Systems of Plane Curves whose Orthogonals form a Similar System. By Prof. Tait.

(*Abstract.*)

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While tracing the lines of motion and the meridian sections of their orthogonal surfaces for an infinite mass of perfect fluid disturbed by a moving sphere:—the question occurred to me "When are such systems similar?" In the problem alluded to, the equations of the curves are, respectively,

$$(r/a)^2 = \cos\theta, \quad \text{and} \quad (r/b)^{\frac{1}{2}} = \sin\theta.$$

It was at once obvious that any sets of curves such as

$$(r/a)^m = \cos\theta \quad \text{and} \quad (r/b)^{\frac{1}{m}} = \sin\theta$$

are orthogonals. But they form *similar* systems only when

$$m^2 = 1.$$

Hence the only sets of similar orthogonal curves, having equations of the above form, are (*a*) groups of parallel lines and (*b*) their electric images (circles touching each other at one point). As the electric images of these, taken from what point we please, simply reproduce the same system, I fancied at first that the solution must be unique:—and that it would furnish an even more remarkable example of limitation than does the problem of dividing space into infinitesimal cubes. (See *Proc.* vol. xix. p. 193.) But I found that I could not prove this proposition; and I soon fell in with an infinite class of orthogonals having the required property. These are all of the type

$$r\frac{d\theta}{dr} = (\tan\theta)^{2m+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

which includes the straight lines and circles already specified. The next to these in order of simplicity among this class is

$$r = a\epsilon^{\frac{1}{2\cos^2\theta}}\cos\theta,$$

$$\text{with} \qquad r = b\epsilon^{\frac{1}{2\sin^2\theta}}\sin\theta.$$

In order to get other solutions from any one pair like this, we must take its electric image from a point whose vector is inclined at $\pi/4$ or $3\pi/4$ to the line of reference. For such points alone make the images similar. And a peculiarity now presents itself, in that the new systems are not directly superposable :—but each is the perversion of the other.

If we had, from the first, contemplated the question from this point of view, an exceedingly simple pair of solutions would have been furnished at once by the obviously orthogonal sets of logarithmic spirals

$$r = a\epsilon^{\theta}, \quad r = b\epsilon^{-\theta};$$

and another by their electric images taken from any point whatever. The groups of curves thus obtained form a curious series of spirals, all but one of each series being a continuous line of finite length whose ends circulate in opposite senses round two poles, and having therefore one point of inflection. The excepted member of each series is of infinite length, having an asymptote in place of the point of inflection. This is in accordance with the facts that :—a point of inflection can occur in the image only when the circle of curvature of the object curve passes through the reflecting centre, and that no two circles of curvature of a logarithmic spiral can meet one another.

We may take the electric images of these, over and over again, provided the reflecting centre be taken always on the line joining the poles. All such images will be cases satisfying the modified form of the problem.

If we now introduce, as a factor of the right hand member of (1), a function of θ which is changed into its own reciprocal (without change of sign) when θ increases by $\pi/2$, we may obtain an infinite number of additional classes of solutions of the original question; and from these, by taking their electric images as above, we derive corresponding solutions of the modified form. We may thus obtain an infinite number of classes of solutions where the equations are expressible in ordinary algebraic, not transcendental, forms.

Thus we may take, as a factor in (1), $\tan^2(\theta+\alpha)$. The general integral is complicated, so take the very particular case of $m=1$, $\alpha=\pi/4$. This gives the curves $r = a\dfrac{\tan\theta\sec\theta}{(1+\tan\theta)^2}\epsilon^{2/(1+\tan\theta)}$. Again, let the factor be $\tan(\theta-\alpha)\tan(\theta+\alpha)$. With $m=1$, and $\tan\alpha = 1/\sqrt{3}$, we get the remarkably simple form $\dfrac{x}{a} = 1 - \dfrac{y^2}{3x^2}$. But such examples may be multiplied indefinitely.