the right-hand member of (4). We have

$$\frac{dY}{dX} = 1 + \frac{dY'}{dX'};$$

but

$$\frac{d^2 Y}{dX^2} = \frac{d^2 Y'}{dX'^2}, \text{ \&c.,}$$

and so in general $Y_r = Y'_r$ except when $r = 1$. Accordingly

$$Y_1^{-(w+i)} F(Y_2, Y_3, \ldots) = (1+Y'_1)^{-(w+i)} F(Y'_2, Y'_3, \ldots)$$
$$= Y_1'^{-(w+i)} F(Y'_2, Y'_3, \ldots) \left(\frac{Y'_1}{1+Y'_1}\right)^{w+i} \quad \ldots\ldots(5).$$

Now let the third substitution be made in the right-hand member of (5); it will evidently become

$$\eta_1^{-(w+i)} F(\eta_2, \eta_3, \ldots)\, \eta_1^{w+i}$$

$\left(\text{where } \eta_r \text{ denotes } \dfrac{d^r\eta}{d\xi^r}\right)$,

that is, $F(\eta_2, \eta_3, \ldots)$ simply.

But now $\quad x = X - Y = X' - (X'+Y') = -\eta,$

$$y = Y \quad = X'+Y' \quad = \xi,$$

so that the effect of the train of three substitutions is to change x into $-y$, and y into x. And, since $\eta_2 = -x_2$, $\eta_3 = -x_3$, &c., therefore $F(\eta_2, \eta_3, \ldots)$ is equal to $(-1)^i F(x_2, x_3, \ldots)$. It has therefore been shown

that $\quad y_1^{-(w+i)} F(y_2, y_3, \ldots) = (-1)^i F(x_2, x_3, \ldots);$

i.e., F is a pure reciprocant.

5. The results of the preceding articles may now be made use of to prove the converse of the proposition mentioned at the end of § 3; viz., that if F is a rational integral homogeneous isobaric function of $y_2, y_3, \ldots$, then, if $VF = 0$, F must be a pure reciprocant.

Let x be changed into $x - y\theta$, as in § 3; then, as already seen, F is changed to $F + \delta F$, where

$$\delta F = \left\{ -y_1^2 \frac{dF}{dy_1} + (w+i)\, F + VF \right\} \theta = (w+i)\, F\theta,$$

the other terms vanishing by hypothesis. Since $\delta y_1 = y_1^2\theta$, this can

be written $\quad \delta \left\{ y_1^{-(w+i)} F \right\} = 0,$*

which shows (§ 4) that F is a pure reciprocant.

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* This result may also be seen from equation (3) of § 3, which shows that, if $F(y_2, y_3, \ldots)$ be transformed by substituting for $y_2, y_3, \ldots$ their values in terms of $x_1, x_2, x_3, \ldots$, and become $x_1^{-(w+i)} \Phi(x_1, x_2, x_3, \ldots)$, then, when $VF = 0$, also $\dfrac{d\Phi}{dx_1} = 0$; that is, Φ does not involve x_1. Accordingly

$$\delta \left\{ y_1^{-(w+i)} F \right\} = \delta\Phi(x_2, x_3, \ldots) = 0.$$

6. Let F now stand again for *any* function of $y_1, y_2, \ldots$; and let an infinitesimal orthogonal change be given to x and y; *i.e.*, let x become $x-y\theta$ and y become $y+x\theta$, where θ is infinitesimal. Then proceeding as in § 3 to find the change in F, we have

$$\omega = \delta y - y_1\,\delta x = (x+yy_1)\,\theta,$$

therefore
$$\delta y_1 = (1+y_1^2)\,\theta,$$

and δy_2, δy_3, &c. have the same values as given in § 3. Thus

$$\delta F = \left\{ (1+y_1^2)\frac{dF}{dy_1} + y_1\left(3y_2\frac{dF}{dy_2} + 4y_3\frac{dF}{dy_3} + \ldots\right) + VF \right\}\theta \;\ldots\ldots(6).$$

In the case, then, where F is an absolute orthogonal reciprocant O,

$$(1+y_1^2)\frac{dO}{dy_1} + y_1\left(2y_1\frac{dO}{dy_1} + 3y_2\frac{dO}{dy_2} + \ldots\right) + VO = 0 \quad \ldots\ldots(7),$$

or, say,
$$U\,.\,O = 0.$$

If O be an orthogonal reciprocant, but no longer an absolute one, then we can make it into an absolute one by dividing it by a suitable power of y_2; if this power be the k^{th}, then

$$U\,.\,O = 3ky_1 O \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

For
$$\begin{aligned} U(Oy_2^{-k}) &= y_2^{-k}UO - ky_2^{-(k+1)}O\,\delta y_2\,\theta^{-1} \\ &= y_2^{-k}UO - 3ky_1 y_2^{-k}O \\ &= y_2^{-k}(UO - 3ky_1 O) \;\ldots\ldots\ldots\ldots\ldots\ldots(9). \end{aligned}$$

7. If F is a function of $y_1, y_2, \ldots$ such that

$$UF = \mu y_1 F,$$

where μ is some constant, then F must be a reciprocant, and an orthogonal one; such, moreover, that $y_2^{-\frac{1}{3}\mu}F$ is an *absolute* orthogonal reciprocant.

This proposition, the converse of that given in equation (7) of § 6, is easily proved. For, if $x-y\theta$, $y+x\theta$ be written for x and y, as in § 6, the change made in $y_2^{-\frac{1}{3}\mu}F$

$$\begin{aligned} &= U(y_2^{-\frac{1}{3}\mu}F) && \text{by (7)} \\ &= y_2^{-\frac{1}{3}\mu}(UF - \mu y_1 F) && \text{by (9)} \\ &= 0, \end{aligned}$$

by hypothesis.

Therefore $y_2^{-\frac{1}{3}\mu}F$ is not altered by an infinitesimal orthogonal change given to x and y; and therefore is not altered by any number of such changes made successively; that is to say, by any orthogonal change in the variables. In other words, it is an absolute orthogonal reciprocant.

8. Let $R(y_1, y_2, \ldots)$ be any reciprocant; let it be made absolute by division by a suitable power of y_2, say the k^{th}. Thus,

$$y_2^{-k}R(y_1, y_2, \ldots) = \pm x_2^{-k}R(x_1, x_2, \ldots),$$

so that, by equation (1) of § 3,

$$\left\{ y_1^2\frac{d}{dy_1} - y_1\left(2y_1\frac{d}{dy_1} + 3y_2\frac{d}{dy_2} + \ldots\right) - V\right\}(y_2^{-k}R) = \pm\frac{d}{dx_1}(x_2^{-k}R).$$

But having regard to the value of $U(y_2^{-k}R)$, as given in (7) of § 6, this may be written

$$\left(\frac{d}{dy_1} - U\right)(y_2^{-k}R) = \pm\frac{d}{dx_1}(x_2^{-k}R),$$

so that $$U(y_2^{-k}R) = y_2^{-k}\frac{dR}{dy_1} \mp x_2^{-k}\frac{dR}{dx_1}.$$

This last equation shows that, if R is an orthogonal reciprocant, $\dfrac{dR}{dy_1}$ must be a reciprocant; and that, conversely, if R is a reciprocant such that $\dfrac{dR}{dy_1}$ is also a reciprocant, then R must be an orthogonal one. These are of course well-known results, due to Professor Sylvester.

9. In § 3, let F stand for $y_1^{-(2n-1)}Y_n$ (Y_n was defined in § 1); then $\Phi = x_n$, thus equation (1) of § 3 will give

$$\left[-y_1^2\frac{d}{dy_1} + y_1\left(2y_1\frac{d}{dy_1} + 3y_2\frac{d}{dy_2} + \ldots\right) + V\right][Y_n y_1^{-(2n-1)}] = 0,$$

or $$\left\{-y_1^2\frac{d}{dy_1} + y_1\left(2y_1\frac{d}{dy_1} + 3y_2\frac{d}{dy_2} + \ldots\right) + V\right\}Y_n = (2n-1)\,y_1 Y_n,$$

or $$-y_1^2\frac{dY_n}{dy_1} + (n-1+2n-2)\,y_1 Y_n + VY_n = (2n-1)\,y_1 Y_n$$

[since Y_n is of degree $n-1$ and weight $2(n-1)$],

which reduces to

$$y_1^2\frac{dY_n}{dy_1} - (n-2)\,y_1 Y_n - VY_n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

10. The equation (10) may be put into a very simple form in the following manner. But at this point it is convenient to abandon the notation used so far, and to take the usual one; I write, then, t in place of y_1, and $a, b, c, \dots$ in place of $y_2, y_3, y_4, \dots$. This done, (10)

takes the form $$t^2\frac{dY_n}{dt}-(n-2)\,tY_n-VY_n=0 \quad\text{.................(11).}$$

Let Y_n be written in the form

$$t^{n-2}A_0+t^{n-3}qA_1+t^{n-4}q^2A_2+\dots+q^{n-2}A_{n-2},$$

where $A_0, A_1, \dots$ are *pure* functions (*i.e.*, they do not involve t), and $q=1$ is a quantity put in to make the expression homogeneous. Then, since Y_n is homogeneous and of degree $n-2$, if considered as a quantic in t and q,

$$t\frac{dY_n}{dt}+q\frac{dY_n}{dq}=(n-2)\,Y_n,$$

therefore $$t^2\frac{dY_n}{dt}-(n-2)\,tY_n+tq\frac{dY_n}{dq}=0,$$

subtracting which from (11) (in which the VY_n must be multiplied by q to make the equation homogeneous), we see that the latter takes the very simple form

$$VY_n=-t\frac{dY_n}{dq} \quad\text{..............................(12).}$$

The effect of the operator U on Y_n may also be noticed; we have

$$U.\,Y_n=(1-t^2)\frac{dY_n}{dt}+3\,(n-1)\,tY_n+VY_n=\frac{dY_n}{dt}+(2n-1)\,tY_n \text{...(13)},$$

substituting for VY_n from (11).

11. It is clear that by means of the Y functions any number of reciprocants can be formed. For, if we take any homogeneous and isobaric function of $Y_m, Y_n, Y_p, \dots$ and add to (or subtract from) it the same function of $y_m, y_n, y_p, \dots$ multiplied by any power of y_1 or t, we have an expression which does not change in value when y and x are written one for the other; *i.e.*, a reciprocant. But there will be a change in *sign* in those expressions which are obtained by subtraction; those obtained by addition will be unaltered even in sign when x and y are interchanged. That is to say, the addition method will give reciprocants of positive character, and the subtraction method reciprocants of negative character.

The simplest set of reciprocants which can be formed in this way are obtained by adding y_n multiplied by any power of y_1 to Y_n, and by subtracting the same expressions. If X_n denote the same function of $x_1, x_2, \ldots$ that Y_n is of $y_1, y_2, \ldots$, we have

$$\begin{aligned} \pm y_n y_1^{\lambda} + Y_n &= \pm \frac{X_n}{x_1^{2n-1}} \frac{1}{x^{\lambda}} + x_n y_1^{2n-1} \\ &= y_1^{2n+\lambda-1} \{ \pm X_n + x_n x_1^{\lambda} \} \\ &= \pm y_1^{2n+\lambda-1} \{ \pm x_n x_1^{\lambda} + X_n \}. \end{aligned}$$

If, then, $y_n t^{\lambda}$ be added to (subtracted from) Y_n, the result is a reciprocant of positive (negative) character, and of index $2n+\lambda-1$.

Writing down the Y's in the ordinary notation,

$$\begin{aligned} Y_1 &= 1, \\ Y_2 &= -a, \\ Y_3 &= -tb+3a^2, \\ Y_4 &= -t^2c+10tab-15a^3, \\ Y_5 &= -t^3d+t^2(15ac+10b^2)-105ta^2b+105a^4, \end{aligned}$$

it is seen at once that, *e.g.*, $-tb+Y_3$ is the Schwarzian, t^2c+Y_4 is $5a$ times the Schwarzian,— $tc+Y_4$ is $2t$ times the post-Schwarzian less the negative reciprocant $15a^3$, while $-b+tY_3$ and $-c+Y_4$ are well-known orthogonal reciprocants, &c., &c. If λ be chosen so as to be equal to $n-2$, we derive the most important species of reciprocants belonging to this class, viz., the homogeneous ones. They form the series

$$\begin{aligned} N_2 &= -2a, \\ N_3 &= -2tb+3a^2, \\ N_4 &= -2t^2c+10tab-15a^3, \text{ \&c.,} \end{aligned}$$

all of negative character; and

$$\begin{aligned} P_3 &= 3a^2, \\ P_4 &= 10tab-15a^3, \\ P_5 &= 15t^2ac+10t^2b^2-105ta^2b+105a^4, \text{ \&c.,} \end{aligned}$$

all of positive character.

N_n and P_n may with fitness be called the *fundamental* mixed homogeneous reciprocants (of negative and positive character respectively) of order n.

12. The equations giving the values of the Y's in terms of y_1, y_2, &c., may be written in the form

$$y_2 = -Y_2,$$
$$y_1 y_3 = -Y_3 + 3y_2^2,$$
$$y_1^2 y_4 = -Y_4 + \text{a function of } y_1 y_3 \text{ and } y_2,$$
$$y_1^3 y_5 = -Y_5 + \text{a function of } y_1^2 y_4,\ y_1 y_3, \text{ and } y_2,$$
&c., &c.,

and, generally,

$$y_1^{n-2} y_n = -Y_n + \text{a function of } y_1^{n-3} y_{n-1},\ y_1^{n-4} y_{n-2},\ \ldots, \text{ and } y_2.$$

Accordingly, by successive substitutions, $y_1^{n-2} y_n$ may be expressed as a function of Y_n, Y_{n-1}, ..., Y_2. It follows that any homogeneous isobaric function f of $y_1, y_2, \ldots y_n$ can, by successive substitutions, be expressed as a function ϕ of $Y_2, Y_3, \ldots Y_n$, divided by some power of Y_1; and, since $Y_1 = 1$, such function can be made homogeneous and isobaric by suitably inserting various powers of Y_1. If

$$f(y_1, y_2, \ldots y_n) = y_1^{-\lambda} \phi(Y_1, Y_2, \ldots Y_n),$$

it is readily seen that $\lambda = w - 2i$, where i, w are the degree and weight of f, considering y_r as of weight r.

For any term $y_m^{\alpha} y_n^{\beta} y_p^{\gamma} \ldots$ in f will give rise (among others) to a term

$$y_1^{-\alpha(m-2)} Y_m^{\alpha}\, y_1^{-\beta(n-2)} Y_n^{\beta}\, y_1^{-\gamma(p-2)} Y_p^{\gamma} \ldots$$

in ϕ. But this is $Y_m^{\alpha} Y_n^{\beta} Y_p^{\gamma} \ldots$ divided by y_1 raised to the power

$$(m\alpha + n\beta + p\gamma \ldots) - 2(\alpha + \beta + \gamma + \ldots),$$

i.e., to the power $w - 2i$.

We may then write

$$y_1^{w-2i} f(y_1, y_2, \ldots y_n) = \phi(Y_1, Y_2, \ldots Y_n) \quad \ldots\ldots\ldots\ldots(14).$$

The expression on the left of (14) is such that its weight is double its degree (as is the case with the Y functions). For the weight is $w - 2i + w$, that is, $2(w-i)$; and the degree is $w - 2i + i$, that is, $w - i$. Consequently ϕ will satisfy the relations

$$(w-i)\,\phi = Y_1 \frac{d\phi}{dY_1} + Y_2 \frac{d\phi}{dY_2} + Y_3 \frac{d\phi}{dY_3} + \&c. \quad \ldots\ldots\ldots\ldots(15),$$

$$2(w-i)\,\phi = Y_1 \frac{d\phi}{dY_1} + 2Y_2 \frac{d\phi}{dY_2} + 3Y_3 \frac{d\phi}{dY_3} + \&c. \quad \ldots\ldots\ldots(16).$$

In particular, if f be a reciprocant R of degree i and weight w, then, as in § 2,

$$R(y_1, y_2, \dots y_n) = \pm y_1^{w+i} R(x_1, x_2, \dots x_n)$$
$$= \pm y_1^{w+i} R(Y_1 y_1^{-1}, Y_2 y_1^{-3}, \dots Y_n y_1^{-(2n-1)})$$
$$= \pm y_1^{w+i} y_1^{-(2w-1)} R(Y_1, Y_2, \dots Y_n),$$

since any term in R such as $y_m^{\alpha} y_n^{\beta} \dots$ gives rise to a term

$$Y_m^{\alpha} y_1^{-(2m-1)\alpha} Y_n^{\beta} y_1^{-(2n-1)\beta} \dots \quad \text{or} \quad Y_m^{\alpha} Y_n^{\beta} y_1^{-(2w-i)};$$

therefore $\quad y_1^{w-2i} R(y_1, y_2, \dots y_n) = \pm R(Y_1, Y_2, \dots Y_n)$

$$= \pm Y_1^{w-2i} R(Y_1, Y_2, \dots Y_n) \dots\dots\dots (17),$$

(since $Y_1 = 1$); *i.e.*, the reciprocant on the left-hand side of (17), when expressed in terms of the Y's, takes exactly the same form, except for a possible change of sign.

As an example, take the reciprocant $y_1 y_4 - 5 y_2 y_3$, of degree 2 and weight 5. We have

$$y_1 y_4 - 5 y_2 y_3 = -y_1^{5+2}(x_1 x_4 - 5 x_2 x_3)$$
$$= -y_1^{7} y^{-(2.5-2)}(Y_1 Y_4 - 5 Y_2 Y_3)$$
$$= -y_1^{-1}(Y_1 Y_4 - 5 Y_2 Y_3);$$

therefore $\quad y_1^2 y_4 - 5 y_1 y_2 y_3 = -(Y_1^2 Y_4 - 5 Y_1 Y_2 Y_3),$

where each expression is of degree 3 and weight 2×3 in its coefficients.

From what has been said above, it is clear that any homogeneous isobaric function of $Y_2, Y_3, \dots Y_n$ (of degree i' and weight w', taking Y_r as of weight r') can be expressed as a function of $y_1, y_2, \dots y_n$ of a similar kind. If this be done, the highest power of y_1 which will occur is the $w' - 2i'^{\text{th}}$. For the highest power of y_1 which occurs in Y_r is y_1^{r-2}; therefore the highest power of y_1 in $Y_m^{\alpha} Y_n^{\beta} \dots$ will be the $(m\alpha + n\beta + \dots) - 2(\alpha + \beta + \dots)^{\text{th}}$; that is, the $w' - 2i'^{\text{th}}$.

13. Referring back to § 10, let us write

$$Y_{n+2} = t^n A_0 + t^{n-1} q A_1 + t^{n-2} q^2 A_2 + \dots + q^n A_n,$$

A_0, A_1, &c. being homogeneous functions of $y_2, y_3, \dots y_n$, and not involving t or y_1 (A_0 is in fact $-y_{n+2}$); and $q = 1$ being inserted to make the expression homogeneous. Then it has been proved that

$$VY_{n+2} = -t\frac{dY_{n+2}}{dq}.$$

But $$VY_{n+2} = t^nVA_0+t^{n-1}qVA_1+t^{n-2}q^2VA_2+\ldots+q^nVA_n,$$

and $$t\frac{dY_{n+2}}{dq} = t^nA_1+2t^{n-1}qA_2+3t^{n-2}q^2A_3+\ldots+ntq^{n-1}A_n.$$

Equating coefficients of the various powers of t, we have

$$\left.\begin{aligned} VA_0 &= -A_1 \\ VA_1 &= -2A_2 \\ VA_2 &= -3A_3, \text{ \&c.} \\ VA_{n-1} &= -nA_n \\ \text{and, finally,}\quad VA_n &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(18),$$

and we may write

$$Y_{n+2} = t^nA_0-t^{n-1}qVA_0+\frac{1}{1.2}t^{n-2}q^2V^2A_0-\ldots+\frac{(-1)^{n-1}}{\lfloor n-1}tq^{n-1}V^{n-1}A_0,$$

where A_0 stands for $-y_{n+2}$; or, symbolically, and replacing $n+2$ by n,

$$Y_n = -t^{n-2}e^{-V\frac{q}{t}}.y_n\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19).$$

The Y functions are therefore of such a kind that, regarded as quantics $(A_0, A_1, \ldots A_n\between t, q)^n$, their coefficients satisfy relations (18) of a kind precisely analogous to those satisfied by covariants in the ordinary theory of the binary quantics—the operator V here replacing the operator $a\delta_b+2b\delta_c+3c\delta_d+$&c. of the latter theory; in fact they are *quasi-covariants*, so to speak. The term A_0 may be called the *source* of the quasi-covariant Y_{n+2}; and, just as in Salmon's *Higher Algebra*, p. 127, it is seen that the source of the product of two quasi-covariants is equal to the product of their sources.

The Y's satisfy also the equation

$$WY_n = t\frac{dY_n}{dt},$$

where $$W = b\delta_b+2c\delta_c+3d\delta_d+\ldots.$$

This is a consequence of Y_n being of degree $n-1$ and weight $2(n-1)$; for

$$(n-1)\,Y_n = (t\delta_t+a\delta_a+b\delta_b+c\delta_c+\ldots)\,Y_n,$$

$$2(n-1)\,Y_n = (t\delta_t+2a\delta_a+3b\delta_b+4c\delta_c+\ldots)\,Y_n,$$

therefore $$0 = (-t\delta_t + b\delta_b + 2c\delta_c + \dots)\, Y_n,$$

or $$WY_n = t\frac{dY_n}{dt} \quad \dots\dots\dots\dots\dots\dots(20).$$

This equation and $$VY_n = -\,t\frac{dY_n}{dq}$$

are the analogues of the equations

$$\Omega F = y\frac{dF}{dx}, \quad OF = x\frac{dF}{dy},$$

satisfied by the covariants of a binary quantic.

14. With the same notation as in § 13, let f be any homogeneous and isobaric function of $A_0, A_1, \dots A_n$. Then

$$Vf(A_0, A_1, \dots A_n) = \frac{df}{dA_0}VA_0 + \frac{df}{dA_1}VA_1 + \dots + \frac{df}{dA_n}VA_n$$
$$= -\left\{A_1\frac{d}{dA_0} + 2A_2\frac{d}{dA_1} + \dots + nA_n\frac{d}{dA_{n-1}}\right\}f \dots\dots(21).$$

Now let Y_{n+2} or $(A_0, A_1, \dots A_n \between t, q)^n$ be regarded as a purely algebraic form, a quantic in t, q of the n^{th} degree, of which A_0, A_1, &c. are the coefficients. Then the vanishing of the right-hand side of (21) is the condition that f should be a seminvariant of the quantic, in the sense of being unaltered if q be changed into $q+\lambda$. For the expression within the brackets is precisely the second (O) of the two well-known operators (Salmon, *Higher Algebra*, § 65) written with non-binomial coefficients. The vanishing of the left-hand side of (21) is the necessary and sufficient condition that f should be a pure reciprocant. It follows that, when f is a seminvariant of

$$(A_0, A_1, \dots \between t, q)^n$$

in the sense explained (and of course also when f is a full invariant of the quantic), then f is a pure reciprocant. And conversely, any pure reciprocant f is at least a seminvariant of the quantic in the sense explained. And evidently, if there be any number of quantics of the form $(A_0, A_1, \dots \between t, q)^n$ of various degrees (corresponding to various Y's), what has been said about invariants and q-seminvariants of one of them will hold good with regard to their joint invariants and q-seminvariants.

Any number of pure reciprocants can therefore be formed from the Y's by regarding any number of these as if they were a system of covariants belonging to a binary quantic, and forming (in any of the ordinary ways known to the theory of binary forms) invariants and q-seminvariants of them.

For example, the discriminant of Y_4 gives $3ac-5b^2$; the resultant of Y_3 and Y_4 gives $9a^2d-45abc+40b^3$; if Y_5 be written

$$\alpha t^3+3\beta t^2q+3\gamma tq^2+\delta q^3,$$

the q-seminvariant $3\beta\gamma\delta-\alpha\delta^2-2\gamma^3$ gives

$$9a^2d-45abc+40b^3, \text{ \&c., \&c.}$$

If in any of the Y's the t and q be replaced by $\frac{d}{dq}$ and $-\frac{d}{dt}$, an operator will be formed whose effect on any of the Y's is to make it into a reciprocant; for example,

$$\left(b\frac{d}{dq}+3a^2\frac{d}{dt}\right)Y_4=-2at\,(3ac-5b^2),$$

and so on. And this last method is only a particular case of one (see Faà de Bruno, *Formes Binaires*, p. 251) by the application of which to any pair of Y's any number of reciprocants ("associated" quasi-covariants) can be generated.

15. The following gives another method whereby pure reciprocants can be formed in any number from the Y functions, and is simpler of application than that of § 14. The idea is an extension of that applied to binary quantics by Mr. Griffiths. Writing

$$Y'_n=p^{n-2}A_0+p^{n-3}qA_1+p^{n-4}q^2A_2+\dots$$

where A_0, A_1, &c. are still functions of a, b, c, &c., and do not involve t, but where p and q now stand for any quantities whatever which are functions of a, b, c, ..., let us see, following Mr. Griffiths' method, whether p and q can be chosen so as to turn Y'_n into a reciprocant. We have

$$\begin{aligned}VY'_n&=p^{n-2}VA_0+p^{n-3}qVA_1+p^{n-4}q^2VA_2+\dots\\&\quad+\frac{dY'_n}{dp}Vp+\frac{dY'_n}{dq}Vq\\&=\frac{dY'_n}{dp}Vp+\frac{dY'_n}{dq}Vq-p^{n-2}A_1-2p^{n-3}qA_2-3p^{n-4}q^2A_3-\text{\&c.}\\&=\frac{dY'_n}{dp}Vp+\frac{dY'_n}{dq}(Vq-p)\ \dots\dots\dots\dots(22).\end{aligned}$$

But if Y'_n is to be a reciprocant, VY'_n must vanish; accordingly the right-hand side of (22) must vanish. It follows that, if quantities p, q can be found to satisfy the relation

$$\frac{dY'_n}{dp} Vp + \frac{dY'_n}{dq}(Vq-p) = 0,$$

then these quantities will, if substituted for t, q in the expression for Y_n, give rise to a reciprocant. We may then take p and q to satisfy

either $$Vp = 0, \quad Vq = p \text{.........................} (23),$$

or $$Vp = 0, \quad \frac{dY'_n}{dq} = 0 \text{.....................} (24),$$

or $$Vq = p, \quad \frac{dY'_n}{dp} = 0 \text{.....................} (25).$$

Of these (23) are the most useful. For, since the equations (23) do not, like (24) and (25), involve Y'_n, it is clear that they will give values of p and q which, when substituted in *any* of the Y's, will give reciprocants; and moreover, since V does not involve δ_t, these reciprocants will all be pure ones.

As a simple example of the application of (23), take $p = 3a^2$, $q = b$; if then we put $3a^2$ for t, and b for q, in the expressions for Y_3, Y_4, Y_5, &c., as given in § 12, we get the series of pure reciprocants

$$0, \quad 3a^3(5b^2-3ac), \quad 3a^4(-9a^2d+45abc-40b^3), \text{ \&c., \&c.}$$

16. Proceeding exactly as in the last paragraph, only taking the orthogonal operator U instead of the operator V,

$$UY'_n = \frac{dY'_n}{dp} Up + \frac{dY'_n}{dq} Uq + p^{n-2}UA_0 + p^{n-3}qUA_1 + \text{\&c.......} (26).$$

Now A_0 is of degree 1, and of weight n,

A_1 „ 2, „ $n+1$,

A_2 „ 3, „ $n+2$, &c.,

therefore $t(2t\delta_t + 3a\delta_a + \ldots)$ operating on A_0, A_1, A_2, &c., gives the results $\quad A_0t(n+1), \quad A_1t(n+3), \quad A_2t(n+5)$, &c.;

therefore $$UA_0 = t(n+1)A_0 + VA_0 = t(n+1)A_0 - A_1,$$

$$UA_1 = t(n+3)A_1 + VA_1 = t(n+3)A_1 - 2A_2,$$

and so on; thus the last part of the right-hand side of (26) is equal to

$$t\left[(n+1)A_0p^{n-2}+(n+3)A_1p^{n-3}q+\ldots\right],$$

$$-\left[A_1p^{n-2}+2A_2p^{n-3}+\ldots\right],$$

that is, to $t\left[(n+1)Y'_n+(2n-4)Y'_n-2p\dfrac{dY'_n}{dp}\right]-p\dfrac{dY}{dq}$,

and therefore

$$UY'_n=\frac{dY'_n}{dp}(Up-2pt)+\frac{dY'_n}{dq}(Uq-p)+3(n-1)tY'_n\ \ldots(27).$$

Now let p, q be chosen so as to satisfy any one of the equations

$$\frac{dY'_n}{dp}=0,\quad Up=2pt,$$

at the same time that it satisfies any one of the two

$$\frac{dY'_n}{dq}=0,\quad Uq=p;$$

then will $$UY'_n=3(n-1)tY'_n,$$

i.e., Y'_n will become an orthogonal reciprocant such that the factor $y_2^{-(n-1)}$ will make it absolute. And, just as in § 15, we see that, if the

two equations $$Up=2pt,\quad Uq=p\ \ldots\ldots\ldots\ldots(28)$$

be chosen, then any values of p and q which satisfy them will make *all* the Y's into orthogonal reciprocants.

As a simple example, take $p=1+t^2$ and $q=t$; then substitute $1+t^2$ for t and t for q respectively in the expressions for Y_2, Y_3, Y_4, &c., in § 11; we obtain the series of orthogonal reciprocants

$$O_2=-a,$$

$$O_3=-(1+t^2)b+3ta^2,$$

$$O_4=-(1+t^2)^2c+10(1+t^2)tab-15t^2a^3,$$

and so on; and these can be made into absolute orthogonals by dividing them by a, a^2, a^3, &c., respectively.

17. The results of § 13 are very convenient in the treatment of mixed homogeneous reciprocants. For instance, we may make use of them to prove and further extend the following theorems due to

Mr. Rogers, viz., that either of the operators

$$V,\quad 2y_1^2\frac{d}{dy_1}-V,$$

acting on a mixed homogeneous reciprocant, generates another mixed homogeneous reciprocant. So far as I know, these theorems have not been rigorously proved before.

Let R be any mixed homogeneous reciprocant of degree i and weight w; R' the same made absolute by a proper power of y_1; so that

$$R'(y_1, y_2, \ldots) = y_1^{w-2i}R(y_1, y_2, \ldots).$$

Then
$$VR' = \Sigma\frac{dR'}{dY_n}VY_n$$
$$= \Sigma\frac{dR'}{dY_n}\left\{y_1^2\frac{dR'}{dy_1}-(n-2)\,y_1Y_n\right\}$$
$$= y_1^2\frac{dR'}{dy_1}\mp y_1\left\{Y_3\frac{dR'}{dY_3}+2Y_4\frac{dR'}{dY_4}+\ldots\right\} \quad\ldots\ldots\ldots(29),$$

where the R' within the brackets on the right is

$$Y_1^{w-2i}R'(Y_1, Y_2, \ldots),$$

and the double sign corresponds to that in (17).

But now, writing R' for ϕ in (15) and (16), and subtracting the double of (15) from (16),

$$-Y_1\frac{dR'}{dY_1}+Y_3\frac{dR'}{dY_3}+2Y_4\frac{dR'}{dY_4}+\ldots=0;$$

substituting from this in (29), we have

$$VR' = y_1\left\{y_1\frac{dR'}{dy_1}\mp Y_1\frac{dR'}{dY_1}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(30),$$

where the $+$ or the $-$ sign is to be taken according as R' is of negative or positive character.

Again, from (30),

$$2y_1^2\frac{dR'}{dy_1}-VR' = y_1\left\{y_1\frac{dR'}{dy_1}\pm Y_1\frac{dR'}{dY_1}\right\} \ldots\ldots\ldots\ldots\ldots(31),$$

and, more generally, if k be any number whatever,

$$(y_1^2\pm y_1^k)\frac{dR'}{dy_1}-VR' = \pm y_1^k\frac{dR'}{dy_1}\pm y_1Y_1\frac{dR'}{dY_1}$$
$$= y_1\left\{\pm y_1^{k-1}\frac{dR'}{dy_1}\pm Y_1^{k-1}\frac{dR'}{dY_1}\right\} \quad\ldots\ldots(32),$$

a result which includes both (30) and (31) as particular cases.

Now by an obvious extension of what has been said in § 11, it is clear that, since the expressions within the brackets on the right of (30), (31), and (32) are symmetrical in the y's and the Y's, they will be reciprocants; and they will of course still be reciprocants when multiplied by y_1. Therefore the expressions on the left of (30), (31), (32) must all be reciprocants; and the first two of these will evidently be homogeneous. Now,

$$V.R' = V.y_1^{w-2i}R = y_1^{w-2i}VR;$$

therefore, if R is a mixed homogeneous reciprocant, VR is also a mixed homogeneous reciprocant. To see that the operator

$$(y_1^2 \pm y_1^k)\frac{d}{dy_1} - V \quad \text{..........................(33)}$$

gives a reciprocant when it acts upon R (*any* homogeneous reciprocant), and not only when it acts on R', we notice that, by a simple application of (17),

$$y_1^2(3y_2y_4 - 5y_3^2) = Y_1^2(3Y_2Y_4 - 5Y_3^2);$$

raising which to a suitable power, and dividing (17) by the result, we find

$$\frac{R(y_1, y_2, \ldots)}{(3y_2y_4 - 5y_3^2)^{\frac{1}{2}w-i}} = \frac{R(Y_1, Y_2, \ldots)}{(3Y_2Y_4 - 5Y_3^2)^{\frac{1}{2}w-i}} \quad \text{..............(34).}$$

Now (32) is equally true if for R' we substitute the expressions in (34); but if we do so, then, since the operator (33) can have no effect on the denominators, we arrive at an equation exactly like (32), but with R in place of R'. It is therefore proved that the operator (33), acting on a mixed homogeneous reciprocant, produces another reciprocant. In the particular cases (30) and (31), where $k = 2$, this reciprocant is homogeneous.

Both Mr. Rogers and myself had already independently noticed that

$$(1 + y_1^2)\frac{d}{dy_1} - V,$$

operating on a mixed homogeneous reciprocant, produces another reciprocant; but the complete theorem (32) is new, so far as I know. Taking the signs on the right of (30) and (31) along with those of (17), and with what has been said in § 11, it is seen that the operator of (30) changes the character of the reciprocant on which it acts; while that of (31) leaves the character unaltered.

18. I add various properties of the functions Y_n, N_n, P_n :—

(1) The sum of the numerical coefficients in the expression for Y_n is

$$(-1)^{n-1}(n-1)\,!$$

This may be seen immediately by writing $y = e^x$ on the right-hand and $x = \log y$ on the left-hand side of the identity

$$x_n = Y_n y_1^{-(2n-1)}.$$

(2) Y_n, Y_{n+1} are connected by the equation

$$Y_{n+1} = y_1 \frac{dY_n}{dx} - (2n-1)\, y_2 Y_n \quad \ldots\ldots\ldots\ldots\ldots\ldots(35).$$

This comes simply from differentiating the same identity, and substituting for x_{n+1} its equivalent $Y_{n+1} y_1^{-(2n+1)}$.

(3) Y_{n+1} may be derived from Y_n by the operator

$$\left(y_1 y_3 - \frac{2n-1}{n-1} y_2^2\right) \frac{d}{dy_2} + \left(y_1 y_4 - \frac{3n-1}{n-1} y_2 y_3\right) \frac{d}{dy_3}$$
$$+ \left(y_1 y_5 - \frac{4n-1}{n-1} y_2 y_4\right) \frac{d}{dy_4} + \&c.$$

This may be deduced from (35) by means of the equations,

$$y_1 \frac{dY_n}{dx} = y_1 \left\{ y_2 \frac{d}{dy_1} + y_3 \frac{d}{dy_2} + \&c. \right\},$$

$$(n-1)\, Y_n = y_1 \frac{d}{dy_1} + y_2 \frac{d}{dy_2} + \&c.,$$

$$2\,(n-1)\, Y_n = y_1 \frac{d}{dy_1} + 2y_2 \frac{d}{dy_2} + \&c.$$

(4) N_n, N_{n+1} are connected by the equation

$$N_{n+1} = y_1 \frac{dN_n}{dx} - (2n-1)\, y_2 N_n - (n+1)\, y_1^{n-2} y_2 y_n \quad \ldots\ldots(36).$$

For, since
$$N_n = Y_n - y_1^{n-2} y_n,$$

$$\frac{dN_n}{dx} = \frac{dY_n}{dx} - (n-2)\, y_1^{n-3} y_2 y_n - y_1^{n-2} y_{n+1},$$

and $$N_{n+1} = Y_{n+1} - y_1^{n-1} y_{n+1};$$

substituting from which in (35), the result (36) follows.

(5) P_n, P_{n+1} are connected by the equation

$$P_{n+1} = y_1 \frac{dP_n}{dx} - (2n-1)\, y_2 P_n + (n+1)\, y_1^{n-2} y_2 y_n.$$

This is found in the same way as (36).

19. The method of § 3 can easily be extended to Mr. Elliott's ternary, &c. reciprocants; but the results are somewhat complicated. Let F be any function of $\frac{dz}{dx}, \frac{dz}{dy}, \frac{d^2z}{dx^2}, \frac{d^2z}{dx\,dy}$, &c., where x, y are independent variables, z a dependent variable. The effect on F of changing x into $x-\theta z$, and y into $y-\phi z$ (where θ and ϕ are infinitesimals), can be expressed without difficulty. If

$$\theta z \frac{dz}{dx} + \phi z \frac{dz}{dy} = \omega,$$

then (see, *e.g.*, Todhunter, *History of the Calculus of Variations*)

$$\delta \frac{dz}{dx} = \frac{d^2z}{dx^2}\theta z + \frac{d^2z}{dx\,dy}\phi z + \frac{d\omega}{dx},$$

$$\delta \frac{dz}{dy} = \frac{d^2z}{dx\,dy}\theta z + \frac{d^2z}{dy^2}\phi z + \frac{d\omega}{dy},$$

and so on; writing (m, n) to denote $\frac{d^{m+n}z}{dx^m\,dy^n}$, the general formula is

$$\delta(m, n) = (m+1, n)\,\theta z + (m, n+1)\,\phi z + \frac{d^{m+n}\omega}{dx^m\,dy^n},$$

and then $$\delta F = \Sigma \frac{dF}{d(m, n)}\,\delta(m, n).$$

Since the changes in x and y are quite arbitrary, and independent of one another, the parts of δF which involve θ and ϕ respectively can be calculated independently. We shall thus find

$$\delta F = \theta\delta_\theta F + \phi\delta_\phi F,$$

where $$\delta_\theta F = \Sigma \frac{dF}{d(m,n)} \left\{ (m+1,\, n)\, z + \frac{d^{m+n}}{dx^m\, dy^n} \left(z \frac{dz}{dx} \right) \right\} \quad \ldots\ldots(37),$$

$$\delta_\phi F = \Sigma \frac{dF}{d(m,n)} \left\{ (m,\, n+1)\, z + \frac{d^{m+n}}{dx^m\, dy^n} \left(z \frac{dz}{dy} \right) \right\} \quad \ldots\ldots(38).$$

If then F become, by transforming it so as to make x the dependent, and y, z the independent variables, a function Φ of $\frac{dx}{dy}$, $\frac{dx}{dz}$, $\frac{d^2x}{dy^2}$, &c., then, exactly as in § 3, it is seen that the partial differential coefficient of Φ with respect to $\frac{dx}{dz}$ is equal to the expression on the right of (37). And, again, if F become by a similar transformation a function Ψ of $\frac{dy}{dx}$, $\frac{dy}{dz}$, $\frac{d^2y}{dx^2}$, &c., then the differential coefficient of Ψ with respect to $\frac{dy}{dz}$ will be equal to the expression on the right of (38).

If F be a reciprocant, it must then clearly satisfy *two* relations of a kind analogous to equation (1) of § 3; and these can be written down without difficulty for the case of any special class of ternary reciprocants. Similar reasoning applies to the case of n-ary reciprocants; these will satisfy $n+1$ independent relations of this kind.

20. Pure ternary reciprocants will then possess a pair of annihilators. Referring to § 3, it is seen that the process of calculating V for ordinary pure reciprocants may be arranged as follows:—

$$\begin{array}{ll|l}
\omega & = yy_1\theta & \\
\omega' & = (yy_2+y_1^2)\,\theta & \\
\omega'' & = (yy_3+3y_1y_2)\,\theta & \\
\omega''' & = (yy_4+4y_1y_3 & +3y_2^2)\,\theta \\
\omega'''' & = (yy_5+5y_1y_4 & +10y_2y_3)\,\theta,
\end{array}$$

and so on; and the part on the right of the vertical line gives θ times V. In precisely the same manner the pair of annihilators for pure ternary reciprocants can be calculated. We have only to write down $z\frac{dz}{dx}$, and differentiate it any number of times for x or y, cutting off after differentiation all terms involving z, $\frac{dz}{dx}$, or $\frac{dz}{dy}$. What remains will give the annihilator corresponding to the change of x into $x-\theta z$. And a similar process applied to $z\frac{dz}{dy}$ will give the second annihilator, that corresponding to the change of y into $y-\phi z$. I have only had the courage to calculate a few terms of the x-annihilator; these I give below. The corresponding terms of the y-annihilator can be derived from them by symmetry.

$$\omega = z\frac{dz}{dx} = z\,(10),$$

$$\frac{d\omega}{dx} = z\,(20)+(10)^2,\quad \frac{d\omega}{dy} = z\,(11)+(10)(01);$$

$$\frac{d^2\omega}{dx^2} = z\,(30)+3\,(10)(20),\quad \frac{d^2\omega}{dy^2} = z\,(12)+2\,(11)(01)+(10)(02),$$

$$\frac{d^2\omega}{dx\,dy} = z\,(21)+2\,(10)(11)+(01)(20);$$

$$\frac{d^3\omega}{dx^3} = 3\,(20)^2+\ldots,$$

$$\frac{d^3\omega}{dx^2\,dy} = 3\,(11)(20)+\ldots,$$

$$\frac{d^3\omega}{dx\,dy^2} = (20)(02)+2\,(11)^2+\ldots,$$

$$\frac{d^3\omega}{dy^3} = 3\,(11)(02)+\ldots;$$

$$\frac{d^4\omega}{dx^4} = 10\,(20)(30)+\ldots,$$

$$\frac{d^4\omega}{dx^3\,dy} = 4\,(11)(30)+6\,(20)(21)+\ldots,$$

$$\frac{d^4\omega}{dx^2\,dy^2} = (02)(30)+3\,(20)(12)+6\,(11)(21)+\ldots,$$

$$\frac{d^4\omega}{dx\,dy^3} = (20)(03)+3\,(02)(21)+6\,(11)(12)+\ldots,$$

$$\frac{d^4\omega}{dy^4} = 4\,(11)(03)+6\,(02)(12)+\ldots;$$

and so on, the omitted part being in each case that involving z or (10) or (01). The annihilator will therefore be

$$3\,(20)^2\frac{d}{d\,(30)}+3\,(11)(20)\frac{d}{d\,(21)}+\left\{(20)(02)+2\,(11)^2\right\}\frac{d}{d\,(12)}$$
$$+3\,(02)(11)\frac{d}{d\,(03)}+4\,(20)(30)\frac{d}{d\,(40)}+\&c.;$$

the coefficient of $\dfrac{d}{d\,(mn)}$ being

$$\frac{d^{m+n}\{z\,(10)\}}{dx^m\,dy^n} - \text{terms in this which involve } z, (10), \text{ or } (01).$$

In a similar manner, by following the method of § 6, the pair of operators for "orthogonal" ternary reciprocants, analogous to the operator U of § 6, might be worked out; the one by writing $x-z\theta$ for x and $z+x\theta$ for z simultaneously, and the second by writing $y-z\phi$ for y and $z+y\phi$ for z simultaneously. But the calculation would be very laborious.

21. The method of § 11 is clearly applicable, *mutatis mutandis*, to ternary reciprocants. As an example, take one of the simplest cases, and let a_1', b_1', c_1', a_1'', b_1'', c_1'' be each expressed in terms of p, q, a_1, b_1, c_1. (For the notation I refer to Mr. Elliott's paper, *Proceedings*, Vol. XVII., p. 172.) It is found that

$$\left.\begin{aligned} -a_1' &= q^2a_1-2pqb_1+p^2c_1 \\ -b_1' &= pb_1-qa_1 \\ -c_1' &= a_1 \end{aligned}\right\} \div p^3,$$

$$\left.\begin{aligned} -a_1'' &= c_1 \\ -b_1'' &= qb_1-pc_1 \\ -c_1'' &= q^2a_1-2pqb_1+p^2c_1 \end{aligned}\right\} \div q^3.$$

Then $a_1'p^3+a_1''q^3-a_1$ and $c_1'p^3+c_1''q^3-c_1$ each give the reciprocant

$$(1+q^2)\,a_1-2pqb_1+(1+p^2)\,c_1,$$

while $b_1'p^3+b_1''q^3-b_1$ gives the reciprocant

$$(1+p+q)\,b_1-qa_1-pc_1.$$

These two reciprocants correspond to those obtained by the addition method of § 11. Others can be formed, involving the imaginary cube roots of unity, corresponding to those found by the subtraction method of § 11. I have not pursued this method further; but it is evidently one which may be expected to yield good results, giving, as it does, the means of forming any number of ternary reciprocants.