456. Investigation of a Simple Formula for Calculating the Successive "Numbers of Bernoulli"

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Hence the general solution is

$$x^2 - \log \sec^2 l = f[x^2 - \log(\sin^2 u/\sin^2 l)]. \quad\text{.....................(x)}$$

Consider the solution

$$\left.\begin{aligned} x &= \surd(\log \sec l), \\ y &= \surd(\log \sec l)\,.\,\cot l\,.\,\sin u. \end{aligned}\right\} \quad\text{.........................(xi)}$$

giving

Before accepting this solution we must determine the limit, as $l$ approaches zero, of $\surd(\log \sec l)\,.\,\cot l$.

$$\text{It} = \text{Lt} \cot l\,.\,\surd[-\log(1 - l^2/2) + \ldots]$$
$$= \tfrac{1}{2}, \text{ after an easy reduction.}$$

The equation to a meridian is

$$x = \surd(\log \sec l),$$

and the equation to a parallel is

$$y = x\,.\,\sin u/\surd(e^{2x^2} - 1).$$

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**456.** [D. 6. c. δ.] *Investigation of a simple formula for calculating the successive "numbers of Bernoulli."*

Let

$$\frac{x}{e^x - 1} = y = a_0 + a_1 x + a_2 \frac{x^2}{\lfloor 2} + \ldots, \quad\text{..........................(1)}$$

where $a_2$, $-a_4$, $a_6$, $-a_8$, ... are Bernoulli's numbers.

Then

$$e^x(y) = y + x;$$
$$\therefore\ e^x(y_1 + y) = y_1 + 1 \quad\text{and}\quad e^x(y_2 + 2y_1 + y) = y_2, \text{ etc.}$$
$$\therefore \text{ putting } x = 0, \quad a_1 + a_0 = a_1 + 1 \ [\text{whence } a_0 = 1], \quad\text{.........................(2)}$$
$$a_2 + 2a_1 + a_0 = a_2 \ [\text{whence } a_1 = -\tfrac{1}{2}], \text{ and so on;}$$

the expansion on the left being of binomial type, with suffixes instead of powers.

If we symbolize $a_2$, $a_3$, $a_4$, ... by $a^2$, $a^3$, $a^4$, etc., and remember that $a_0 = 1$, the formulae, excluding the first, become

$$(a+1)^2 = a^2, \quad (a+1)^3 = a^3, \quad (a+1)^4 = a^4, \text{ etc.} \quad\text{................(3)}$$

Hence we obtain the difference equations,

$$a(a+1)^2 = a^2(a-1), \quad a(a+1)^3 = a^3(a-1), \quad a(a+1)^4 = a^4(a-1), \text{ etc.}; \ldots(4)$$

and, again taking differences,

$$a^2(a+1)^2 = a^2(a-1)^2, \quad a^2(a+1)^3 = a^3(a-1)^2, \text{ etc.}, \quad\text{...............(5)}$$

and, generally,

$$a^r(a+1)^s = a^s(a-1)^r, \quad\text{.................................(6)}$$

with the exception of (2) and any processes involving (2).

These are avoided if $s$ is greater than 1.

The series of equations

$$a^2(a+1)^2 = a^2(a-1)^2, \quad a^3(a+1)^3 = a^3(a-1)^3, \quad \ldots\ldots \quad a^r(a+1)^r = a^r(a-1)^r,$$

shows that all the odd $a$'s, except $a_1$, are zero; and the equations

$$a(a+1)^2 = a^2(a-1), \quad a^2(a+1)^3 = a^3(a-1)^2, \ \ldots\ldots,$$
$$\text{i.e. } a^n(a+1)^{n+1} = a^{n+1}(a-1)^n,$$

determine in succession the values of $a_2$, $a_4$, ... $a_{2n}$.

Thus, from $a(a+1)^2 = a^2(a-1)$ we have $a_3 + 2a_2 + a_1 = a_3 - a_2$;

$$\therefore\ 3a_2 = -a_1; \quad \therefore\ a_2 = \tfrac{1}{6};$$

and, from the next equation, we obtain

$$a_5 + 3a_4 + 3a_3 + a_2 = a_5 - 2a_4 + a_3, \quad \text{also } a_3 = 0;$$
$$\therefore\ 5a_4 + a_2 = 0; \quad \therefore\ a_4 = -\tfrac{1}{30}.$$

When $n > 1$, we can use the general equation,

$$a^n(a+1)^{n+1}=a^{n+1}(a-1)^n, \text{ and omit the odd } a\text{'s};$$

whence, writing $C_r$ for ${}^nC_r$ and $C_r'$ for ${}^{n+1}C_r$, we have

$$C_1'a_{2n}+C_3'a_{n-2}+C_5'a_{n-4}+\ldots+C_1a_{2n}+C_3a_{n-2}+C_5a_{n-4}+\ldots=0\,;\ldots\ldots.(7)$$

*.g.* if $n=2, 3, 4, 5, 6$ in succession,

$$(3+2)a_4+(1+0)a_2=0, \qquad \therefore\ a_{11}=-\tfrac{1}{30};$$

$$(4+3)a_6+(4+1)a_4=0, \qquad \therefore\ a_6=\tfrac{5}{7} \text{ of } \tfrac{1}{30}=\tfrac{1}{42};$$

$$(5+4)a_8+(10+4)a_6+(1+0)a_4=0, \qquad \therefore\ a_8=\tfrac{1}{9}\{-\tfrac{1}{3}+\tfrac{1}{30}\}=-\tfrac{1}{30};$$

$$(6+5)a_{10}+(20+10)a_8+(6+1)a_6=0, \qquad \therefore\ a_{10}=\tfrac{1}{11}\{1-\tfrac{1}{6}\}=\tfrac{5}{66};$$

$$(7+6)a_{12}+(35+20)a_{10}+(21+6)a_8+(1+0)a_6=0,$$

$$\therefore\ a_{12}=\tfrac{1}{13}\{-\tfrac{25}{6}+\tfrac{9}{10}-\tfrac{1}{42}\}=-\tfrac{691}{2730}.$$

The positive values of these fractions, beginning with $a_2$, are Bernoulli's numbers, generally denoted by $B_1, B_2, \ldots$. The above results give them as far as $B_6$. The next one, $B_7$, is $\tfrac{7}{6}$; after that they are heavy numbers.

The advantage of formula (7), as a basis of calculation, over the formula (3), which is usually taken as the basis, is two-fold:

(1) The calculation of any coefficient $a_{2n}$, when $n$ is even, depends on $\tfrac{1}{2}n$ previous coefficients instead of $n+1$ coefficients; and, when $n$ is odd, it depends on $\tfrac{1}{2}(n-1)$ coefficients instead of $n+1$.

(2) The multipliers needed in (7) are much lighter than those of (3), as they consist of pairs of binomial coefficients of degrees $n+1$ and $n$ instead of binomial coefficients of degree $2n+1$.

For example, $a_{18}$ is calculated from 4 terms, with binomial multipliers of the 9th and 10th degrees, instead of from 10 terms with binomial multipliers of the 19th degree.

If desired, formula (7) may be written in the form

$$(2n+1)a_{2n}+\frac{2n-1}{3}C_2\,.\,a_{2n-2}+\frac{2n-3}{5}C_4\,.\,a_{2n-4}+\ldots,$$

but it is rather easier to work in its original form, and it is probably easiest to *remember* in the unreduced symbolic form,

$$a^n(a+1)^{n+1}=a^{n+1}(a-1)^n.$$

A. LODGE.

**457.** [L¹. a.] In Russell's *Pure Geometry* (1893, p. 47) it is "proved" that a pair of points is a conic by reciprocating a pair of straight lines. Now a pair of straight lines, $AOB$, $COD$, is a conic only when considered as the limit of a hyperbola, *i.e.* if the order of description be $AOCDOBA$ or $AODCOBA$. The tangent to such a conic rotates at $O$, and the true reciprocal is seen to be either the finite straight line which is the limit of an ellipse, or the straight line with a finite gap which is the limit of a hyperbola. The paradox that two points, though a conic, cannot be obtained as a section of a cone, is thus avoided.

A similar argument applies to the statement in Askwith's *Analytical Geometry* (1908, p. 397), that the tangential equation of the second degree represents two points when it reduces to the form

$$(Al+Bm+Cn)(Al'+B'm+C'n)=0.$$

C. W. ADAMS.

**458.** [D. 6. c.] *On certain coefficients connected with the expansions of* $(e^x-1)^n$, $(xD)^nf(x)$, *and* $(x+1)(x+2)(x+3)\ldots(x+n)$.

Let $f, f_1, f_2, \ldots$ denote any function of $x$ and its 1st, 2nd, ... differential coefficients.