ON THE CONTINUATION OF THE HYPERGEOMETRIC SERIES

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[Received September 11th, 1917.—Read November 1st, 1917.]

1. The idea of the continuation of a series is algebraic in origin, so that, from a theoretical point of view, the continuation of a series $P(z)$ should be obtainable by expanding it in powers of $z-z_1$ by Taylor's theorem, and then expanding the series again in powers of $z-z_2$, and so on. But this process generally presents very great difficulty, and the continuations are usually obtained by contour integration or the known results as to the theory of the solution of differential equations. It would seem, however, to be of interest to obtain the continuation of a few series by algebraic processes only.

2. In the year 1902 I obtained by *ordinary algebraic expansion* the continuations of two series which have one singular point, viz., the binomial and logarithmic series; and of two series which have two singular points, viz., the series for arc tan x and arc sin x (*Proc. London Math. Soc.*, Ser. 1, Vol. xxxv, pp. 388–416).

The object of this paper is to apply the same method to the case next in order of complexity, viz. to the hypergeometric series, which has three singular points. The analytical difficulties in carrying through the work are different in kind to those which present themselves in the series mentioned above.

3. It is sufficient to prove the equation

$$\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)\,F(\alpha,\,\beta,\,\alpha+\beta-\gamma+1,\,1-x)$$

$$= \Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(1-\gamma)\,\Gamma(\alpha+\beta-\gamma+1)\,F(\alpha,\,\beta,\,\gamma,\,x)$$

$$+\left[\begin{array}{r}\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)\,\Gamma(\gamma-1)\,\Gamma(\alpha+\beta-\gamma+1)\,x^{1-\gamma}\\ \times F(\alpha-\gamma+1,\,\beta-\gamma+1,\,2-\gamma,\,x)\end{array}\right], \quad \text{(I)}$$

because the other relations of the same kind can be obtained from it by the aid of Euler's identities and appropriate transformations of the variable and the parameters. Purely algebraic proofs of Euler's identities will be found in the *Messenger of Mathematics*, New Series, No. 524, December 1914. I have been unable to find Euler's or any other purely algebraic proof of Euler's identities.

4. The six fundamental solutions of the differential equation satisfied by the hypergeometric series can be taken as

(i) $F(\alpha, \beta, \gamma, x)$,

(ii) $x^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x)$,

(iii) $F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x)$,

(iv) $(1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1, 1-x)$,

(v) $x^{-\alpha} F(\alpha, \alpha-\gamma+1, \alpha-\beta+1, x^{-1})$,

(vi) $x^{-\beta} F(\beta, \beta-\gamma+1, \beta-\alpha+1, x^{-1})$.

These are all finite at the singular points in their respective domains if the real parts of α, β, $1-\gamma$ and $\gamma-\alpha-\beta$ are positive.

The demonstration is greatly simplified if we suppose this to be the case, and if moreover we suppose the parameters to be real. This makes α, β, γ and $\gamma-\alpha-\beta$ all positive proper fractions. It will be seen that the coefficients of every power of x in each of the six fundamental solutions are in this case positive.

5. The following notation will be used in this paper.

The product $a(a+1)(a+2) \ldots (a+r-1)$ will be denoted by a_r.

And
$$G\begin{pmatrix} \alpha, & \beta, & \\ \gamma, & \delta, & s \end{pmatrix}$$

will denote the sum of the series

$$1+\sum_{p=1}^{s} (\alpha_p \beta_p)/(\gamma_p \delta_p). \tag{II}$$

Then $G\begin{pmatrix} \alpha, & \beta, & \\ 1, & \gamma, & \infty \end{pmatrix}$ is the series usually denoted by $F(\alpha, \beta, \gamma, 1)$, and we have

$$G\begin{pmatrix} \alpha, & \beta, & \\ 1, & \gamma, & \infty \end{pmatrix} = \{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)\} / \{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)\}, \tag{III}$$

provided that the real part of $\gamma-\alpha-\beta$ is positive.

6. In the *Proceedings of the London Mathematical Society*, 1907, pp. 335–338, it is shown by purely algebraic methods that

$$G\begin{pmatrix} a, \beta, & \\ 1, \gamma, & s\end{pmatrix} = \{\Gamma(\gamma)\,\Gamma(\gamma-a-\beta)\}/\{\Gamma(\gamma-a)\,\Gamma(\gamma-\beta)\}$$
$$-\{(a_{s+1}\beta_{s+1})/(s!\,\gamma_{s+1})\}\,(\gamma-a-\beta)^{-1}\,G\begin{pmatrix}\gamma-a, & \gamma-\beta, & \\ \gamma-a-\beta+1, & \gamma+s+1, & \infty\end{pmatrix}. \quad \text{(IV)}$$

7. In the *Quarterly Journal of Pure and Applied Mathematics*, 1910, Vol. 41, pp. 128–135, there is an algebraic proof by Mr. Whipple and the writer of this paper of the theorem that

$$(\delta-1)^{-1}\,G\begin{pmatrix} a, \beta, & \\ \gamma, \delta, & \infty\end{pmatrix}$$
$$= (\gamma+\delta-a-\beta-1)^{-1}\,G\begin{pmatrix}\gamma-a, & \gamma-\beta, & \\ & \gamma, \ \gamma+\delta-a-\beta, & \infty\end{pmatrix}, \quad \text{(V)}$$

provided that the real parts of δ and of $\gamma+\delta-a-\beta$ are each greater than unity.

8. The theorem that

$$(a+b)_n = \sum_{r=0}^{n} {}_nC_r a_{n-r} b_r \quad \text{(VI)}$$

can be deduced from Vandermonde's theorem or proved independently as follows.

The theorem is obviously true for $n = 2$ and 3.

Assuming it to be true for any integral value of n, multiply each side by $(a+b+n)$.

The left-hand side becomes $(a+b)_{n+1}$.

And since
$$(a+b+n)\,a_{n-r}\,b_r$$
$$= (a+n-r)\,a_{n-r}\,b_r + (b-r)\,a_{n-r}\,b_r$$
$$= a_{n-r+1}\,b_r + a_{n-r}\,b_{r+1},$$

the right-hand side becomes

$$\sum_{r=0}^{n} {}_nC_r(a_{n-r+1}\,b_r + a_{n-r}\,b_{r+1})$$
$$= \sum_{r=0}^{n} ({}_nC_r + {}_nC_{r-1})\,a_{n+1-r}\,b_r + {}_nC_n\,a_0\,b_{n+1}$$
$$= \sum_{r=0}^{n+1} {}_{n+1}C_r\,(a_{n+1-r}\,b_r);$$

therefore $$(a+b)_{n+1} = \sum_{r=0}^{n+1} {}_{n+1}C_r\, a_{n+1-r} b_r,$$

and now the theorem follows by induction.

9. The series $$F(\alpha,\ \beta,\ \alpha+\beta-\gamma+1,\ 1-x),$$

and $$x^{1-\gamma} F(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma,\ x),$$

cannot be expressed in series of positive integral powers of x.

Consider therefore the first $(n+1)$ terms of the first of these series.

Expanding each term in powers of x, it will be found that the coefficient of x^r is

$$[(-1)^r \alpha_r \beta_r / \{r!\,(\alpha+\beta-\gamma+1)_r\}] \times G\begin{pmatrix}\alpha+r, & \beta+r, & \\ 1, & \alpha+\beta-\gamma+1+r, & n-r\end{pmatrix}. \quad \text{(VII)}$$

Applying equation (IV) to transform the series G in (VII), the whole expression (VII) can be rèplaced by another consisting of two parts.

The first of these parts is

$$[(-1)\, r\alpha_r \beta_r / \{r!\,(\alpha+\beta-\gamma+1)_r\}] \times \Gamma(\alpha+\beta-\gamma+1+r)\,\Gamma(1-\gamma-r)/\{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)\},$$

which reduces to

$$[\alpha_r \beta_r / \{r!\,\gamma_r\}]\,\Gamma(1-\gamma)\,\Gamma(\alpha+\beta-\gamma+1)/\{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)\}, \quad \text{(VIII)}$$

and thus accounts for the coefficient of x^r in the term containing $F(\alpha,\ \beta,\ \gamma,\ x)$ in (I), after that equation has been divided by the coefficient of $F(\alpha,\ \beta,\ \alpha+\beta-\gamma+1,\ 1-x)$ on the left.

10. The second part of the coefficient of x^r is

$$[(-1)^{r+1} \alpha_r \beta_r / \{r!\,(\alpha+\beta-\gamma+1)_r\}]$$
$$\times [(\alpha+r)_{n-r+1}\,(\beta+r)_{n-r+1} / \{(n-r)!\,(\alpha+\beta-\gamma+1+r)_{n-r+1}\}]$$
$$\times (1-\gamma-r)^{-1}\, G\begin{pmatrix}\alpha-\gamma+1, & \beta-\gamma+1, & \\ 2-\gamma-r, & \alpha+\beta-\gamma+n+2, & \infty\end{pmatrix}. \quad \text{(IX)}$$

The series in (IX) satisfies the condition of convergence.

It is necessary in what follows to separate it into two parts, taking the first $n+1$ terms in the first part, and the remaining terms in the second part.

The expression (IX) can be reduced to the form

$$[\alpha_{n+1}\beta_{n+1}/\{(\alpha+\beta-\gamma+1)_{n+1}(\gamma-1)_{n+1}\}] \quad \text{(X)}$$

$$\times[(-1)^{r+1}(\gamma-1)_{n+1}/\{r!\,(n-r)!\}] \quad \text{(XI)}$$

$$\times(1-\gamma-r)^{-1}G\begin{pmatrix}\alpha-\gamma+1, & \beta-\gamma+1, & \\ 2-\gamma-r, & \alpha+\beta-\gamma+n+2, & \infty\end{pmatrix}. \quad \text{(XII)}$$

The $(p+1)$-th term in the series (XII) is

$$(\alpha-\gamma+1)_p(\beta-\gamma+1)_p/\{(\alpha+\beta-\gamma+n+2)_p(1-\gamma-r)_{p+1}\}. \quad \text{(XIII)}$$

Now it can be shown by induction, or by the method of partial fractions that

$$p!/a_{p+1} = \sum_{s=0}^{p}(-1)^s\,{}_pC_s/(a+s);$$

therefore $\quad p!/(1-\gamma-r)_{p+1} = \sum_{s=0}^{p}(-1)^s\,{}_pC_s/(1-\gamma-r+s$

therefore

$$(\alpha-\gamma+1)_p(\beta-\gamma+1)_p/\{(\alpha+\beta-\gamma+n+2)_p(1-\gamma-r)_{p+1}\}$$
$$=[(\alpha-\gamma+1)_p(\beta-\gamma+1)_p/\{(\alpha+\beta-\gamma+n+2)_p\,p!\}]$$
$$\times\left[\sum_{s=0}^{p}(-1)^s\,{}_pC_s/(1-\gamma-r+s)\right].$$

Consequently the terms in the series (XII), which give rise to terms containing $(1-\gamma-r+k)^{-1}$, are those for which $p \geqslant k$.

As, however, we consider in the first instance only the first $n+1$ terms of the series (IX), we must take $p = k,\ k+1,\ \ldots,\ n$. Also $s = k$.

If now we write t_p for

$$(\alpha-\gamma+1)_p(\beta-\gamma+1)_p/\{p!\,(\alpha+\beta-\gamma+n+2)_p\},$$

then the terms obtainable from the first $(n+1)$ terms in (XII) containing

$(1-\gamma-r+k)^{-1}$ are

$$(-1)^k(1-\gamma-r+k)^{-1}({}_kC_\kappa t_\kappa+{}_{\kappa+1}C_\kappa t_{\kappa+1}+{}_{\kappa+2}C_\kappa t_{\kappa+2}+\ldots+{}_nC_\kappa t_n)$$

$$=\left\{\begin{matrix}(-1)^k(1-\gamma-r+k)^{-1}(a-\gamma+1)_k(\beta-\gamma+1)_k/\{k!\,(a+\beta-\gamma+n+2)_k\} \\ \times G\begin{pmatrix}a-\gamma+1+k, & \beta-\gamma+1+k, & \\ & 1, \quad a+\beta-\gamma+n+2+k, & n-k\end{pmatrix}\end{matrix}\right\}. \tag{XIV}$$

Hence the portion of the second part of the coefficient of x^r which contains $(1-\gamma-r+k)^{-1}$ is the whole expression (XIV) multiplied by the product of the expressions (X) and (XI).

Now $\quad (\gamma-1)_{n+1}=(-1)^k(\gamma-k-1)_{n+1}(\gamma-k+n)_k/(2-\gamma)_k.$

Making use of this, the portion of the second part of the coefficient of x^r which contains $(1-\gamma-r+k)^{-1}$ is

$$\begin{aligned}&a_{n+1}\beta_{n+1}/\{(a+\beta-\gamma+1)_{n+1}(\gamma-1)_{n+1}\} \\ &\times(a-\gamma+1)_k(\beta-\gamma+1)_k/\{k!\,(2-\gamma)_k\} \\ &\times(\gamma-k+n)_k/(a+\beta-\gamma+n+2)_k \\ &\times(\gamma-k-1)_{n+1}/(1-\gamma-r+k) \\ &\times(-1)^{r+1}/\{r!\,(n-r)!\} \\ &\times G\begin{pmatrix}a-\gamma+1+k, & \beta-\gamma+1+k, & \\ & 1, \quad a+\beta-\gamma+n+2+k, & n-k\end{pmatrix}.\end{aligned} \tag{XV}$$

If the series G in (XV) were continued to infinity, the condition for convergence would be $k<n+\gamma$.

By hypothesis γ is a positive proper fraction and the greatest value of k is n. So the series continued to infinity is convergent.

Let us suppose *for the moment* that we replace the series G in (XV) by its sum to infinity. Since all the coefficients in G are positive, this amounts to an addition to the value of G. We shall investigate the effect of this addition at a later stage.

The sum of the series G in (XV) supposed extended to infinity is

$$\Gamma(a+\beta-\gamma+n+2+k)\,\Gamma(n+\gamma-k)/\{\Gamma(a+n+1)\,\Gamma(\beta+n+1)\}.$$

This is equal to

$$\begin{aligned}&\Gamma(a+\beta-\gamma+n+2)\,\Gamma(n+\gamma)/\{\Gamma(a+n+1)\,\Gamma(\beta+n+1)\} \\ &\times(a+\beta-\gamma+n+2)_k/(n+\gamma-k)_k.\end{aligned}$$

Supposing then G in (XV) to be extended to infinity, (XV) is replaced by

$$\alpha_{n+1}\beta_{n+1}/\{(\alpha+\beta-\gamma+1)_{n+1}(\gamma-1)_{n+1}\}$$
$$\times\Gamma(\alpha+\beta-\gamma+n+2)\,\Gamma(n+\gamma)/\{\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)\}$$
$$\times(\alpha-\gamma+1)_k(\beta-\gamma+1)_k/\{k!\,(2-\gamma)_k\}$$
$$\times(-1)^r(\gamma-k-1)_{n+1}/\{r!\,(n-r)!\,(\gamma-k-1+r)\}$$
$$=\Gamma(\alpha+\beta-\gamma+1)\,\Gamma(\gamma-1)/\{\Gamma(\alpha)\,\Gamma(\beta)\}$$
$$\times(\alpha-\gamma+1)_k(\beta-\gamma+1)_k/\{k!\,(2-\gamma)_k\}$$
$$\times(-1)^r(\gamma-k-1)_{n+1}/\{r!\,(n-r)!\,(\gamma-k-1+r)\}. \qquad \text{(XVI)}$$

The expression (XVI) is that portion of the coefficient of x^r which contains $(\gamma-k-1+r)^{-1}$.

The next step is to sum the product of x^r and the expression (XVI) from $r=0$ to $r=n$.

It is therefore sufficient to calculate

$$\sum_{r=0}^{n}(-1)^r x^r(\gamma-k-1)_{n+1}/\{r!\,(n-r)!\,(\gamma-k-1+r)\}. \qquad \text{(XVII)}$$

11. To determine its meaning, let us expand $x^{1-\gamma}$ in powers of $1-x$, it being assumed that $|1-x|<1$.

The first $(n+1)$ terms of the expansion are

$$1+(\gamma-1)(1-x)+(\gamma-1)_2(1-x)^2/2!+\ldots+(\gamma-1)_n(1-x)^n/n!.$$

Arranging them in a series of ascending powers of x we get

$$\sum_{r=0}^{n}(-1)^r x^r(\gamma-1)_{n+1}/\{r!\,(n-r)!\,(\gamma-1+r)\}.$$

From this the expression (XVII) can be deduced by changing γ into $\gamma-k$.

Hence the expression (XVII) is seen to be the equivalent of the first $n+1$ terms of the expansion of

$$x^{1-\gamma+k},$$

in a series of powers of $1-x$.

To see whether we can replace the expression (XVII) by $x^{1-\gamma+k}$, we must investigate the value of the terms following the $(n+1)$-th. We therefore require a superior limit for

$$|(\gamma-k-1)_{n+1}(1-x)^{n+1}/(n+1)!+(\gamma-k-1)_{n+2}(1-x)^{n+2}/(n+2)!+\dots \text{ to } \infty|$$

$$=|(\gamma-k-1)_{n+1}(1-x)^{n+1}/(n+1)!|$$

$$\times\left|1+\frac{\gamma-k+n}{n+2}(1-x)+\frac{(\gamma-k+n)_2}{(n+2)_2}(1-x)^2+\dots\right|.$$

Now, since $k<n+\gamma$, the coefficients of the powers of $1-x$ are all less than 1, and therefore the series last written is less than

$$|1+|1-x|+|1-x|^2+\dots|;$$

and therefore less than $\qquad 1/\{1-|1-x|\}$.

Moreover, as n increases,

$$|(\gamma-k-1)_{n+1}(1-x)^{n+1}/(n+1)!|$$

tends to zero.

Hence, provided that $k<n+\gamma$, we can replace the expression (XVII) by $x^{1-\gamma+k}$.

Hence the second part of the coefficient of x^r, if summed from $r=0$ to n, contains the term

$$\Gamma(\alpha+\beta-\gamma+1)\,\Gamma(\gamma-1)/\{\Gamma(\alpha)\,\Gamma(\beta)\}$$
$$\times(\alpha-\gamma+1)_k(\beta-\gamma+1)_k/\{k!\,(2-\gamma)_k\}$$
$$\times x^{1-\gamma+k},$$

provided that it is admissible to replace the series G in (XV) by its sum to infinity.

This gives the coefficient of $x^{1-\gamma+k}$ in the term containing

$$x^{1-\gamma}F(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma,\ x)$$

in (I), after that equation has been divided by the coefficient of

$$F(\alpha,\ \beta,\ \alpha+\beta-\gamma+1,\ 1-x).$$

12. There remain still outstanding two points.

(i) What is the effect on the result of the addition to the series G in (XV) of the terms from the $(n-k+1)$-th to infinity?

(ii) What is the effect of neglecting the terms in the series G in (XII) from the $(n+1)$-th to infinity?

13. As regards the first point it is clear that if the number k (which does not exceed n) tends to infinity with n, then the effect of replacing

$$G\begin{pmatrix} \alpha-\gamma+1+k, & \beta-\gamma+1+k, & \\ & 1, \ \alpha+\beta-\gamma+n+2+k, & n-k \end{pmatrix}$$

by its sum to infinity may add largely to the value of the series G in (XV); but the total arrived at, viz.

$$\begin{aligned} &\Gamma(\alpha+\beta-\gamma+1)\,\Gamma(\gamma-1)/\{\Gamma(\alpha)\,\Gamma(\beta)\} \\ &\times(\alpha-\gamma+1)_k\,(\beta-\gamma+1)_k/\{k!\,(2-\gamma)_k\} \\ &\times x^{1-\gamma+k}, \end{aligned}$$

is small; and therefore the effect on the whole series to be calculated is small, and by sufficiently increasing k can be made as small as we please. Since n can be made as large as we please, it is possible to make k as large as we please.

14. If, however, k does not tend to infinity with n, it is necessary to show that the sum added to the G in (XV) is small when n is large.

Now the ratio of the sum of the terms added to the total arrived at can be obtained from equation (IV) by dividing the second term on its right-hand side by its first term, and then replacing α, β, γ, s by $\alpha-\gamma+1+k$, $\beta-\gamma+1+k$, $\alpha+\beta-\gamma+n+2+k$, $n-k$ respectively.

This ratio is

$$\begin{aligned} &\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)/\{\Gamma(\alpha+\beta-\gamma+2+n+k)\,\Gamma(n+\gamma-k)\} \\ &\times\left[\frac{(\alpha-\gamma+1+k)_{n-k+1}\,(\beta-\gamma+1+k)_{n-k+1}}{\{(n-k)!\,(\alpha+\beta-\gamma+2+n+k)_{n-k+1}\}}\right] \\ &\times(\gamma+n-k)^{-1}\,G\begin{pmatrix} \alpha+n+1, & \beta+n+1, & \\ \gamma+n-k+1, & \alpha+\beta-\gamma+3+2n, & x \end{pmatrix}. \end{aligned}$$

The series G satisfies the condition of convergence.

From (V) it follows that

$$\begin{aligned} &(\alpha+\beta-\gamma+2n+2)^{-1}\,G\begin{pmatrix} \alpha+n+1, & \beta+n+1, & \\ \gamma+n-k+1, & \alpha+\beta-\gamma+2n+3, & x \end{pmatrix} \\ &=(n-k+1)^{-1}\,G\begin{pmatrix} \gamma-\alpha-k, & \gamma-\beta-k, & \\ n+\gamma-k+1, & n-k+2, & x \end{pmatrix}. \end{aligned}$$

Hence the ratio under discussion is

$$\Gamma(a+n+1)\,\Gamma(\beta+n+1)/\{\Gamma(a+\beta-\gamma+2+n+k)\,\Gamma(n+\gamma-k)\}$$

$$\times\left[\frac{(a-\gamma+k+1)_{n-k+1}\,(\beta-\gamma+k+1)_{n-k+1}}{\{(n-k)!\,(a+\beta-\gamma+n+2+k)_{n-k+1}\}}\right]$$

$$\times(a+\beta-\gamma+2n+2)/(n-k+1)$$

$$\times(\gamma+n-k)^{-1}\,G\begin{pmatrix}\gamma-a-k, & \gamma-\beta-k, & \infty\\ n+\gamma-k+1, & n-k+2, & \end{pmatrix}$$

$$=\Gamma(a+n+1)\,\Gamma(\beta+n+1)/\{\Gamma(n+\gamma-k+1)\,\Gamma(a+\beta-\gamma+2n+2)\}$$

$$\times(a-\gamma+k+1)_{n-k+1}\,(\beta-\gamma+k+1)_{n-k+1}/(n-k+1)!$$

$$\times G\begin{pmatrix}\gamma-a-k, & \gamma-\beta-k, & \infty\\ n+\gamma-k+1, & n-k+2, & \end{pmatrix}$$

$$=\quad G\begin{pmatrix}\gamma-a-k, & \gamma-\beta-k, & \infty\\ n+\gamma-k+1, & n-k+2, & \end{pmatrix}$$

$$\times\Gamma(a+n+1)\,\Gamma(\beta+n+1)\,\Gamma(a-\gamma+n+2)\,\Gamma(\beta-\gamma+n+2),$$

divided by

$$\left[\begin{matrix}\Gamma(n-k+2)\,\Gamma(n+\gamma-k+1)\,\Gamma(a+\beta-\gamma+2n+2)\\ \times\Gamma(a-\gamma+k+1)\,\Gamma(\beta-\gamma+k+1)\end{matrix}\right].$$

Now, since k does not tend to infinity with n, the series G is near to unity.

It is therefore sufficient to find the value to which

$$\left.\begin{matrix}\Gamma(a+n+1)\,\Gamma(\beta+n+1)\,\Gamma(a-\gamma+n+2)\,\Gamma(\beta-\gamma+n+2)\\ \text{divided by}\\ \Gamma(n-k+2)\,\Gamma(n+\gamma-k+1)\,\Gamma(a+\beta-\gamma+2n+2)\end{matrix}\right\}\quad\text{(XVIII)}$$

tends as n increases.

Now omitting the factor $(2\pi)^{\frac{1}{2}}$, which is not here material, we may replace $\Gamma(x)$, when x is large, by $e^{-x}x^{x-\frac{1}{2}}$; and therefore $\Gamma(a+n)$, when n is large, by

$$e^{-a-n}\,(a+n)^{a+n-\frac{1}{2}},$$

i.e.
$$e^{-a-n}n^{n+a-\frac{1}{2}}\left(1+\frac{a}{n}\right)^{n+a-\frac{1}{2}}$$

Now $\left(1+\dfrac{a}{n}\right)^{n+a-\frac{1}{2}}$ tends to e^a as n tends to infinity.

Consequently we can replace $\Gamma(a+n)$ by $e^{-n}n^{n+a-\frac{1}{2}}$, and similarly $\Gamma(a+2n)$ by $e^{-2n}(2n)^{2n+a-\frac{1}{2}}$. Hence the expression (XVIII) can be replaced by

$$n^{2k+\alpha+\beta-2\gamma+\frac{1}{2}} \div 2^{2n+\alpha+\beta-\gamma+\frac{3}{2}},$$

which is small since k does not tend to infinity with n.

It should be observed that the $n-k+1$ terms of the series G in (XV), which go to make up a portion of the coefficient of $x^{1-\gamma+k}$ in

$$x^{1-\gamma}F(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma,\ x),$$

are contributed by the first $n+1$ terms of the series

$$F(\alpha,\ \beta,\ \alpha+\beta-\gamma+1,\ 1-x).$$

However large k may be, we can take n so large that the coefficient of $x^{1-\gamma+k}$ is obtained to any required degree of accuracy.

This completes the examination of the first outstanding point noted in § 12, and shows that we may replace the $n-k+1$ terms of the series G in (XV) by the sum of that series extended to infinity.

15. It remains to examine the effect on the result of neglecting the terms after the $(n+1)$-th in the series (XII).

The $(p+1)$-th term of that series is

$$(\alpha-\gamma+1)_p\,(\beta-\gamma+1)_p/\{(\alpha+\beta-\gamma+n+2)_p\,(1-\gamma-r)_{p+1}\}.$$

Now
$$(-1)^r/(1-\gamma-r)_{p+1}=1/\{\gamma_r\,(1-\gamma)_{p+1-r}\}.$$

Hence, r being less than n, p greater than n, and γ a positive proper fraction, it appears that $(-1)^r/(1-\gamma-r)_{p+1}$ is positive.

Consequently every term in the expression

$$\sum_{r=0}^{n}\left[(-1)^r Rx^r/\{r!\,(n-r)!\}\right],$$

where

$$R=\left[\sum_{p=n+1}^{\infty}(\alpha-\gamma+1)_p\,(\beta-\gamma+1)_p/\{(\alpha+\beta-\gamma+n+2)_p\,(1-\gamma-r)_{p+1}\}\right]$$

is such that the coefficient of each power of x is positive.

Moreover, since $|x| < 1$, the numerical value of the whole expression cannot exceed the value obtained by putting $x = 1$.

We have therefore to calculate the value of

$$\sum_{r=0}^{n} [(-1)^r R/\{r!\,(n-r)!\}],$$

where

$$R = \left[\sum_{p=n+1}^{\infty} (\alpha-\gamma+1)_p\,(\beta-\gamma+1)_p/\{(\alpha+\beta-\gamma+n+2)_p\,(1-\gamma-r)_{p+1}\}\right]. \tag{XIX}$$

Now

$$\sum_{r=0}^{n} [(-1)^r/\{r!\,(n-r)!\}][1/(1-\gamma-r)_{p+1}]$$

$$= \frac{1}{n!} \sum_{r=0}^{n} {}_nC_r(-1)^r/(1-\gamma-r)_{p+1}$$

$$= \frac{1}{n!} \sum_{r=0}^{n} (-1)^r\,{}_nC_r(1-\gamma-n)_{n-r}\,(2-\gamma-r+p)_r/(1-\gamma-n)_{p+1+n}$$

$$= \sum_{r=0}^{n} {}_nC_r(1-\gamma-n)_{n-r}(\gamma-p-1)_r/\{n!\,(1-\gamma-n)_{p+1+n}\}$$

$$= (-n-p)_n/\{n!\,(1-\gamma-n)_{p+1+n}\},$$

by (VI).

Hence the expression (XIX) is

$$\sum_{p=n+1}^{\infty} (\alpha-\gamma+1)_p(\beta-\gamma+1)_p(-n-p)_n/\{(1-\gamma-n)_{p+1+n}(\alpha+\beta-\gamma+n+2)_p\}, \tag{XX}$$

divided by $n!$.

Now the whole expression to be evaluated is the product of (XX) by

$$-\alpha_{n+1}\beta_{n+1}/\{\alpha+\beta-\gamma+1)_{n+1}\,(\gamma-1)_{n+1}\}$$
$$\times\,(\gamma-1)_{n+1}/n!.$$

But the expression (XX) is

$$(\alpha-\gamma+1)_{n+1}(\beta-\gamma+1)_{n+1}(-2n-1)_n/\{(1-\gamma-n)_{2n+2}(\alpha+\beta-\gamma+n+2)_{n+1}\} \tag{XXI}$$

multiplied by

$$1+\sum_{q=1}^{\infty}\left[\frac{(a-\gamma+n+2)_q\,(\beta-\gamma+n+2)_q\,(2n+2)_q}{\{(n+2)_q\,(3-\gamma+n)_q\,(a+\beta-\gamma+2n+3)_q\}}\right]. \qquad \text{(XXII)}$$

This series satisfies the condition of convergency.

All its terms are positive. Moreover,

$$a+\beta-\gamma+2n+3 > 2n+2,$$

since $$1 > \gamma-a-\beta;$$

therefore $$(a+\beta-\gamma+2n+3)_q > (2n+2)_q.$$

Hence the series (XXII) is less than the series

$$1+\sum_{q=1}^{\infty}(a-\gamma+n+2)_q\,(\beta-\gamma+n+2)_q/\{(n+2)_q\,(3-\gamma+n)_q\}.$$

This series, which is

$$G\begin{pmatrix} a-\gamma+n+2, & \beta-\gamma+n+2, & \infty \\ 3-\gamma+n, & n+2, & \end{pmatrix},$$

satisfies the condition of convergency.

If we apply to it the equation (V), it can be replaced by

$$(n+1)(\gamma-a-\beta)^{-1}\,G\begin{pmatrix} 1-a, & 1-\beta, & \infty \\ 3-\gamma+n, & \gamma-a-\beta+1, & \end{pmatrix}.$$

Hence the whole expression to be evaluated is

$$-a_{n+1}\beta_{n+1}/\{(a+\beta-\gamma+1)_{n+1}\,(\gamma-1)_{n+1}\}$$
$$\times(\gamma-1)_{n+1}/n!$$
$$\times(a-\gamma+1)_{n+1}(\beta-\gamma+1)_{n+1}(-2n-1)_n/\{(1-\gamma-n)_{2n+2}(a+\beta-\gamma+n+2)_{n+1}$$
$$\times(n+1)(\gamma-a-\beta)^{-1}\,G\begin{pmatrix} 1-a, & 1-\beta, & \infty \\ 3-\gamma+n, & \gamma-a-\beta+1, & \end{pmatrix}.$$

The factor $$-a_{n+1}\beta_{n+1}/\{(a+\beta-\gamma+1)_{n+1}\,(\gamma-1)_{n+1}\},$$

when n tends to infinity, tends to

$$\frac{\Gamma(a+\beta-\gamma+1)\,\Gamma(\gamma-1)}{\Gamma(a)\,\Gamma(\beta)}.$$

The G series tends to 1, and $(\gamma-a-\beta)^{-1}$ is finite.

Hence it remains only to show that

$$(n+1)(\gamma-1)_{n+1}/n!$$
$$\times(\alpha-\gamma+1)_{n+1}(\beta-\gamma+1)_{n+1}(-2n-1)_n,$$

divided by $\{(\alpha+\beta-\gamma+n+2)_{n+1}(1-\gamma-n)_{2n+2}\}$,

tends to zero as n tends to infinity.

This reduces to

$$-(\alpha-\gamma+1)_{n+1}(\beta-\gamma+1)_{n+1}/\{(2-\gamma)_{n+1}(\alpha+\beta-\gamma)_{n+1}\}$$
$$\times\quad(\alpha+\beta-\gamma)_{n+1}/(\alpha+\beta-\gamma+n+2)_{n+1}$$
$$\times\quad(2n+1)!/(n!\,n!). \qquad \text{(XXIII)}$$

The factor on the first line tends to

$$-\Gamma(2-\gamma)\Gamma(\alpha+\beta-\gamma)/\{\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)\},$$

as n tends to infinity.

The product of the factors on the second and third lines of (XXIII) is

$$\left[\frac{\Gamma(\alpha+\beta-\gamma+n+1)\,\Gamma(\alpha+\beta-\gamma+n+2)\,\Gamma(2n+2)}{[\Gamma(\alpha+\beta-\gamma)\,\Gamma(\alpha+\beta-\gamma+2n+3)\{\Gamma(n+1)\}^2]}\right]. \qquad \text{(XXIV)}$$

And now, as in Art. 14, replacing $\Gamma(a+n)$ by $e^{-n}n^{n+a-\frac{1}{2}}$, and $\Gamma(a+2n)$ by $e^{-2n}(2n)^{2n+a-\frac{1}{2}}$, and putting for brevity

$$\gamma-\alpha-\beta=\delta,$$

the expression (XXIV) is replaced by

$$2^{\delta-1}n^{-\delta}/\Gamma(-\delta),$$

which tends to zero as n tends to infinity, since δ is positive.

From this it follows that the terms in the G series in (XII) after the $(n+1)$-th may be neglected.

This completes the demonstration of the equation (I).