During the year 1901 the following presents to the Library have been sent, in the first instance, to Prof. Love, to be indexed for the "International Catalogue of Scientific Literature":—

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. XLV., Parts 1–4; Vol. XLVI., Part 1.

"Proceedings of the Edinburgh Mathematical Society," Vol. XIX.

"Transactions of the Institution of Naval Architects—Memoirs of the Spring and Summer Meetings of 1901."

"Transactions of the Institution of Engineers and Shipbuilders in Scotland," Vol. XLIV., Parts 1–3.

"Journal of the Institute of Actuaries," Vol. XXXV., Part 6; Vol. XXXVI., Parts 2, 3.

"Transactions of the Insurance and Actuarial Society of Glasgow," Ser. 5, Nos. 1–6.

Copies of the following were also sent especially for the purpose of the Catalogue:—

"The Mathematical Gazette," Vol. II., No. 1 and Nos. 26–30.

"The Educational Times," Vol. LIV., Nos. 477–488.

"Journal of the Royal Statistical Society," Vol. LXIV., Parts 1–3.

On Boussinesq's Problem. *By* Horace Lamb.

Read and received February 13th, 1902.

The particular problem here referred to is that of finding the displacements produced in a semi-infinite isotropic solid by pressures applied normally to the plane boundary. This was first solved by Boussinesq,* independent investigations have been given by Hertz† and Cerruti,‡ and quite recently a very ingenious solution has been published by Prof. Michell in the Society's *Proceedings*.§ It may appear that there is hardly room for further discussion of the sub-

* In the *Comptes Rendus*, Vols. LXXXVI.–LXXXVIII. (1878–9); see also his book *Applications des Potentiels*, &c., Paris, 1885.

† *Crelle*, Vol. XCII. (1881); reprinted in *Werke*, Leipzig, 1895, Vol. I., p. 155.

‡ *R. Accad. dei Lincei, Mem. fis. mat.*, t. XIII. (1882). An account of Boussinesq's and Cerruti's investigations, which include also the effect of tangential stresses on the surface, is given in Love's *Elasticity*, Vol. I., p. 248.

§ Vol. XXXI., p. 183 (1899).

ject, but the following method is perhaps worth notice, as being perfectly simple and straightforward, and requiring only the knowledge of one or two integral properties of Bessel's functions. It is suggested by H. Weber's method of treating various potential problems.*

The solid is supposed to be bounded by the plane $z = 0$, and to extend to infinity on the side for which $z > 0$. We aim first at finding the effect of a distribution of surface pressure which is symmetrical about the origin, but otherwise arbitrary.

In a usual notation, the Cartesian equations to be satisfied in the interior of the solid are

$$\left.\begin{aligned} \nabla^2 u = -\frac{\lambda+\mu}{\mu}\frac{d\theta}{dx}, \quad \nabla^2 v = -\frac{\lambda+\mu}{\mu}\frac{d\theta}{dy}, \quad \nabla^2 w = -\frac{\lambda+\mu}{\mu}\frac{d\theta}{dz} \\ \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz} = 0 \end{aligned}\right\}. \qquad (1)$$

Since there is symmetry about Oz, we introduce cylindrical coordinates, and write

$$x = \varpi \cos\omega, \quad y = \varpi \sin\omega, \quad u = q\cos\omega, \quad v = q\sin\omega.$$

The equation $\nabla^2\theta = 0$, which is implied in (1), then takes the form

$$\frac{d^2\theta}{d\varpi^2}+\frac{1}{\varpi}\frac{d\theta}{d\varpi}+\frac{d^2\theta}{dz^2} = 0;$$

this is satisfied by

$$\theta = Ae^{-mz}J_0(m\varpi), \qquad (3)$$

where m is supposed positive in order to secure finiteness for $z = \infty$. To find the corresponding values of q and w we have

$$\left.\begin{aligned} \frac{d^2w}{d\varpi^2}+\frac{1}{\varpi}\frac{dw}{d\varpi}+\frac{d^2w}{dz^2} = \frac{\lambda+\mu}{\mu}Ame^{-mz}J_0(m\varpi) \\ \frac{d^2q}{d\varpi^2}+\frac{1}{\varpi}\frac{dq}{d\varpi}-\frac{q}{\varpi^2}+\frac{d^2q}{dz^2} = -\frac{\lambda+\mu}{\mu}Ame^{-mz}J_0'(m\varpi) \end{aligned}\right\}. \qquad (4)$$

If we assume that, as regards dependence on ϖ,

$$w \propto J_0(m\varpi), \quad q \propto J_1(m\varpi),$$

* See Gray and Mathews, *Bessel Functions*, or H. Weber, *Part. Diff.-Gleichungen d. math. Physik*, Brunswick, 1900-01. The second volume of the latter work has just appeared; it contains yet another solution of Boussinesq's problem, p. 188.

we have
$$\left.\begin{aligned}\frac{d^2w}{dz^2}-m^2w&=\frac{\lambda+\mu}{\mu}Ame^{-mz}\\ \frac{d^2q}{dz^2}-m^2q&=\frac{\lambda+\mu}{\mu}Ame^{-mz}\end{aligned}\right\}, \tag{5}$$

where the factors involving ϖ are omitted. Hence we obtain

$$\left.\begin{aligned}w&=\left(-\frac{\lambda+\mu}{2\mu}Az+\frac{B}{m}\right)e^{-mz}J_0(m\varpi)\\ q&=\left(-\frac{\lambda+\mu}{2\mu}Az+\frac{C}{m}\right)e^{-mz}J_1(m\varpi)\end{aligned}\right\}. \tag{6}$$

Since
$$\theta=\frac{dq}{d\varpi}+\frac{q}{\varpi}+\frac{dw}{dz}, \tag{7}$$

the constants A, B, C are not independent, being connected by the relation
$$B-C=-\frac{\lambda+3\mu}{2\mu}A. \tag{8}$$

Subject to this condition, the formulæ (6) constitute a typical solution of our equations (1), in the case of symmetry. The corresponding values of the surface stresses are

$$\left.\begin{aligned}p_{zz}&=\lambda\theta+2\mu\frac{dw}{dz}=-\mu\,(A+2B)\,J_0(m\varpi)\\ p_{z\varpi}&=\mu\left(\frac{dq}{dz}+\frac{dw}{d\varpi}\right)=-\left\{\tfrac{1}{2}(\lambda+\mu)A+\mu(B+C)\right\}J_1(m\varpi)\end{aligned}\right\}, \tag{9}$$

where z has been put $=0$ after the differentiations. Hence (6) will give the displacements produced by prescribed surface stresses of the types
$$p_{zz}=PJ_0(m\varpi),\quad p_{z\varpi}=QJ_1(m\varpi), \tag{10}$$

provided
$$A+2B=-\frac{P}{\mu},\quad \frac{\lambda+\mu}{2\mu}A+B+C=-\frac{Q}{\mu}. \tag{11}$$

Hence, and from (8), we find

$$\left.\begin{aligned}\theta&=\frac{P-Q}{\lambda+\mu}e^{-mz}J_0(m\varpi),\\ w&=\left[-\frac{P-Q}{2\mu}z+\frac{1}{m}\left\{-\frac{\lambda+2\mu}{2\mu(\lambda+\mu)}P+\frac{Q}{2(\lambda+\mu)}\right\}\right]e^{-mz}J_0(m\varpi)\\ q&=\left[-\frac{P-Q}{2\mu}z+\frac{1}{m}\left\{\frac{P}{2(\lambda+\mu)}-\frac{\lambda+2\mu}{2\mu(\lambda+\mu)}Q\right\}\right]e^{-mz}J_1(m\varpi)\end{aligned}\right\}. \tag{12}$$

Since the hypothesis of a symmetrical *tangential* surface traction does not lead to anything very interesting, we now put $Q = 0$. On the other hand, we may generalize the above solution by putting $P = \phi(m)\, dm$, and integrating from 0 to ∞. Hence, corresponding to the surface stresses

$$p^{zz} = \int_0^\infty J_0(m\varpi)\, \phi(m)\, dm, \quad p_{z\varpi} = 0, \tag{13}$$

we have

$$\left.\begin{aligned} \theta &= \frac{1}{\lambda+\mu}\int_0^\infty e^{-mz} J_0(m\varpi)\, \phi(m)\, dm \\ w &= -\frac{1}{2\mu}\int_0^\infty z e^{-mz} J_0(m\varpi)\, \phi(m)\, dm \\ &\qquad - \frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\int_0^\infty e^{-mz} J_0(m\varpi)\, \phi(m)\, \frac{dm}{m} \\ q &= -\frac{1}{2\mu}\int_0^\infty z e^{-mz} J_1(m\varpi)\, \phi(m)\, dm \\ &\qquad + \frac{1}{2(\lambda+\mu)}\int_0^\infty e^{-mz} J_1(m\varpi)\, \phi(m)\, \frac{dm}{m} \end{aligned}\right\} . \tag{14}$$

To make (13) represent any arbitrary (symmetrical) distribution of normal traction, we have only to determine $\phi(m)$ suitably. This is effected by means of the formula

$$f(\varpi) = \int_0^\infty J_0(m\varpi)\, m\, dm \int_0^\infty f(\lambda)\, J_0(m\lambda)\, \lambda\, d\lambda, \tag{15}$$

viz., if $f(\varpi)$ be the given surface value of p_{zz}, we must write

$$\phi(m) = m \int_0^\infty f(\lambda)\, J_0(m\lambda)\, \lambda\, d\lambda. \tag{16}$$

Thus, to find the effect of a concentrated normal pressure at the origin, we may suppose that $f(\varpi)$ vanishes for all but infinitesimal values of ϖ, when it becomes infinite in such a manner that

$$\int_0^\infty f(\varpi)\, 2\pi\varpi\, d\varpi = -1;$$

this makes

$$\phi(m) = -\frac{m}{2\pi}.$$

The evaluation of the integrals in (14) then follows at once from the known theorem

$$\int_0^\infty e^{-mz} J_0(m\varpi)\, dm = \frac{1}{\sqrt{(\varpi^2+z^2)}} = \frac{1}{r}, \tag{17}$$

where r denotes distance from the origin. Differentiating this with respect to z and ϖ respectively, we infer

$$\int_0^\infty e^{-mz} J_0(m\varpi)\, m\,dm = \frac{z}{r^3}, \tag{18}$$

$$\int_0^\infty e^{-mz} J_1(m\varpi)\, m\,dm = \frac{\varpi}{r^3}. \tag{19}$$

Again, integrating (19) with respect to z, and determining the additive constant so as to make the result vanish for $z = \infty$, we have

$$\int_0^\infty e^{-mz} J_1(m\varpi)\, dm = \frac{\varpi}{r(r+z)}. \tag{20}$$

Substituting in (14), we find

$$\left.\begin{aligned} \theta &= -\frac{1}{2\pi(\lambda+\mu)}\,\frac{z}{r^3} \\ w &= \frac{1}{4\pi\mu}\,\frac{z^2}{r^3} + \frac{\lambda+2\mu}{4\pi\mu(\lambda+\mu)}\,\frac{1}{r} \\ q &= \frac{1}{4\pi\mu}\,\frac{z\varpi}{r^3} - \frac{1}{4\pi(\lambda+\mu)}\,\frac{\varpi}{r(r+z)} \end{aligned}\right\}, \tag{21}$$

which are the known results for this case.* For the component stresses at any point we deduce

$$\left.\begin{aligned} p_{zz} &= -\frac{3}{2\pi}\,\frac{z^3}{r^5}, \quad p_{z\varpi} = -\frac{3}{2\pi}\,\frac{z^2\varpi}{r^5} \\ p_{\varpi\varpi} &= -\frac{3}{2\pi}\,\frac{z\varpi^2}{r^5} + \frac{\mu}{2\pi(\lambda+\mu)}\,\frac{1}{r(r+z)} \end{aligned}\right\}. \tag{22}$$

These formulæ have been discussed by Boussinesq and others; but one or two simple results relating to the case of incompressibility ($\lambda = \infty$) may be noted. In the first place the differential equation

* See, for example, Love, *Elasticity*, Vol. I., p. 270.

of the "isostatic" lines in a meridian plane, viz.,

$$\frac{p_{zz}\,dz+p_{z\varpi}\,d\varpi}{dz}=\frac{p_{z\varpi}\,dz+p_{\varpi\varpi}\,d\varpi}{d\varpi}, \tag{23}$$

reduces to

$$(\varpi\,d\varpi+z\,dz)(z\,d\varpi-\varpi\,dz)=0; \tag{24}$$

so that the curves in question consist of concentric circles and radial straight lines.* Again, the values of q and w can be expressed in terms of a displacement-function ψ, viz., we have

$$q=\frac{1}{\varpi}\frac{d\psi}{dz},\quad w=-\frac{1}{\varpi}\frac{d\psi}{d\varpi}, \tag{25}$$

where

$$\psi=\frac{1}{4\pi\mu}\frac{\varpi^2}{r}; \tag{26}$$

the equation of the lines of displacement is therefore

$$\varpi^2=ar.$$

These curves start from the boundary at right angles, and then curve outwards; they have inflexions at distances $\frac{3}{2}a$ from the origin, and ultimately became parabolic. When the substance is not incompressible the lines of displacement are inclined towards the axis at the surface, making an angle $\tan^{-1}(\lambda+2\mu)/\mu$ with it.

Boussinesq has also investigated the case where a perfectly rigid circular cylinder of finite radius (a) presses normally (as *e.g.*, by its own gravity when the surface is horizontal) against the surface. The conditions to be satisfied are then

$$w=\text{const.}\quad \text{for}\quad z=0,\ \varpi<a,$$

and

$$p_{zz}=0\qquad \text{for}\quad z=0,\ \varpi>a.$$

The function $\phi(m)$ which occurs in (13) and (14) must therefore satisfy the conditions

$$\int_0^\infty J_0(m\varpi)\,\phi(m)\,dm=0\qquad [\varpi>a], \tag{27}$$

$$\int_0^\infty J_0(m\varpi)\,\phi(m)\frac{dm}{m}=\text{const.}\quad [\varpi<a]. \tag{28}$$

It is known that

$$\int_0^\infty J_0(m\varpi)\sin ma\,dm=\frac{1}{\sqrt{(a^2-\varpi^2)}},\quad \text{or}\quad 0, \tag{29}$$

* Compare the sketch given by Hertz, *Werke*, Vol. I., p. 185.

and $$\int_0^\infty J_0(m\varpi)\frac{\sin ma}{m}\,dm = \tfrac{1}{2}\pi, \quad \text{or} \quad \sin^{-1}\frac{a}{\varpi}, \tag{30}$$

the first or second value being taken, in each case, according as $\varpi \lesseqgtr a$. The conditions (27), (28) are therefore satisfied by

$$\phi(m) = C\sin ma,$$

which gives an aggregate normal pressure

$$W = -2\pi aC.$$

The surface displacements are accordingly, in terms of W,

$$\left.\begin{aligned} w_0 &= \frac{\lambda+2\mu}{4\pi\mu(\lambda+\mu)}\frac{W}{a}\int_0^\infty J_0(m\varpi)\frac{\sin ma}{m}\,dm \\ q_0 &= -\frac{1}{4\pi(\lambda+\mu)}\frac{W}{a}\int_0^\infty J_1(m\varpi)\frac{\sin ma}{m}\,dm \end{aligned}\right\}; \tag{31}$$

whence $$\left.\begin{aligned} w_0 &= \frac{\lambda+2\mu}{8\mu(\lambda+\mu)}\frac{W}{a} \\ q_0 &= -\frac{W}{4\pi(\lambda+\mu)a}\frac{a-\sqrt{(a^2-\varpi^2)}}{\varpi} \end{aligned}\right\}[\varpi<a], \tag{32}$$

and $$\left.\begin{aligned} w_0 &= \frac{\lambda+2\mu}{8\mu(\lambda+\mu)}\frac{W}{a}\frac{2}{\pi}\sin^{-1}\frac{a}{\varpi} \\ q_0 &= -\frac{W}{4\pi(\lambda+\mu)a}\frac{a}{\varpi} \end{aligned}\right\}[\varpi>a].* \tag{33}$$

In this problem the pressure on the surface of contact increases from the centre to the circumference (where it is infinite) according to the law given by (29). The effect of a pressure distributed *uniformly* over a circular area of radius a is obtained by making $f(\lambda) = -1$ from 0 to a, and $= 0$ from a to ∞, in (16). If W be the total pressure, we find for the surface displacements

$$\left.\begin{aligned} w_0 &= \frac{\lambda+2\mu}{4\pi\mu(\lambda+\mu)}\frac{W}{a}\int_0^\infty J_0(m\varpi)\,J_1(ma)\frac{dm}{m} \\ q_0 &= -\frac{1}{4\pi(\lambda+\mu)}\frac{W}{a}\int_0^\infty J_1(m\varpi)\,J_1(ma)\frac{dm}{m} \end{aligned}\right\}. \tag{34}$$

* The evaluation of the second integral in (31) is effected by multiplying both sides of (29) by $\varpi\,d\varpi$ and integrating with respect to ϖ.

The definite integrals can be evaluated in the form of infinite series by means of the formulæ

$$\int_0^\infty e^{-mz} J_0(m\varpi)\, m^n dm = \frac{n!}{r^{n+1}} P_n\left(\frac{z}{r}\right), \tag{35}$$

$$\int_0^\infty e^{-mz} J_1(m\varpi)\, m^n dm = \frac{(n-1)!}{r^{n+2}} P_n'\left(\frac{z}{r}\right) \varpi, \tag{36}$$

where P_n is the symbol of the ordinary zonal harmonic. These identities follow easily from (17).* We thus find

$$w_0 = \frac{\lambda+2\mu}{4\pi\mu(\lambda+\mu)} \frac{W}{a} F\left(\tfrac{1}{2}, -\tfrac{1}{2}, 1, \frac{\varpi^2}{a^2}\right), \tag{37}$$

for $\varpi < a$, and

$$w_0 = \frac{\lambda+2\mu}{4\pi\mu(\lambda+\mu)} \frac{W}{\varpi} F\left(\tfrac{1}{2}, \tfrac{1}{2}, 2, \frac{a^2}{\varpi^2}\right), \tag{38}$$

for $\varpi > a$.†

‡ [The foregoing analysis can be adapted to the study of the deformations of an infinite *plate* produced by normal forces applied to its boundaries; the results, however, do not appear to admit of easy reduction.

The most interesting case is that of pure flexure, where the middle surface is unextended. We assume therefore

$$\theta = A \sinh mz\, J_0(m\varpi), \tag{39}$$

the origin being taken in the middle surface. We find

$$\left.\begin{aligned} w &= \left(-\frac{\lambda+\mu}{2\mu} Az \sinh mz + \frac{B}{m}\cosh mz\right) J_0(m\varpi) \\ q &= \left(\frac{\lambda+\mu}{2\mu} Az \cosh mz + \frac{C}{m}\sinh mz\right) J_1(m\varpi) \end{aligned}\right\}, \tag{40}$$

* [They are known results; see Hobson, *Proc. Lond. Math. Soc.*, Vol. xxv., pp. 72, 73.]

† The method of this paper is, of course, not restricted to the case of symmetry about the axis of z; and it would doubtless be possible to work out in a similar manner the effect of a concentrated tangential surface traction. It can hardly be claimed, however, that the method is specially appropriate to this case.

‡ [Added March 15, 1902, in accordance with a suggestion made by Prof. Love at the meeting when the paper was read.]

with the condition $$B+C = \frac{\lambda+3\mu}{2\mu} A. \tag{41}$$

If h denote the half-thickness, we have, at the surfaces $z = \pm h$,

$$p_{zz} = \pm \tfrac{1}{2} P_{,} J_0(m\varpi), \quad p_{z\varpi} = 0, \tag{42}$$

provided

$$\left.\begin{aligned} -\left(\sinh mh + \frac{\lambda+\mu}{\mu} mh \cosh mh\right) A + 2B \sinh mh &= \tfrac{1}{2} P \\ \frac{\lambda+\mu}{2\mu} (\cosh mh + 2mh \sinh mh) - (B-C) \cosh mh &= 0 \end{aligned}\right\}. \tag{43}$$

These equations, combined with (41), give A, B, C. In particular, for the deflection w_0 of the middle surface we find

$$w_0 = \frac{\frac{\lambda+2\mu}{\lambda+\mu} \cosh mh + \sinh mh}{\sinh 2mh - 2mh} \, \frac{P}{2m} J_0(m\varpi). \tag{44}$$

This can be generalized as before by writing

$$P = \phi(m)\, dm,$$

and integrating with respect to m between the limits 0 and ∞; and, if the total normal force per unit area (supposed divided equally between the two faces) be denoted by $f(\varpi)$, the value of $\phi(m)$ is as in (16); whence

$$w_0 = \tfrac{1}{2} \int_0^\infty \frac{\frac{\lambda+2\mu}{\lambda+\mu} \cosh mh + \sinh mh}{\sinh 2mh - 2mh} J_0(m\varpi)\, dm \int_0^\infty f(\lambda) J_0(m\lambda) \lambda \, d\lambda. \tag{45}$$

As a particular case we may suppose the plate to be horizontal, and to be supported along the circumference of a circle ($r = a$), whilst a load W is applied at the origin. We find

$$w_0 = \frac{1}{4\pi} \int_0^\infty \frac{\frac{\lambda+2\mu}{\lambda+\mu} \cosh mh + \sinh mh}{\sinh 2mh - 2mh} \left\{1 - J_0(ma)\right\} J_0(m\varpi)\, dm. \tag{46}$$]