

## MINOR CONTRIBUTION.

THE DISTRIBUTION OF ALTERNATING CURRENTS IN  
CYLINDRICAL WIRES.

BY ERNEST MERRITT.

WHEN an alternating current flows along a wire the distribution of current is not uniform throughout the cross-section, as it would be in the case of a steady current. Experiment and theory both show that with alternating currents the current density is always greatest at the surface. As the frequency of alternation is increased, the difference between the current density at the surface of the wire and that at points in the interior becomes more marked, and finally, when frequencies are reached of the order of Hertz' vibrations, the current is confined to an extremely thin layer on the surface. In such cases the conductivity depends rather upon the surface of the conductor than upon its cross-section.

This so-called throttling effect has been known for many years. The fact that such phenomena might occur was indeed suggested by Maxwell. Attention was attracted to the subject by the experiments of Hughes on self-induction, and its theory has been very completely worked out by Heaviside, Rayleigh, Kelvin, and J. J. Thomson. The conclusions have also been verified, at least from a qualitative standpoint, by the experiments of Hertz, Lodge, and others.

The general interest which this phenomenon of throttling has aroused needs no better evidence than the mention of the names of the eminent men who have devoted their time to its investigation. It is, in fact, evident that the phenomenon is from several points of view an important one. In the first place, it affords an excellent illustration of the manner in which electrical disturbances are propagated, and accentuates the fact that such disturbances proceed from the ether into the wire, instead of being transmitted by the conductor itself. On the other hand, as soon as frequencies in excess of a few hundred periods per second are reached, the fact that the whole cross-section of the conductor is not utilized in the conduction of alternating currents has important practical bearings. In the case of iron wires the throttling effect, and the resulting increase in the effective

resistance offered by such wires to alternating currents, become noticeable at very low frequencies.

Although, as above stated, the theory of the subject has received complete and varied treatment, numerical computations seem to have been made in only a few cases. It is generally known that this concentration of the current at the surface of a wire will occur, but it is a matter of some difficulty to find data which will enable one to tell just how great the throttling effect will be in any specified case. I have encountered this difficulty several times during the past few years, and at length decided to make the computations necessary for plotting a few curves, so as to be able to represent the results in such a form as will give a concrete idea of the nature and magnitude of the effect. The belief that the need of such numerical results has also been felt by others must serve as the excuse for this article.

In general, the computation of the distribution of current across the cross-section of a wire is so complicated as to be almost beyond the scope of analytical treatment. In a few special cases, however, the formulas involved may be reduced to a comparatively simple form. One of these cases is that of two flat conductors, so close together that they may be treated as plane conducting sheets. This case has been discussed by Lord Kelvin. Another case, and one which approaches more nearly to practical conditions, is that of a cylindrical wire at such a distance from surrounding bodies that the magnetic force in its immediate neighborhood may be looked upon as approximately the same at all points equally distant from the wire. In this case, from the symmetry of the conditions, we are authorized to assume a symmetrical distribution of current throughout the conductor, so that if  $r$  represents the distance from the center,  $x$  the position in the wire measured in the direction of the axis from some arbitrary zero point, and  $t$  the time, the current density will be a function of these three quantities only. The condition of symmetry about the axis of the wire can be accurately reached only by making the return circuit in the form of a hollow cylinder completely surrounding the wire; but in the case of a circuit formed of two wires whose distance apart is fifteen or twenty times the radius, the condition of symmetry is approached with sufficient closeness for ordinary purposes.

In discussing the theory of the subject most writers have started with Maxwell's equations for the electro-magnetic field, have modified these to suit the conditions of symmetry about an axis, and have then written down and discussed the solution of the resulting differential equation. In presenting the subject to beginners it seems to me better not to assume Maxwell's equations as already derived, but to start from the fundamental physical laws upon which these equations themselves rest, and then to

apply these laws to the case in question. To obtain the differential equation by this method is perhaps in appearance not so direct as to make use of Maxwell's equations; but by referring back to the fundamental principles involved the physical side of the problem is likely to be better understood, and the process of solving the problem is less apt to be looked upon as a mere manipulation of symbols. The discussion which follows is intended as a brief outline of a method for solving the problem, which I think may be used to advantage with a class whose members are more familiar with physical principles than with the methods of physical mathematics.

In addition to the condition of symmetry about the axis of the wire, we have at our disposal for solving the problem of distribution of current the following three laws:

1. The line integral of the magnetic force about any conductor which carries a current is equal to  $4\pi$  times the current. Or, as it is often stated, "the work done in carrying unit pole once around the current  $i$  is equal to  $4\pi i$ ." This law, proven experimentally for the case of currents flowing in wires, may be extended, at least tentatively, to the case of currents in three dimensions. The fact that conclusions based upon this extension have been experimentally verified, justifies the assumption that the law applies. A mathematical expression of the law, with especial reference to the problem in hand, may be derived as follows:

Consider a small portion of the cross-section of the wire, bounded by arcs of circles whose radii are  $r$  and  $r + dr$ , and by the radial lines  $OA'$ ,  $OB'$ . If the current density is  $u$ , the total current flowing through this small area  $AA'B'B$  is  $ur\theta dr$ .

The magnetic force at  $A$  is at right angles to  $OA$  (since the lines of force are circles about  $O$ ) and has the same value at all points along  $AB$ . Let the magnetic force at  $A$  be denoted by  $T$ .  $T$  is then a function of  $r$ , and its value at points along  $A'B'$  will be  $T + \frac{\partial T}{\partial r} dr$ . The work done by the magnetic forces when unit pole is carried around the path  $AA'B'B$  is therefore

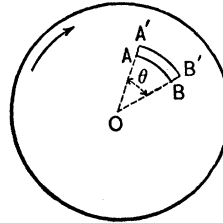


Fig. 1.

$$\left(T + \frac{\partial T}{\partial r} dr\right)(r + dr)\theta - T r \theta,$$

and this, in accordance with the law stated, is equal to  $4\pi$  times the current through  $AA'B'B$ . We thus have

$$\begin{aligned} \left(T + \frac{\partial T}{\partial r} dr\right)(r + dr)\theta - T r \theta &= 4\pi ur\theta dr. \\ \therefore 4\pi u &= \frac{\partial T}{\partial r} + \frac{1}{r} T. \end{aligned} \quad (1)$$

2. The line integral of the electric intensity around any circuit is equal to the rate of decrease of the magnetic induction through the circuit. Or in other words, the electromotive force induced in any circuit is equal to the rate of decrease of the flux of induction through that circuit. This law also has received direct experimental verification only in the case of circuits of finite magnitude, but its application has been extended, tentatively as before, to the case of infinitely small circuits.

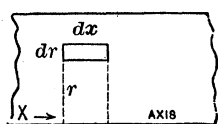


Fig. 2.

To apply it to the present case, consider a section of the wire formed by a plane passing through the axis, and determine the line integral of the electric intensity around the rectangle  $drdx$ . Let the electric intensity in the direction of  $x$  be denoted by  $P$ , and that in the direction of  $r$  by  $R$ ; we then have

$$Rdr + \left(P + \frac{\partial P}{\partial r} dr\right)dx - \left(R + \frac{\partial R}{\partial x} dx\right)dr - Pdx = \frac{d}{dt} \cdot \mu T dr dx,$$

where  $\mu$  denotes the permeability of the material composing the wire, and  $\mu T dr dx$  therefore represents the total flux of induction through the area  $drdx$ . This reduces to the equation

$$\mu \frac{dT}{dt} = \frac{\partial P}{\partial r} - \frac{\partial R}{\partial x}.$$

If the specific conductivity of the wire is  $C$ , we have, as a consequence of Ohm's law,

$$u = CP, \quad \rho = CR,$$

where  $\rho$  denotes the component current density in the direction of  $r$ . On substituting for  $P$  and  $R$  in the equation above, we finally obtain

$$\mu C \frac{dT}{dt} = \frac{\partial u}{\partial r} - \frac{\partial \rho}{\partial x}. \quad (2)$$

3. In the interior of a conductor there can be no accumulation of electricity; or in other words, if we consider any small portion of a conductor, the amount of electricity flowing into this portion must be equal to the amount leaving it.

Consider any small volume bounded by the cylindrical surfaces whose radii are  $r$  and  $r + dr$ , and by two planes perpendicular to the axis of the wire. The quantity of electricity entering this volume during the time  $dt$  is  $r\theta dr u dt + r\theta dx \rho dt$ . That leaving the volume by the opposite faces is

$$r\theta dr \left(u + \frac{\partial u}{\partial x} dx\right) dt + (r + dr)\theta dx \left(\rho + \frac{\partial \rho}{\partial r} dr\right) dt.$$

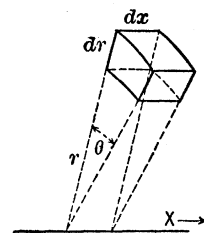


Fig. 3.

Since these quantities must be equal, we have

$$r\theta \frac{\partial u}{\partial x} dr dx dt + \rho \theta dr dx dt + r\theta \frac{\partial \rho}{\partial r} dr dx dt = 0.$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\rho}{r} + \frac{\partial \rho}{\partial r} = 0. \quad (3)$$

We thus obtain three equations which must be satisfied by the various quantities involved, and these equations are seen to be nothing more than the analytical statement of the three physical laws quoted above. By elimination a condition is obtained which must be satisfied by the current density at all points. Elimination may be accomplished as follows:

Remembering that  $x$  and  $r$  are independent, equations (3) and (2) may be written

$$\frac{\partial}{\partial r}(r\rho) = -r \frac{\partial u}{\partial x}; \quad (3')$$

$$-\frac{\partial}{\partial x}(r\rho) = -r \frac{\partial u}{\partial r} + \mu C \frac{d}{dt}(rT). \quad (2')$$

If we differentiate (3') with reference to  $x$ , and (2') with reference to  $r$ , and add, we obtain

$$r \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} + r \frac{\partial^2 u}{\partial x^2} - \mu C \frac{\partial}{\partial r} \frac{d(rT)}{dt} = 0. \quad (4)$$

By differentiating (1) with regard to  $t$ , we obtain

$$4\pi r \frac{du}{dt} = \frac{d}{dt} \frac{\partial(rT)}{\partial r}. \quad (1')$$

$T$  may therefore be eliminated between (1') and (4), giving finally

$$4\pi\mu C \frac{du}{dt} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2}. \quad (5)$$

The distribution of the current across the cross-section must therefore be such that the current density at all points satisfies the condition expressed in the above differential equation. As it is our intention to obtain a solution only for cases where the frequency of alternation is relatively small, viz. less than 100,000 per second, the equation might be still further simplified by the omission of the term in  $x$ ; for it is clear that in circuits of moderate dimensions the current will have practically the same value at a given instant at all points along the wire. This is equivalent to saying that  $u$  is independent of  $x$ , and that the terms involving  $x$  may therefore be neglected. But by omitting this term we not merely restrict (apparently) the range of application of the resulting equations, but also lose sight of the manner in which the disturbance is propagated along the

wire. It therefore seems better to obtain a solution of equation (5) as it stands, and not to introduce approximations until later.

From physical considerations it is a matter of no great difficulty to determine the form which the solution of (5) must take in the present case. The experiments of Hertz have shown that if a periodic electromotive force is impressed at some point in a circuit, the disturbance will travel along the wire as a series of waves. The simplest possible expression for  $u$  would therefore be

$$u = M \cos 2\pi \left( \frac{t}{\tau} - \frac{x}{\lambda} \right),$$

where  $\tau$  is the period and  $\lambda$  the wave length. On account of the resistance of the wire, we should expect the waves to gradually diminish in intensity as their distance from the origin increases. This requires a factor of the form  $e^{-ax}$  in the expression for  $u$ . Again, the amplitude and phase of the current wave will probably depend upon the distance of the point considered from the axis of the wire. To convince ourselves of this fact, we may think of the wire as a bundle of small conducting filaments, each one of these filaments forming a branch of the circuit in parallel with all the rest. The current will then divide among the different branches in a manner depending upon their mutual and self induction. From analogy with cases of linear circuits we may at once conclude that the different currents will differ both in amplitude and phase. Returning to a consideration of the actual wire, it thus appears that the current density at any point will vary harmonically with the time, but that the amplitude and phase will be functions of the distance of the point from the center. The current density may therefore be expressed in the following form :

$$u = e^{-ax} \left[ M \sin 2\pi \left( \frac{t}{\tau} - \frac{x}{\lambda} \right) + N \cos 2\pi \left( \frac{t}{\tau} - \frac{x}{\lambda} \right) \right], \quad (6)$$

in which  $M$  and  $N$  are functions of  $x$ .

On substituting this expression for  $u$  in (5) and equating the coefficients of the sine and cosine terms separately to zero, two equations are obtained from which  $M$  and  $N$  can be determined. This method has been followed by Kelvin.

In this case, as in many other cases of the solution of differential equations, it is an advantage to proceed by a different method — a method which does not indicate so clearly the physics of the problem, but which makes the analytical work of solving the equation less tedious, and whose result is obtained in a form which lends itself more readily to numerical computation.

Remembering Euler's exponential expression for the sine and cosine, viz.

$$e^{-i\theta} = \cos \theta - i \sin \theta,$$

it is clear that (6) may be written in exponential form. If we write

$$\frac{2\pi}{\tau} = p, \quad \frac{2\pi}{\lambda} = m,$$

we may put  $u$  equal to the *real part* of

$$S e^{-\alpha x - i(p\tau - mx)}, \quad (7)$$

where  $S$  is a function, in general complex, of  $r$ . The real part of  $S$  is evidently equal to the  $N$  of equation (6), while the imaginary part of  $S$  is equal to  $iM$ . If the expression above is substituted for  $u$  in equation (5), this becomes

$$\frac{\partial^2 S}{\partial r^2} + \frac{1}{r} \frac{\partial S}{\partial r} + (4\pi i \mu C p + (im - \alpha)^2) S = 0. \quad (8)$$

The expression (7) is therefore a solution of (5) provided the function  $S$  satisfies the conditions indicated in equation (8). Since the coefficients of the original equation are all real, the real and imaginary parts of (7) must separately satisfy equation (6). We are therefore justified in using the formal solution (7) as long as there is any saving in manipulation by so doing, and may then take the real part alone as the solution of the actual physical problem.

To determine the values of  $\alpha$  and  $m$  corresponding to any given frequency requires the dimensions and physical constants of the return circuit to be considered. A detailed discussion of several important cases has been given by J. J. Thomson.<sup>1</sup>

For the purposes of the present problem, however, it is not necessary to determine  $\alpha$  and  $m$ ; for both are small compared with the term  $4\pi\mu C p$ . Since  $m$  involves the reciprocal of the wave length, it will be a small fraction unless the frequency is extremely great ( $10^8$  or more per second). Experience teaches us that  $\alpha$ , whose value depends upon the rate of damping of the waves, will also be small;  $p$ , on the other hand, will be large. The quantity  $(im - \alpha)^2$  may then be neglected,<sup>2</sup> and (8) becomes

$$\frac{\partial^2 S}{\partial r^2} + \frac{1}{r} \frac{\partial S}{\partial r} + 4\pi i \mu C p S = 0. \quad (9)$$

The above is a form of Bessel's equation. If we write  $4\pi\mu C p = \phi^2$ , the general solution is<sup>3</sup>

$$S = K J_0(\phi r \sqrt{i}) + K' Y_0(\phi r \sqrt{i}), \quad (10)$$

<sup>1</sup> Recent Researches in Electricity and Magnetism, p. 262.

<sup>2</sup> Note that this leads to the same result as though the term in  $\alpha$  had been omitted in equation (5).

<sup>3</sup> See Gray and Mathews' Treatise on Bessel Functions; or Byerly, Fourier's Series.

where  $K$  and  $K'$  are arbitrary constants. The function  $Y_0$ , however, becomes infinite when  $r = 0$ . This would mean an infinite current density at the center of the wire, which is evidently impossible.  $K'$  must therefore be equal to zero, and the solution reduces to

$$S = KJ_0(\phi r\sqrt{i}),$$

where  $J_0$  is defined by the series :

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots$$

The formal solution of equation (5) is therefore

$$u = KJ_0(\phi r\sqrt{i})\epsilon^{-ax-i(pt-mx)}. \quad (11)$$

If we put

$$J_0(\phi r\sqrt{i}) = X - iY,$$

this becomes

$$u = K\epsilon^{-ax}(X - iY)\{\cos(pt - mx) - i\sin(pt - mx)\}.$$

As already explained, the solution of the actual physical problem is given by the *real* part of this expression. The proper expression for the current density at a distance  $r$  from the center of the wire is therefore

$$u = KX\epsilon^{-ax}\cos(pt - mx) - KY\epsilon^{-ax}\sin(pt - mx). \quad (12)$$

Since we are concerned only with the distribution over the cross-section of the wire, any convenient value, as zero, may be assigned to  $x$ . If in addition we make the maximum value of  $u$  at the center of the wire equal to unity, the expression for the current density may be put into the following form :

$$u = X\cos pt - Y\sin pt,$$

or

$$u = u_0\cos(pt + \theta),$$

where

$$u_0 = \sqrt{X^2 + Y^2},$$

$$\tan \theta = \frac{Y}{X},$$

and

$$X - iY = J_0(\phi r\sqrt{i}) = J_0(\sqrt{i}\sqrt{4\pi\mu C\rho r^2}).$$

It is thus seen that the maximum value of the current density, and also the phase of the current at any distance  $r$  from the axis, depends upon  $X$  and  $Y$  only; while  $X$  and  $Y$  in turn depend only upon the value of  $4\pi\mu C\rho r^2$ . In recording the results of numerical computation it is therefore convenient to tabulate the values of  $u_0$  and  $\theta$  corresponding to a series of values of the expression  $4\pi\mu C\rho r^2$ . By giving proper values to the constants  $\mu$ ,  $C$ ,  $\rho$ , and  $r$ , such a table may be made to give numerical results for any special case within its range. The values of  $J_0(q\sqrt{i})$  up to  $q = 6$



are given by Gray and Mathews,<sup>1</sup> while a table extending, with greater intervals, to  $q = 20$ , has been computed by Mr. Magnus Maclean.<sup>2</sup> By the aid of the values for  $X$  and  $Y$  given by these writers, I have computed  $u_0$  and  $\theta$  for different values of  $q$ , where  $q = 4\pi\mu Cpr^2 = 8\pi^2\mu Cnr^2$ ,  $n$  being the frequency. The results are given in the accompanying table.

TABLE I.

$q$	$X$	$Y$	$u_0$	$\theta$ (in degrees).	$\frac{R_n}{R_s}$
0.	1.000	0.000	1.000	0.	1.000
0.5	0.999	0.062	1.001	3.5	1.000
1.	0.984	0.250	1.015	14.3	1.000
1.5	0.921	0.558	1.077	31.2	1.026
2.	0.752	0.972	1.229	52.3	1.080
2.5	0.400	1.457	1.511	74.6	1.175
3.	— 0.221	1.938	1.96	96.5	1.318
3.5	— 1.194	2.283	2.58	117.6	1.492
4.	— 2.563	2.293	3.44	138.1	1.678
4.5	— 4.299	1.686	4.62	158.6	1.863
5.	— 6.230	+ 0.116	6.23	178.9	2.043
5.5	— 7.973	— 2.790	8.45	199.3	2.219
6.	— 8.858	— 7.335	11.5	219.6	2.394
8.	+ 20.97	— 35.02	40.8	300.6	3.096
10.	— 138.8	+ 56.37	150.	382.1	3.794
15.	— 2970.	— 2952.	4190.	584.8	5.573
20.	+11500.	+47580.	48900.	796.4	7.325

EXPLANATION OF TABLE.

$q = \sqrt{4\pi\mu Cpr^2} = \sqrt{8\pi^2\mu Cnr^2}$ , where  $\mu$  is the permeability,  $C$  the specific conductivity, and  $r$  the distance of the point considered from the center of the wire.  $n$  represents the frequency. Then  $u_0$  is the value of the maximum current density, the current density at the center being taken equal to unity.  $\theta$  is the phase difference between the current at the distance  $r$  from the center and the current along the axis of the wire.

$\frac{R_n}{R_s}$  is the ratio between the resistance offered to alternating currents of frequency  $n$ , and the resistance offered by the same wire to steady currents.

$$X - iY = J_0(q\sqrt{i}).$$

As an illustration of the use of this table, consider the following special case: A copper wire 1 cm. in diameter carries an alternating current

<sup>1</sup> Treatise on Bessel Functions, Table IV.

<sup>2</sup> See Kelvin, Mathematical and Physical Papers, Vol. III., p. 493.

whose frequency is 2000 per sec. How does the current density at the surface compare with that at the center?

We have in this case,

$$\mu = 1, C = \frac{1}{1600}, n = 2000, r = 0.5,$$

$$q = \sqrt{8\pi^2\mu Cnr^2} = 4.97.$$

The value of  $u_0$  corresponding to  $q = 4.97$  is very nearly 6.1. The maximum current density at the surface of the wire is thus six times as great as the maximum value at the center. The phase difference  $\theta$  is in this case nearly  $178^\circ$ . It thus appears that the current at the center of the wire lags behind the current at the surface by almost  $180^\circ$ ; so that when the surface current is at its maximum in the positive direction the current at the center of the wire has just reached its maximum in the negative direction.

By giving different values, ranging from 0 to 0.5, to  $r$ , the value of  $u_0$  may be found for all points in the wire. The dotted curve in Fig. 4 has been plotted from results obtained in this way, distances from the center of the wire being used as abscissas, and the corresponding values of  $u_0$  as ordinates. To avoid interpolation,  $n$  has been taken equal to 2020, thus making  $q = 5$  at the surface.

While this curve indicates fairly well the magnitude of the throttling effect, it fails to give an entirely satisfactory picture of the actual condition of affairs in the wire. On account of the varying phase angle  $\theta$ , the current density does not reach its greatest value at the same instant at all points of the cross-section, so that the dotted curve in Fig. 4 does not show the actual distribution of current at any definite time. To find the distribution at any given instant, we must assign the proper value to  $u_0$  and  $\theta$  in the expression

$$u = u_0 \cos\left(\frac{2\pi t}{\tau} + \theta\right),$$

and thus determine  $u$  for different values of  $r$ , but for the *same* value of  $t$ . If these results are plotted, we obtain what might be termed an instantaneous view of the current distribution. Three such curves are shown in Fig. 4. The first shows the condition of affairs at the instant that the surface current has reached its positive maximum. At a distance of 2.8 mm. from the center of the wire the current at this instant is zero, while at all points still nearer the axis the current flows in the opposite direction. It is clear that so far as outside magnetic effects are concerned, the current in the interior of the wire tends to neutralize the effect of the surface current. In this case the presence of the interior portion of the conductor is not merely of small utility; it is actually a disadvantage. The second

curve in Fig. 4 gives the distribution at the instant that  $u$  at the surface is passing through zero in the negative direction. In Curve III the surface current is approaching its negative maximum.

For a frequency of 2000 the values of  $\theta$  are too small to bring out clearly the manner in which the current is propagated into the wire. By taking  $n = 8080$ , which makes  $q = 10$  at the surface of a copper wire 1 cm. in diameter, we obtain a case where all the phenomena of throttling are much more marked. The maximum current density at the surface is now 150 times as great as that at the axis, while the phase difference between the two currents is  $382^\circ$ . The variation of  $u_r$  with  $r$  in this case is shown

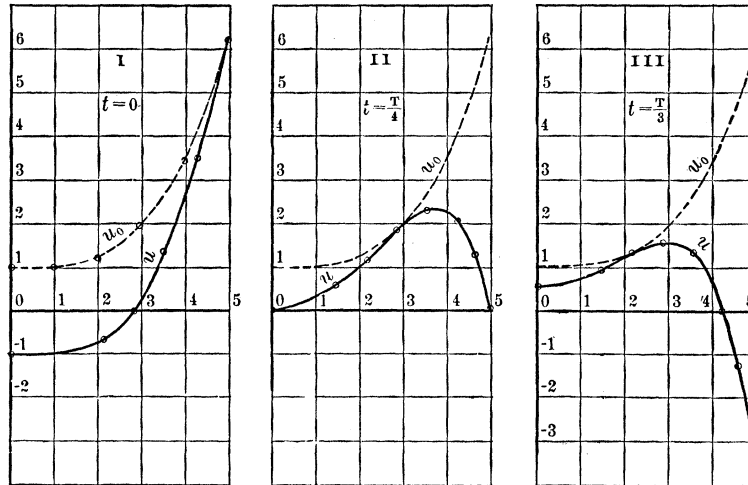


Fig. 4.

in Fig. 5, IV. The dotted curve in the same figure gives the value of  $\theta$  at different distances from the axis. It will be observed that the latter curve, with the exception of the region near the center, is practically straight. We conclude from this that the current density at points not too near the axis lags behind the surface current by an angle which is proportional to the depth below the surface. The disturbance which constitutes the current thus travels inward from the surface of the wire in the form of a wave, whose velocity of propagation remains practically constant until the wave has almost reached the center. The amplitude, on the other hand, diminishes very rapidly as the depth below the surface increases. The curves plotted in Fig. 5 have been drawn to illustrate this point. In Curve I we have the distribution at the instant when the surface current is passing through zero in the positive direction. Curve II shows the condition

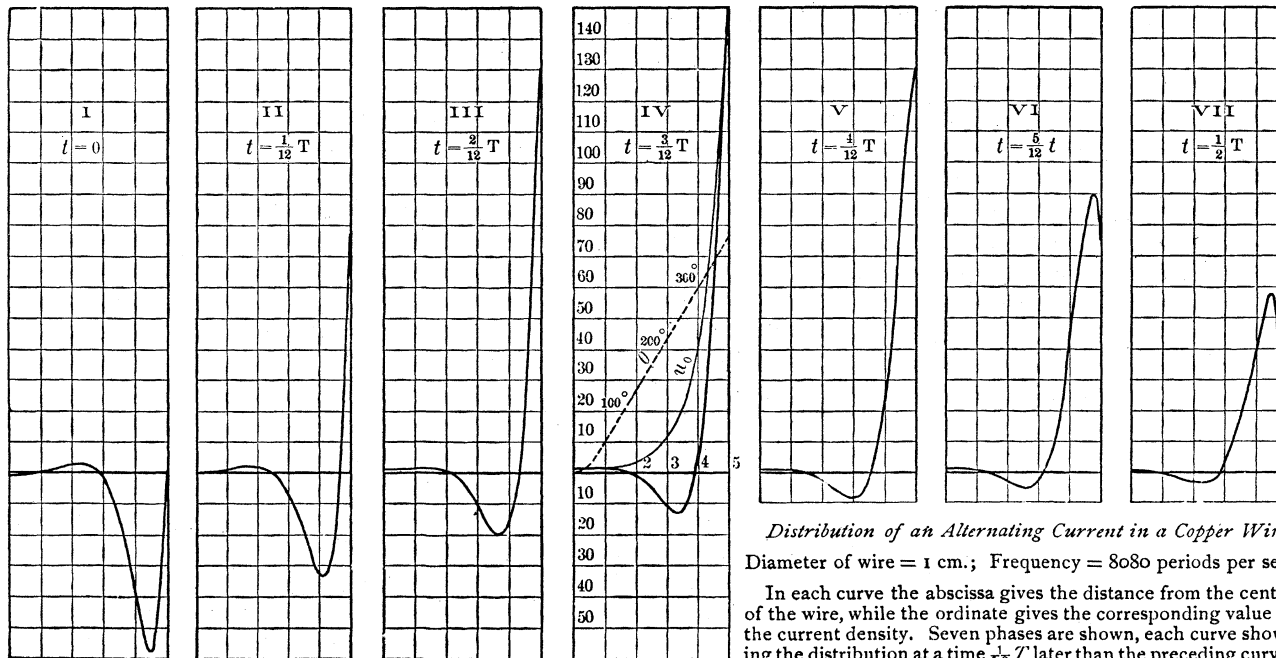


Fig. 5.

*Distribution of an Alternating Current in a Copper Wire.*  
Diameter of wire = 1 cm.; Frequency = 8080 periods per sec.

In each curve the abscissa gives the distance from the center of the wire, while the ordinate gives the corresponding value of the current density. Seven phases are shown, each curve showing the distribution at a time  $\frac{1}{12}T$  later than the preceding curve.

of affairs  $\frac{1}{2}$  period later; Curve III gives the distribution at the end of another  $\frac{1}{2}$  period interval; and so on. Figure 5 thus shows seven successive phases of the disturbance in the wire, and enables the phenomenon to be followed through half an oscillation. It may be added that the next half oscillation will differ from that shown only in the fact that the sign of  $u$  is reversed.

Only a brief inspection of Fig. 5 is necessary in order to appreciate the manner in which the current is propagated into the wire. The wave of current, starting in Curve I near the surface of the wire, may be followed through its successive stages as it proceeds, with rapidly diminishing amplitude, toward the axis. These diagrams also enable the velocity of propagation to be determined. If we compare Curves I and VII, we see that the zero point of the current wave has moved from  $r = 5$  mm. to  $r = 2.82$  mm. The wave has therefore traveled a distance of 2.18 mm. in  $\frac{1}{16160}$  sec. This corresponds to a velocity of 35 meters per second.

In general, the velocity with which the disturbance is propagated inward will depend upon the frequency of the current as well as upon the material of the wire. The velocity will not be strictly constant unless the diameter of the wire is infinite; but for points in the wire for which  $q$  exceeds 2 it will be so nearly constant that its variation is negligible. A general expression for the velocity is

$$V = \sqrt{\frac{n}{\mu C}}.$$

This expression is obtained on the assumption that  $r = \infty$ ; *i.e.* for a plane conducting sheet.<sup>1</sup> That it holds with a high degree of approximation even for rather small wires may be verified by the use of the table given above.

There results from this concentration of current at the surface a real change in the resistance of the wire — real in the sense that it corresponds to an increased development of heat, and not merely to a change in the impedance of the circuit. The central portion of the wire no longer contributes its share to the conductivity, so that the resistance is practically that of a thin cylindrical tube. In the last column of the table on p. 55, I have tabulated the values given by Kelvin<sup>2</sup> for the effective resistance offered by cylindrical wires to alternating currents. The resistance  $R_n$  for an alternating current of frequency  $n$  is here expressed in terms of the resistance  $R_0$ , which the same wire offers to steady currents. In Fig. 6 these results are shown graphically for the case of a copper wire 1 cm. in diameter.

<sup>1</sup> See Kelvin, "Anti-effective Copper," London Electrician, Vol. 25, p. 512, 1890.

<sup>2</sup> Mathematical and Physical Papers, Vol. III., p. 492.

On account of the high permeability of iron the various effects of throttling are shown by wires of that material even when the periodicity is quite low. If we assume a permeability of 1500, which is a low value for soft iron, the curves of Fig. 5 would apply to an iron wire 1 cm. in diameter when the frequency is only 32 per second. In the case of the magnetic metals, however, the value of  $\mu$  cannot be treated as constant. The permeability in such cases is a function of the current, and, if hysteresis is



Fig. 6.

appreciable, of the rate of change of the current also. The results derived above can therefore not be expected to show an exact agreement with experiment in the case of iron wires, although they doubtless give a rough approximation to the truth. The subject of the distribution of alternating currents in wires of the magnetic metals is in fact one which deserves further experimental study. It can hardly be doubted that such an investigation would be a valuable contribution to our knowledge of magnetism.