ON THE DIFFERENTIATION OF A SURFACE INTEGRAL AT A POINT OF INFINITY

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1. The present paper is a study of a particular case of differentiation of a surface integral taken over a curved surface. The subject of integration is assumed to depend upon the coordinates of a particular point O in the surface and to have an infinity at O, and the differentiation contemplated corresponds to displacement of O in the surface. The resulting formula is applied to some surface integrals which are important in potential theory.

A Theorem on Surface and Contour Integrals.

2. We begin by proving a lemma with regard to surface integrals which may perhaps be described as a theorem in calculus of variations.

Consider a curved surface S and an area of integration upon it bounded by a curve T; it is assumed that the surface has no singular points in the area of integration. Let f be a function of position on the surface.

Consider an infinitesimal correspondence between points on the surface, namely P' corresponding to P. Denote PP' by δs, and let the correspondence depend upon an infinitesimal parameter $\delta\lambda$ in such a way that for each point P the ratio $\delta s/\delta\lambda$ tends, for $\delta\lambda \to 0$, to a definite value, say $ds/d\lambda$, a function of position of P.

For a selected correspondence of this character let the range of P be the area enclosed by T, and let the corresponding range of P' be the area enclosed by a curve T' on the surface, so that T' is the curve corresponding to T. In general the curve T' lies near to T, and the greater part of the area within T is also within T'.

Let dS be an element of area at P, and dS' the corresponding element of area at P'. If f_P be the value of f at P,

$$f_P dS = f_P (dS/dS') dS',$$

and so
$$\int^{T} f\,dS = \int^{T'} f'\,dS',$$

where the first integral is over the area bounded by T, the second is over the area bounded by T', and

$$f'_{P'} = (f\,dS/dS')_P.$$

It is clear, however, that if we express $\int^{T} f\,dS$ in terms of P' instead of P we get the same value provided the boundary is properly chosen. In fact
$$\int^{T} f_P\,dS = \int^{T} f_{P'}\,dS',$$

where the notation implies that P' ranges over the area bounded by T; hence
$$\int^{T'} f_P\,dS = \int^{T'} f_{P'}\,dS' - \int_{T}^{T'} f_{P'}\,dS',$$

the second integral on the right being through the strip between T' and T, dS' being reckoned positive or negative according as it is outside or inside the curve T.

Thus, putting $f_P + (df/ds)_P\,\delta s$ for $f_{P'}$, and assuming the properties of f to be such that the closeness of this approximation can be ensured to any arbitrary degree of accuracy by taking $\delta\lambda$ sufficiently small, we have, by equating the two different modified expressions for the original integral,

$$\int^{T'} \left(f_P \frac{dS}{dS'}\right) dS' = \int^{T'} \left\{f_P + \left(\frac{df}{ds}\right)_P \delta s\right\} dS' - \int_{T}^{T'} \left\{f_P + \left(\frac{df}{ds}\right)_P \delta s\right\} dS'.$$

If in this we neglect small quantities of higher order than δs, and note that in the strip between T and T' the element dS' may be put equal to $ds\,\delta s \cos(\delta s, \nu)$, where ν is the outward normal to the contour in the tangent plane to the surface and ds is an element of arc of the contour, we get, to an approximation which can be rendered arbitrarily close,

$$\int^{T'} \left\{f_P\left(1 - \frac{dS}{dS'}\right) + \left(\frac{df}{ds}\right)_P \delta s\right\} dS' = \int_{T} f_P\,\delta s \cos(\delta s, \nu)\,ds.$$

Dividing by $\delta\lambda$ and noticing that the degree of approximation is unaltered on the left-hand side by taking the element of area as dS and the boundary as T, we get

$$\int^{T} \left\{f\left(\frac{dS' - dS}{dS\,\delta\lambda}\right) + \left(\frac{df}{ds}\right)\frac{\delta s}{\delta\lambda}\right\} dS = \int_{T} f\,\frac{\delta s}{\delta\lambda} \cos(\delta s, \nu)\,ds,$$

as an equality which is approximate to an arbitrary degree of closeness depending on the value of $\delta\lambda$.

Proceeding to the limit for $\delta\lambda \to 0$, and denoting $\underset{\delta\lambda\to 0}{\text{Lim}} \left(\frac{dS'-dS}{dS\,\delta\lambda}\right)$ by θ, we get the result

$$\int^{T} \left(f\theta + \frac{df}{ds}\frac{ds}{d\lambda}\right) dS = \int_{T} f \frac{ds}{d\lambda} \cos(\delta s, \nu)\, ds. \tag{1}$$

3. The theorem of the preceding article is a somewhat general relation between a surface integral and a line integral taken round the boundary edge of the surface. Its significance may be illustrated by two special cases.

First, let the surface be a plane surface and let the infinitesimal correspondence be a translation of the whole area of integration, as if rigid, in its own plane. Such a correspondence is given by $\delta\lambda = \delta x$, δs equal and parallel to δx, and $\theta = 0$. The theorem then yields the equality

$$\int \frac{\partial f}{\partial x}\, dS = \int f \cos(x, \nu)\, ds,$$

a familiar result.

Second, let the surface be a curved surface on which there is a set of orthogonal coordinates p, q such that the element of arc is given by

$$ds^2 = P^2 dp^2 + Q^2 dq^2.$$

Let the correspondence consist in an increase of the p coordinate of every point by the same amount δp. Then θ is $\frac{1}{PQ}\frac{\partial}{\partial p}(PQ)$, and the theorem takes the form

$$\int \frac{1}{PQ}\frac{\partial}{\partial p}(PQf)\, dS = \int fP \cos(p, \nu)\, ds, \tag{2}$$

another well known result.

Differentiation of a Surface Integral.

4. Passing now to the differentiation problem, we consider a surface integral taken over the portion of a curved surface S which is bounded by a curve T, and suppose the subject of integration f to be a function not only of the position of the point P at which the element of area dS is

situated but also of the position of a certain point O in the surface. When necessary this may be emphasised by writing the subject of integration f_{OP}.

We suppose O to be in the region enclosed by T, and if the function f has an infinity at O the area of integration must be separated from O by a surrounding cavity ϵ whose dimensions are made to tend to vanishing.

We want to differentiate, with respect to any vanishing displacement of O in the surface, the integral $\int^T f_{OP}\,dS$, that is

$$\operatorname*{Lim}_{\epsilon\to 0}\int_\epsilon^T f_{OP}\,dS.$$

Let us consider a displacement of O to a neighbouring point O', and denote OO' by $\delta\lambda$.

Let us associate with the passage from O to O' an infinitesimal correspondence of the kind discussed in Art. 2, which makes O' correspond to O, P' to P, T' to T, and gives a cavity ϵ' round O' corresponding to ϵ round O.

The incremental ratio of the integral is

$$\frac{1}{\delta\lambda}\left[\operatorname*{Lim}_{\epsilon'\to 0}\int_{\epsilon'}^T f_{O'P}\,dS-\operatorname*{Lim}_{\epsilon\to 0}\int_\epsilon^T f_{OP}\,dS\right],$$

$$=\frac{1}{\delta\lambda}\left[\operatorname*{Lim}_{\epsilon'\to 0}\int_{\epsilon'}^{T'} f_{O'P'}\,dS'-\operatorname*{Lim}_{\epsilon\to 0}\int_\epsilon^T f_{OP}\,dS-\int_T^{T'} f_{O'P'}\,dS'\right],$$

which, if we denote dS'/dS by χ and bear in mind the infinitesimal correspondence, may be written

$$\frac{1}{\delta\lambda}\left[\operatorname*{Lim}_{\epsilon\to 0}\int_\epsilon^T (f_{O'P'}\chi-f_{OP})\,dS-\int_T^{T'} f_{O'P'}\,dS'\right].$$

In the former of the integrals in this expression it is to be noted that $f_{O'P'}$ is regarded as a function of the position of P; it has an infinity for $P\to O$ because then $P'\to O'$, but it has no infinity at O'. If χ is a function free from peculiarities at O (and we shall assume it continuous at all points in the area bounded by T), then the function $f_{O'P'}\chi$ has an infinity at O of exactly the same character as the infinity of f_{OP} at O. Thus the integrals $\int_\epsilon^T f_{O'P'}\chi\,dS$ and $\int_\epsilon^T f_{OP}\,dS$ are subject to the same conditions as to convergence; if the latter is convergent either absolutely or for some particular mode of vanishing of the cavity ϵ, the former also is convergent either absolutely or for the same mode of vanishing of the cavity as the case may be.

Assuming convergence of the integral to be differentiated, we may accordingly close the cavity in the above formula, and we get that the incremental ratio equals

$$\frac{1}{\delta\lambda}\left[\int^{T}(f_{O'P'}\chi - f_{OP})\,dS - \int_{T}^{T'} f_{O'P'}\,dS'\right]. \tag{3}$$

When the nature of the (P, P') correspondence has been settled as well as the particular arc through O upon which O' is to lie, both $f_{O'P'}$ and χ are functions of the position of P which depend on the parameter OO', *i e.*, $\delta\lambda$. Moreover when O' coincides with O, since the correspondence becomes an identity, $f_{O'P'}\chi$ has the value f_{OP}. Hence by the theorem of mean value, assuming it to be applicable,

$$f_{O'P'}\chi_{O'P'} - f_{OP} = \delta\lambda\left[\frac{D}{d\,\delta\lambda}(f\chi)\right]_{O''},$$

the right-hand side having the value corresponding to some point O'' lying between O and O'. The differentiation here denoted by the symbol D is one in which, for a given P, the points O' and P' (or O'' and P'') move simultaneously, varying as the correspondence varies quantitatively but not qualitatively with the variation of the parameter $\delta\lambda$.

Thus the first term of the incremental ratio is

$$\int^{T}\left(\frac{Df}{d\,\delta\lambda}\chi + f\frac{D\chi}{d\,\delta\lambda}\right)_{O''} dS. \tag{4}$$

Making now a supposition which must be regarded as provisional pending further study, we assume that the function, say $\phi_{O''}$, which constitutes the subject of integration in this formula is such (i) that $\int \phi_O\,dS$ is convergent, and (ii) that we can, by choosing $\delta\lambda$ sufficiently small, make $\int(\phi_{O''} - \phi_O)\,dS$ less than any arbitrarily assigned small quantity, so that

$$\int \phi_{O''}\,dS \to \int \phi_O\,dS \quad \text{as} \quad O'' \to O.$$

Since $(\chi - 1)/\delta\lambda \to \theta$ for $\delta\lambda \to 0$, $(D\chi/d\,\delta\lambda)_O = \theta$; also $\chi_O = 1$. Substituting these in ϕ_O, we see that our provisional conclusion is

$$\operatorname*{Lim}_{\delta\lambda\to 0}\frac{1}{\delta\lambda}\int^{T}(f_{O'P'}\chi - f_{OP})\,dS = \int\left(\frac{Df}{d\,\delta\lambda} + f\theta\right)_O dS. \tag{5}$$

5. Let us now look more closely into the assumptions provisionally made at the conclusion of the previous article. In this connexion two things are of importance, namely the properties of the (P, P') correspondence, and the nature of the function f.

As regards the correspondence we may postulate that it shall be continuous throughout the whole of the area enclosed by T, so that not only is the ratio of δs to $\delta\lambda$ a continuous function, but also the direction of δs varies continuously. It has already been postulated that when P is at O δs is the same as $\delta\lambda$. Hence the continuity of the correspondence involves that, as P approaches O, δs tends to equality and parallelism with $\delta\lambda$ as to a limit. Further we assume that the correspondence is regular, so that the difference between δs and $\delta\lambda$, whether in magnitude or in direction, tends to smallness of the order of OP as $P \to O$. Obviously this might not be possible if O were a conical point or point of other singularity on the surface, but we have already excluded such points from consideration.

As regards the nature of f, it seems desirable to restrict the manner of its dependence on the position of O. Denoting the coordinates of O by (x_0, y_0, z_0) and those of P by (x, y, z), we shall suppose that f is the product of two factors of which the first is a function of $(x-x_0, y-y_0, z-z_0)$, while the second is a function of (x, y, z). In fact

$$f = g(x-x_0, y-y_0, z-z_0)\, h(x, y, z).$$

We shall further suppose that $h(x, y, z)$ is continuous at O.

With these postulates before us, let us consider the integral on the right-hand side of equation (5). The function θ has no infinity or other discontinuity at O, and so $\int (f\theta)_O dS$ is convergent in the same manner as $\int f dS$. The other part of the subject of integration involves the operator $(D/d\delta\lambda)_O$, which for the sake of brevity we may denote by ϑ_O. In the limit for $P \to O$, ϑ_O represents a rate of pure translation in the tangent plane at O corresponding to increase of $\delta\lambda$, $\delta\lambda$ itself being zero since O' is at O, as is implied by the suffix $(_O)$; let this limit operator of pure translation be denoted by t_O. Since $(x-x_0, y-y_0, z-z_0)$ are unaffected by translation, if k be any function of $(x-x_0, y-y_0, z-z_0)$ which is continuous and so free from infinity at O, $t_O k = 0$; in fact the continuity and regularity of the correspondence make $D(x-x_0)/d\delta\lambda$, etc., small of the order of OP or r, so that $\vartheta_O k = rt'_O k$, where t'_O is an operator which does not introduce any infinity at O.

As the function g has an infinity at O it is not safe to take for granted that the operation of ϑ_O upon it necessarily obeys the same law as its operation upon k. Suppose g to be of the form $kr^{-\mu}$, k being as described above and μ being positive. Then

$$\vartheta_O kr^{-\mu} = r^{-\mu}\vartheta_O k \quad -\mu kr^{-\mu-1}\vartheta_O r$$

$$= r^{-\mu+1}t'_O k - \mu kr^{-\mu}t'_O r.$$

Hence in this case $\vartheta_O g$ has just the same kind of infinity at O as has g itself; a similar argument would generally apply to infinities depending on negative powers of $x - x_0$, etc. The occurrence in f of a negative power of the distance of P from the tangent plane at O, though not formally covered by the above argument, could be included by a slight extension, account being taken of the definite curvature of the surface at O.

Now $$\vartheta_O f = h\vartheta_O g + g\vartheta_O h,$$

and as we have supposed h to be continuous at O this shews that $\vartheta_O f$ has the same kind of infinity at O as has f. Consequently the integral

$$\int \phi_O\, dS \quad \text{or} \quad \int \left(\frac{Df}{d\,\delta\lambda} + f\theta\right)_O dS$$

has the same convergence as $\int f dS$.

Passing now to $\int \phi_{O''} dS$, we consider first the effect upon f of the operator $(D/d\,\delta\lambda)_{O''}$ or $\vartheta_{O''}$, bearing in mind that the value of f, though associated for purpose of integration with dS at P, is determined by the positions of O'' and P''. As regards the effect of the operator on P'' we see that for $P'' \to O''$ (which involves $P \to O$) the operator tends to a limit operator $t_{O''}$ which represents a rate of pure translation in the tangent plane to the surface at O''. And since $x_{P''} - x_{O''}$, etc., are unaffected by translation, if k be any function of these arguments which is continuous and so free from infinity for $P'' \to O''$, (which involves, when the function is associated with dS at P, continuity for $P \to O$), $t_{O''}k = 0$; in fact the continuity and regularity of the correspondence make $\vartheta_{O''}(x_{P''} - x_{O''})$, etc., small of the order of $O''P''$ or r'', *i.e.*, of the order of r, so that

$$\vartheta_{O''}k = rt'_{O''}k$$

where $t'_{O''}$ is an operator which does not introduce any infinity for $P'' \to O''$.

The function $g_{O''}$ has an infinity for $P'' \to O''$ and this may be treated by precisely the same reasoning as has been applied to g_O, r'' taking the place of r in the formulæ, and it being borne in mind that r and r'' are of the same order of smallness and tend simultaneously to zero. The conclusion is that $\vartheta_{O''} g_{O''}$ and $g_{O''}$ have, when regarded as functions of the position of P'' the same kind of infinity at O'', and when regarded as functions localised at P the same kind of infinity at O. Consequently $\vartheta_{O''} f_{O''}$ when localised at P has the same kind of infinity at O as $f_{O''}$ or f_O.

The function χ, being continuous, does not introduce any complication into $\phi_{O''}$ as compared with ϕ_O, and so we conclude that

$$\int \phi_{O''} dS \quad \text{or} \quad \int \left(\frac{Df}{d\delta\lambda}\chi + f\frac{D\chi}{d\delta\lambda}\right)_{O''} dS$$

has the same convergence as $\int f dS$. This conclusion has been arrived at without reference to the particular values of OO'' or OO', and is not invalidated by any changes in these values.

Now the assumptions with regard to $\phi_{O''}$ made at the end of the last article were important for the following reason. If at all points in the area of integration ϕ is a continuous function of $\delta\lambda$ it is generally possible, by taking $\delta\lambda$ sufficiently small, to make $|\phi_{O''} - \phi_O|$ less than an arbitrarily assigned small quantity κ, and therefore also $\left|\int (\phi_{O''} - \phi_O)\, dS\right|$ less than an arbitrarily assigned small quantity κ'. If, however, after a value of $\delta\lambda$ has been fixed, P in the course of integration has to approach a point of infinity, there is a danger that the function $\phi_{O''} - \phi_O$ may become so great as to invalidate an inequality of the type

$$\left|\int (\phi_{O''} - \phi_O) dS\right| < \kappa'.$$

But the assumptions and reasoning of the present article have provided against this contingency, for we have shewn that if $\int f dS$ is convergent then both $\int \phi_{O''} dS$ and $\int \phi_O dS$ are convergent, so that necessarily $\int (\phi_{O''} - \phi_O)\, dS$ is convergent. This convergence is independent of $\delta\lambda$ and cannot be modified or nullified by any arbitrary diminution of $\delta\lambda$. Hence the inequality stands, and the formula (5) is justified, provided the transformation and the function f comply with the restrictions set out at the beginning of the present article.

6. The second term of the incremental ratio as set out in formula (3) is a surface integral over the strip between T and T', which strip tends to vanishing as $\delta\lambda \to 0$. Taking the element of area as extending across the strip, so that it is ultimately a parallelogram having for base the element of arc ds of T and for altitude $\delta s \cos(\delta s, \nu)$ where ν is the outward normal to T, we see that

$$\int_T^{T'} f_{O'P'}\, dS' \to \int_T f_{OP}\ \delta s\ \cos(\delta s, \nu)\, ds$$

$$\to \delta\lambda \int_T f_{OP} \frac{ds}{d\lambda} \cos(\delta s, \nu)\, ds.$$

Putting together the two terms of the incremental ratio, dividing by $\delta\lambda$, and proceeding to the limit for $\delta\lambda \to 0$, we get

$$\frac{d}{d\lambda}\int^T f_{OP}\, dS = \int^T (\vartheta_O f_O + f_O \theta)\, dS - \int_T f \frac{ds}{d\lambda} \cos(\delta s, \nu)\, ds. \tag{6}$$

Now let us apply the theorem of Art. 2 to the function f in the area bounded externally by T and internally by a cavity η surrounding the point O. We get

$$\int_T f \frac{ds}{d\lambda} \cos(\delta s, \nu)\, ds - \int_\eta f \frac{ds}{d\lambda} \cos(\delta s, \nu)\, ds = \int_\eta^T \left(\frac{df}{ds}\frac{ds}{d\lambda} + f\theta\right) dS, \tag{7}$$

where, for the curve η, ν is outward.

Eliminating the line-integral round T from these two equations (6) and (7), and passing to the limit for $\eta \to 0$, we find

$$\frac{d}{d\lambda}\int^T f\, dS = \lim_{\eta\to 0}\int_\eta^T \left(\vartheta f - \frac{df}{ds}\frac{ds}{d\lambda}\right) dS - \lim_{\eta\to 0}\int f \frac{ds}{d\lambda} \cos(\delta s, \nu)\, ds. \tag{8}$$

It is to be noted that ϑf is a differentiation of f corresponding to a simultaneous transfer of O to O' and P to P', while $\frac{df}{ds}\frac{ds}{d\lambda}$ is a differentiation corresponding to transfer of P to P' only. Hence $\vartheta f - \frac{df}{ds}\frac{ds}{d\lambda}$ is the differentiation which corresponds to displacement of O to O' leaving P undisturbed. Call this $\partial f/\partial\lambda$. Also we note that, by the continuity of the transformation, $ds/d\lambda \to 1$ for points on η as $\eta \to 0$. So our final formula for the differentiation is

$$\frac{d}{d\lambda}\int^T f\, dS = \lim_{\eta\to 0}\left\{\int_\eta^T \frac{\partial f}{\partial\lambda}\, dS - \int_\eta f \cos(\delta\lambda, \nu)\, ds\right\}. \tag{9}$$

This is the general result, but we must not fail to note that in passing from equations (6) and (7) to equation (8) it has been tacitly assumed that $\int (\vartheta f + f\theta)\, dS$ may be replaced by $\operatorname*{Lim}_{\eta \to 0} \int_\eta (\vartheta f + f\theta)\, dS$. This is clearly legitimate without restriction upon the form of the cavity η if $\int f\, dS$, and therefore also $\int (\vartheta f + f\theta)\, dS$ are absolutely convergent. But if $\int f\, dS$ is semi-convergent, so that it can only be rendered definite by specifying a particular kind of vanishing cavity ϵ, the argument of Art. 5 implies that the same cavity must be associated with $\int (\vartheta f + f\theta)\, dS$; consequently in this case it is necessary in formula (9) to make η the same as ϵ.

Illustrations in Potential Theory.

7. A simple illustration of the use of the differentiation formula which has been obtained above is afforded by the potential and force integrals of a surface concentration of gravitating matter of surface-density σ. We put $f = \sigma r^{-1}$, noting that σ is a function of the position of P only while r^{-1} is a function of the coordinates of P relative to O. The potential V is $\int \sigma r^{-1} dS$, an absolutely convergent integral, and we differentiate for a displacement of O typified by $d\lambda$. The result is

$$\frac{dV}{d\lambda} = \operatorname*{Lim}_{\eta \to 0} \left\{ \int_\eta^T \sigma \frac{\partial}{\partial \lambda} \left(\frac{1}{r} \right) dS - \int_\eta \frac{\sigma}{r} \cos(\delta\lambda, \nu)\, ds \right\}.$$

Denoting the force-component at O, when surrounded by the cavity η, by $F(\eta)$, we see that our result is

$$\frac{dV}{d\lambda} = \operatorname*{Lim}_{\eta \to 0} \left\{ F(\eta) - \int_\eta \frac{\sigma}{r} \cos(\delta\lambda, \nu)\, ds \right\}. \tag{10}$$

The force-integral is semi-convergent, and the difference between the potential gradient $dV/d\lambda$ and the force-component $\operatorname{Lim} F(\eta)$ is

$$-\operatorname*{Lim}_{\eta \to 0} \int \sigma r^{-1} \cos(\delta\lambda, \nu)\, ds,$$

which is generally the same as

$$-\sigma_0 \operatorname*{Lim}_{\eta \to 0} \int r^{-1} \cos(\delta\lambda, \nu)\, ds.$$

The value of this difference clearly depends upon the form of the vanishing cavity η. It is zero if the cavity vanishes as a circle with O as centre, or as any curve having symmetry about the line through O perpendicular to $d\lambda$ in the tangent plane.

8. A second illustration is afforded by the potential integral of a double sheet of strength τ. The main facts as to the convergence and discontinuity of this integral are well known, but before proceeding to the differentiation formula it seems worth while to give a fresh discussion of these, in the hope that the method employed may be found an interesting alternative to the methods in use elsewhere.

The cosines of the normal on the positive side of the double sheet at a point P, (x, y, z), being denoted by (l, m, n), the potential at a point O, not necessarily in the sheet, is

$$V = \int \tau \left(l \frac{\partial}{\partial x} + m \frac{\partial}{\partial y} + n \frac{\partial}{\partial z} \right) \frac{1}{r} \, dS,$$

r being OP. Now $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \frac{1}{r}$, considered as a vector localised at P, is a solenoidal vector and therefore is the curl of some other vector or family of vectors. Such a vector is easily specified when we think of grad $(1/r)$ as corresponding hydrodynamically to symmetrical flow from the point O. We take O as the origin of a set of spherical polar coordinates (r, θ, ϕ), and we try a vector of magnitude Ω perpendicular to the plane of the angle θ, in the direction of ϕ increasing. The flow through the circle ($r =$ const., $\theta =$ const.) is $-2\pi(1-\cos\theta)$, and this must equal the line-integral of Ω, that is $2\pi r \sin\theta\, \Omega$; so

$$\Omega = -\frac{1}{r} \frac{1-\cos\theta}{\sin\theta} = -\frac{1}{r} \tan \tfrac{1}{2}\theta. \tag{11}$$

Having guessed this vector Ω it is easy to verify that it really has the property

$$\operatorname{curl} \Omega = \operatorname{grad} (1/r). \tag{12}$$

We note that Ω has an infinity for $\theta = \pi$, independent of r, as well as the infinity $r = 0$.

Considering a region of the surface S bounded by a curve T and not containing any infinity of Ω, assuming the existence of orthogonal curvilinear coordinates p, q as in Art. 3, and indicating the components of Ω tangential to the curve T and the curves of increase of p and q by suffixes t, p, q respectively, we apply the second theorem of Art. 3, and so get

$$\int_T \tau \Omega_t ds = \int \tau \{-\Omega_p \cos(q, \nu) + \Omega_q \cos(p, \nu)\} ds$$
$$= \int \frac{1}{PQ} \left\{ \frac{\partial}{\partial p}(\tau Q \Omega_q) - \frac{\partial}{\partial q}(\tau P \Omega_p) \right\} dS$$
$$= \int \tau (\text{curl}\, \Omega)_N dS + \int \left(\Omega_q \frac{1}{P} \frac{\partial \tau}{\partial p} - \Omega_p \frac{1}{Q} \frac{\partial \tau}{\partial q} \right) dS,$$

where the suffix N refers to the normal to the surface. The first integral of the last formula equals $\int \tau (\text{grad } 1/r)_N dS$, and so is the potential at O of the part of the double sheet bounded by T. The second has for subject of integration the normal component of the vector product $[\text{grad } \tau, \Omega]$.

It will be convenient to regard the boundary T as made up of two curves, an outer edge which we shall henceforth call T and an inner boundary ϵ surrounding any point at which Ω has an infinity. Our formula now takes the form

$$V = \int_T \tau \Omega_t ds - \int_\epsilon \tau \Omega_t ds - \int_\epsilon^T [\text{grad } \tau, \Omega]_N dS, \tag{13}$$

the two line-integrations being taken in the same sense, namely that corresponding to increase of ϕ.

If O is not in the surface the only infinities of Ω occur at points where the negative direction of the axis of spherical polar coordinates chosen for the origin O cuts the surface, for at these points $\theta = \pi$. Let Q be such a point, ϵ the cavity round it, and P a point in the region of integration near to Q. At P it is clear that Ω is great of the order of $1/QP$, so that the surface integral converges absolutely as $\epsilon \to 0$. Also, from the original definition of Ω, $\int_\epsilon \Omega_t ds$ equals all of the symmetrical flux from O except that part which passes through ϵ, and so tends to -4π whatever be the manner of vanishing of ϵ; consequently

$$\int_\epsilon \tau \Omega_t ds \to \tau_Q \int \Omega_t ds \to -4\pi \tau_Q,$$

τ_Q being reckoned positive when in the sense from Q to O. Thus, if O is not in the surface

$$V = 4\pi \Sigma \tau_Q + \int_T \tau \Omega_t ds - \int^T [\text{grad } \tau, \Omega]_N dS. \tag{14}$$

If O is in the surface (at an ordinary point of the surface), $\theta = \frac{1}{2}\pi$, and Ω tends to infinity of the order $1/r$. The surface integral is therefore absolutely convergent. The integral $\int_\epsilon \Omega_t ds$ tends to equal half the total flux from O, that is -2π, whatever the shape of ϵ, and so

$$\int_\epsilon \tau\Omega\, ds \rightarrow -2\pi\tau_O.$$

Thus V has a definite value, wherever O may be, but has a discontinuity at the surface. For, as O approaches the surface in a direction making an acute angle with the negative sense of the axis of spherical polar coordinates for O, it gets nearer to a point Q which supplies a term $4\pi\tau_Q$ in V. When O is at Q this term disappears, being replaced by $2\pi\tau_Q$: and when O crosses the surface the term $2\pi\tau_Q$ disappears also; and during these changes all the other parts of V vary continuously.

9. Passing now to the differentiation of V, we notice that f or $\tau(l\partial/\partial x+m\partial/\partial y+n\partial/\partial z)(1/r)$, though not as it stands the product of two factors of the g and h types, is however the sum of three terms each of which separately is such a product. Hence the theorem of Art. 6 is applicable and we have, using the notation of Art. 7,

$$\frac{dV}{d\lambda} = \operatorname*{Lim}_{\eta\to 0} \left\{ F(\eta) - \int_\eta \tau \left(l\frac{\partial}{\partial x}+m\frac{\partial}{\partial y}+n\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right) \cos(\delta\lambda, \nu)\, ds \right\}. \quad (15)$$

The force-integral is semi-convergent, and the potential gradient $dV/d\lambda$ differs from the force by

$$-\operatorname*{Lim}_{\eta\to 0} \int_\eta \tau\Sigma \left(l\frac{\partial}{\partial x}\right)\left(\frac{1}{r}\right) \cos(\delta\lambda, \nu)\, ds,$$

which is the same as

$$-\tau_0 \int_\eta \Sigma \left(l\frac{\partial}{\partial x}\right)\left(\frac{1}{r}\right) \cos(\delta\lambda, \nu)\, ds,$$

or, if we put ζ for the distance of a point P from the tangent plane at O,

$$\tau_0 \int_\eta \zeta r^{-2} \cos(\delta\lambda, \nu)\, ds. \quad (16)$$

Projecting on the tangent plane at O, taking as axes the principal directions of curvature corresponding to principal radii ρ_1 and ρ_2, putting $(\cos\theta, \sin\theta)$ for the cosines of OP, (L, M) for the cosines of $\delta\lambda$, and (l', m') for the cosines of the normal to the projection η' of η, the difference between potential gradient and force takes the form

$$\tfrac{1}{2}\tau_0 \int_\eta \left(\frac{\cos^2\theta}{\rho_1} + \frac{\sin^2\theta}{\rho_2}\right)(Ll'+Mm')\, ds. \quad (17)$$

This is different for different vanishing forms of η. If η' be circular, with O as centre, $l' = \cos\theta$, $m' = \sin\theta$, and the line integral has zero limit.