On a Canonical Reduction of Bilinear Forms (Part II.), *with special consideration of Congruent Reductions.* *By* T. J. I'A. Bromwich.

Communicated June 14th, 1900. Received, in revised form, October 3rd, 1900.

The following paper is concerned mainly with reductions in which the coefficients of the substitutions on the two sets of variables are either the same (congruent substitutions) or are conjugate imaginaries. Passing to the details of the paper; I give first a brief *résumé* of the general reduction-process, and add a slight extension of a theorem due to Frobenius. This is then used to reduce a single alternate form by a congruent substitution.

The next and longest division of the paper deals with the simultaneous reduction (by congruent substitutions) of a symmetric and an alternate form. So far as I know, this reduction has only been carried out by Kronecker (in the paper subsequently quoted by the letters *Kr.*); and Kronecker's direct object is the reduction of a single bilinear form by a congruent substitution. Thus Kronecker's reduced forms are not always the simplest from the present point of view. Frobenius (*Berliner Sitzungsberichte*, 1896, p. 7) has shown that, if *any* substitutions P, Q (independent of λ) can be found, such that

$$P(\lambda A-B)\,Q=\lambda C-D,$$

and if, further, A, C be both symmetric or both alternate (the same holding for B, D), then R can be derived from P, Q so that

$$R'(\lambda A-B)\,R=\lambda C-D.$$

Thus, if the invariant-factors of $|\lambda A-B|$ and $|\lambda C-D|$ are the same, a *congruent* substitution can be found to transform $(\lambda A-B)$ into $(\lambda C-D)$. But, apparently, this method cannot be extended so as to cover the analogous theory for conjugate imaginary substitutions, which would be applied to a pair of Hermite's forms. I have, accordingly, discussed a direct process for finding R in a way that can be applied in both cases.

The reduced forms so obtained have been applied to prove certain theorems of Herr Alfred Loewy's on automorphic substitutions, both of real quadratic forms and of Hermite's forms.

1. *General Account of the Reduction of two Bilinear Forms.*

Let A, B be the two forms, and let the form reciprocal to $(\lambda A - B)$* be

$$(\lambda A - B)^{-1} = \sum_{r=1}^{\alpha} P_{r-1} \lambda^{r-1} + \sum_{r=1}^{\beta} Q_r \lambda^{-r} + \sum_{c} \sum_{r=1}^{\gamma} C_r (\lambda - c)^{-r},$$

where $(\lambda - c)$ is a typical factor of the determinant $|\lambda A - B|$, and α, β, γ are the indices of the first invariant-factors (*Elementartheiler*) of $|\lambda A - B|$ corresponding to $\lambda = \infty$, $\lambda = 0$, $\lambda = c$ respectively. In this expression for $(\lambda A - B)^{-1}$ all the terms P will disappear if $|A| \neq 0$, and all the Q's if $|B| \neq 0$.

Multiply up by $(\lambda A - B)$, and we have, in general,

$$AP_r - BP_{r+1} = 0 = AQ_{r+1} - BQ_r = AC_{r+1} + (cA - B) C_r,$$

but

$$E = AQ_1 - BP_0 + \sum_c AC_1.$$

The same equations hold if the order of the products is reversed, thus

$$P_r A - P_{r+1} B = 0, \text{ \&c.}$$

From these it follows that

$$A = (AQ_1 - BP_0) A + \sum_c AC_1 A$$

$$= AQ_1 A - BP_1 B + \sum_c AC_1 A,$$

and

$$B = (AQ_1 - BP_0) B + \sum_c AC_1 B$$

$$= AQ_2 A - BP_0 B + \sum_c (AC_2 A + cAC_1 A).$$

If, now, we consider the form $A(\lambda A - B)^{-1} A$, and expand in powers of $(\lambda - c)$, it is clear that the coefficients of $1/(\lambda - c)$ and $1/(\lambda - c)^2$ will be respectively $AC_1 A$ and $AC_2 A$. But Weierstrass† and Stickelberger‡ have shown how to arrange $A(\lambda A - B)^{-1} A$ in a form which can be readily expanded in powers of any factor of

---

* For the definitions of the product of two forms and of the form reciprocal to a given form, see Frobenius (*Crelle*, Vol. LXXXIV., p. 1); or Muth, *Elementartheiler* (Leipzig, 1899). Short accounts will be found in previous papers by the author (*Proc. Lond. Math. Soc.*, pp. 78, 158, above). E is the unit form $(= \Sigma x_r y_r)$.

† *Berliner Monatsberichte*, 1868, p. 310; *Gesammelte Werke*, Vol. II., p. 19.

‡ *Crelle*, Vol. LXXXVI., 1878, p. 20; this paper will be denoted in future by *St.*, for brevity, as we shall frequently have occasion to quote it.

$|\lambda A - B|$ (see also p. 326 below); and so from each expansion we obtain a group of terms in A and B, viz., AC_1A and $AC_2A + cAC_1A$.

The same method still holds if $c = 0$, for, if $A(\lambda A - B)^{-1}A$ be expanded in powers of λ, the coefficients of $(1/\lambda)$, $(1/\lambda)^2$ are respectively AQ_1A, AQ_2A; which have been proved to form parts of A, B respectively. But the case is rather different for $c = \infty$; to deal with this we may first consider $-B(\lambda A - B)^{-1}B$ and expand in powers of $(1/\lambda)$; then the coefficient of λ and the term independent of λ are $-BP_1A$ and $-BP_0B$ respectively, which form parts of our expressions for A and B.

Another method of dealing with this case depends on expanding $B(\mu B - A)^{-1}B$ in powers of μ and picking out the coefficients of $(1/\mu^2)$, $(1/\mu)$. To see this we note that

$$(\mu B - A)^{-1} = -\frac{1}{\mu}\left(\frac{A}{\mu} - B\right)^{-1}$$

$$= -\frac{1}{\mu}\left[\sum_{r=0}^{\alpha-1} P_r\mu^{-r} + \sum_{r=1}^{\beta} Q_r\mu^r + \sum_c \sum_{r=1}^{\gamma} C_r\mu^r(1-\mu c)^{-r}\right].$$

So, on expanding in powers of μ, we find $-P_1$, $-P_0$ as the coefficients of $(1/\mu)^2$, $(1/\mu)$ in the expansion of $(\mu B - A)^{-1}$; hence the coefficients of $(1/\mu)^2$, $(1/\mu)$ in the expansion of $B(\mu B - A)^{-1}B$ in powers of μ are $-BP_1B$, $-BP_0B$ respectively, which are parts of A and B.

Thus, on the whole, we can find the values of A, B by expanding $A(\lambda A - B)^{-1}A$ in powers of the factors $(\lambda - c)$ of $|\lambda A - B|$ and picking out the coefficients of $1/(\lambda - c)$, $1/(\lambda - c)^2$ for each c; and, in case $|A| = 0$, we have also to expand $B(\mu B - A)^{-1}B$ in powers of μ, and pick out the coefficients of $(1/\mu)^2$, $(1/\mu)$.

This modification of Weierstrass's solution of the problem was given in part in a short note (*Proc. Lond. Math. Soc.*, p. 158, above), where more details will be found on the determination of the indices of those invariant-factors of $|\lambda A - B|$ which correspond to the infinite roots if $|A| = 0$; it will be sufficient to state here that they are the same as the indices of the invariant-factors of $|\mu B - A|$ to base μ.

In the foregoing it is, of course, assumed that the determinant $|\lambda A - B|$ does not vanish identically, or the reciprocal form $(\lambda A - B)^{-1}$ would not exist. The examination of the so-called "singular" case, when $|\lambda A - B| = 0$, will be deferred to § 2.

I proceed to an account of Stickelberger's transformation of

$A(\lambda A - B)^{-1}A$, and I shall give an extension of it, suggested by a theorem for quadratic forms due to Frobenius.

Notation.—I employ the following general scheme of symbols—

$$W_{k-1} = \begin{vmatrix} a_{11}, & \dots, & a_{1n}, & v_1', & \dots, & v_1^{k-1}, & \eta_1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ a_{n1}, & \dots, & a_{nn}, & v_n', & \dots, & v_n^{k-1}, & \eta_n \\ u_1', & \dots, & u_n', & 0, & \dots, & 0, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ u_1^{k-1}, & \dots, & u_n^{k-1}, & 0, & \dots, & 0, & 0 \\ \xi_1, & \dots, & \xi_n, & 0, & \dots, & 0, & 0 \end{vmatrix}.$$

If $\eta_1, \dots, \eta_n$ be replaced by $v_1^k, \dots, v_n^k$, the value so obtained is called U_k; if $\xi_1, \dots, \xi_n$ be replaced by $u_1^k, \dots, u_n^k$, the value is V_k; and, if both sets be replaced in this way, the result is Δ_k (*cf. St.*, § 1).

In these symbols, the u's and v's are arbitrary constants whose indices do not refer to powers, but are to be considered as additional suffixes; the ξ's and η's are linear functions of the variables $x_1, \dots, x_n$, $y_1, \dots, y_n$ respectively, whose exact form will appear later.

Then, as proved by Stickelberger (*St.*, § 1),

$$W_k \Delta_{k-1} = W_{k-1}\Delta_k - U_k V_k,$$

for W_{k-1}, Δ_k, U_k, V_k are first minors of W_k corresponding to the four zeros which are complementary to Δ_{k-1}. Thus

$$\frac{W_{k-1}}{\Delta_{k-1}} = \frac{W_k}{\Delta_k} + \frac{U_k V_k}{\Delta_k \Delta_{k-1}},$$

and so

$$\frac{W_0}{\Delta_0} = \sum_{k=1}^{n} \frac{U_k V_k}{\Delta_k \Delta_{k-1}};$$

for clearly $W_n \equiv 0$. This result is a generalization of Darboux's* for the case of quadratic forms. Its importance in the present investigation is due to the fact that $(-W_0/\Delta_0)$ is the form reciprocal to $A = \Sigma a_{rs}\xi_r\eta_s$ or is A^{-1}.

Frobenius † has shown that two consecutive terms of the series

* *Liouville's Journal*, Vol. XIX. (2me sér.), 1874, p. 347 [p. 354, formula (17)].

† *Berliner Sitzungsberichte*, 1894, pp. 241, 407; reprinted in *Crelle*, Vol. CXIV., 1895, p. 187. The theorem in question is (7) on p. 249 (*SB.*), p. 196 (*Cr.*). Frobenius also verifies that the sum $(U_k V_k/\Delta_k\Delta_{k-1}) + (U_{k+1}V_{k+1}/\Delta_{k+1}\Delta_k)$ can be expressed in this form.

$(U_k V_k/\Delta_k \Delta_{k-1})$ can be combined if so desired; this is done by finding an expression for the difference $(W_{k-1}/\Delta_{k-1} - W_{k+1}/\Delta_{k+1})$. Frobenius's investigation relates to quadratic forms; and I shall now extend his method to bilinear forms, making a slight generalization by using bordered determinants in the place of minors.

Applying Sylvester's theorem * (on determinants of minors) in the form given by Frobenius,† we have

$$W_{k+1}\Delta_{k-1}^2 = \begin{vmatrix} (k, k), & -(k, k+1), & V_k \\ -(k+1, k), & (k+1, k+1), & -V_k' \\ U_k, & -U_k', & W_{k-1} \end{vmatrix},$$

where (r, s) denotes the value of W_{k-1} when $\xi_1, \ldots, \xi_n$ are replaced by $u_1^r, \ldots, u_n^r$, and $\eta_1, \ldots, \eta_n$ by $v_1^s, \ldots, v_n$; so that $(k, k) = \Delta_k$; also U_k', V_k' are analogous to U_k, V_k, but with $u^{k+1}, \ldots, u_n^{k+1}$ in place of $u_1^k, \ldots, u_n^k$, and a similar change in the v's. To see the correctness of this equation we have only to notice that the elements of the determinant so written are all second minors of W_{k+1}, corresponding to the last nine zeros of W_{k+1}; and these zeros are complementary to Δ_{k-1}. We have also

$$\Delta_{k-1}\Delta_{k+1} = \begin{vmatrix} (k, k), & -(k, k+1) \\ -(k+1, k), & (k+1, k+1) \end{vmatrix}$$

for (k, k), $-(k, k+1)$, ... are first minors of Δ_{k+1}.

Thus, expanding the determinants, we find

$$W_{k+1}\Delta_{k-1}^2 = W_{k-1}\Delta_{k-1}\Delta_{k+1} - (k+1, k+1)\, U_k V_k - (k, k)\, U_k' V_k' + (k+1, k)\, U_k' V_k + (k, k+1)\, U_k V_k';$$

or, dividing by $\Delta_{k-1}^2 \Delta_{k+1}$,

$$\frac{W_{k-1}}{\Delta_{k-1}} - \frac{W_{k+1}}{\Delta_{k+1}} = \frac{1}{\Delta_{k-1}^2 \Delta_{k+1}} \big[(k+1, k+1)\, U_k V_k + (k, k)\, U_k' V_k' - (k+1, k)\, U_k' V_k - (k, k+1)\, U_k V_k' \big],$$

which is the extension of Frobenius's theorem.

We may accordingly replace the two consecutive terms $(U_k V_k/\Delta_k \Delta_{k-1}) + (U_{k+1} V_{k+1}/\Delta_{k+1}\Delta_k)$ in our expression for W_0/Δ_0 by the quantity on the right of the last equation.

* *Phil. Mag.*, April, 1851.

† In § 1 of the paper just quoted; or *Crelle*, Vol. LXXXVI., 1878, p. 44, § 3.

Special cases of this theorem have been given also by Kronecker* and Gundelfinger† for quadratic forms.

For our future investigations we shall employ these results in general when a_{rs} is replaced by $(\lambda a_{rs} - b_{rs})$; and U_k, V_k, Δ_k, W_k will generally be used to mean their values when this change is made in them also, and $(\xi_1, \ldots, \xi_n)$, $(\eta_1, \ldots, \eta_n)$ are replaced by $\left(\frac{\partial A}{\partial y_1}, \ldots, \frac{\partial A}{\partial y_n}\right)$, $\left(\frac{\partial A}{\partial x_1}, \ldots, \frac{\partial A}{\partial x_n}\right)$, respectively. For when these substitutions are made we have

$$A(\lambda A - B)^{-1} A = -W_0/\Delta_0 = -\Sigma(U_k V_k)/(\Delta_k \Delta_{k-1}),$$

and in this form we can expand $A(\lambda A - B)^{-1} A$ as explained. It will also be necessary to use the corresponding symbols with $\left(\frac{\partial B}{\partial y_1}, \ldots, \frac{\partial B}{\partial y_n}\right)$, $\left(\frac{\partial B}{\partial x_1}, \ldots, \frac{\partial B}{\partial x_n}\right)$ in place of $(\xi_1, \ldots, \xi_n)$, $(\eta_1, \ldots, \eta_n)$ in order to evaluate the expansion of $B(\mu B - A)^{-1} B$ in powers of μ; but the two investigations are distinct and no confusion need arise.

It will be seen from Stickelberger's paper (*St.*, §§ 1, 2, 5) that in this method the u's and v's cannot be chosen entirely arbitrarily; but that in general the u's and v's may be symmetrical ($u_r^s = v_r^s$) with one special case of exception, when it may happen that for a special value of k we may not have $u_r^k = v_r^k$ in Δ_k; we then avoid Δ_k by using our extension of Frobenius's theorem. For Stickelberger proves that Δ_{k-1} and Δ_{k+1} are not altered by this exception.

2. *Application of Weierstrass's Methods to the "Singular" Case.*

Kronecker (*Berliner Monatsberichte*, March, 1874, p. 156; *Gesammelte Werke*, Bd. I., p. 381) has remarked that the reduction of a "singular" family of bilinear forms could be effected by applying Weierstrass's method to a family obtained by suitably modifying one of the original forms. Kronecker gives the necessary modification, but does not complete the reduction; and, so far as I know, no account of the whole investigation has yet been published.

We have seen in a previous paper (*Proc. Lond. Math. Soc.*, p. 88, above) that, if $|\lambda A - B|$ and all its minors up to the

---

* *Berliner Monatsberichte*, 1874, p. 59, § II.; or *Gesammelte Werke*, Vol. I., p. 349. It is given more completely in § 1 of his paper on p. 397 of the *Monatsberichte* (*Werke*, Vol. I., p. 423); the latter paper will be quoted as *Kr.*

† *Crelle*, Vol. XCI., 1881, p. 221, Lemma 2.

$(k-1)$th be identically zero, some of the kth minors not vanishing, then Δ_k (formed by bordering $|\lambda A - B|$ with u's and v's as above explained) will break up into the product of three determinants. Let the two sets of u's and v's be determined so that for

$$u_s^\epsilon = p_s^\epsilon \quad (\epsilon = 1, 2, \ldots, k;\ s = 1, 2, \ldots, n)$$

we have the one determinant* taking the form

$$\begin{vmatrix} \lambda^{a_1}, & 0, & 0, & \ldots, & 0 \\ 0, & \lambda^{a_2}, & 0, & \ldots, & 0 \\ 0, & 0, & \lambda^{a_3}, & \ldots, & 0 \\ \ldots & \ldots & \ldots & & \ldots \\ 0, & 0, & 0, & \ldots, & \lambda^{a_k} \end{vmatrix},$$

while for $u_s^\epsilon = \pi_s^\epsilon$ the determinant takes a similar form with a diagonal of unities.†

In the same way, for $v_s^\epsilon = q_s^\epsilon$ or κ_s^ϵ, we are to have another determinant in the shape

$$\begin{vmatrix} \lambda^{\beta_1}, & 0, & \ldots, & 0 \\ 0, & \lambda^{\beta_2}, & \ldots, & 0 \\ \ldots & \ldots & & \ldots \\ 0, & 0, & \ldots, & \lambda^{\beta_k} \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 1, & 0, & \ldots, & 0 \\ 0, & 1, & \ldots, & 0 \\ \ldots & \ldots & & \ldots \\ 0, & 0, & \ldots, & 1 \end{vmatrix}.$$

Kronecker's modification of A consists in adding terms; in fact we take

$$C = A + \sum_{\epsilon=1}^{k} t_\epsilon (p_1^\epsilon y_1 + \ldots + p_n^\epsilon y_n)(q_1^\epsilon x_1 + \ldots + q_n^\epsilon x_n),$$

and apply Weierstrass's process to the family $(\lambda C - B)$. The first step is to form the series of determinants $\Delta_1, \Delta_2, \ldots, \Delta_k$ obtained from $|\lambda C - B|$ by bordering it with π's and κ's. We proceed to examine their invariant-factors. To do this we note that

$$\sum_{s=1}^{n} (\lambda c_{rs} - b_{rs}) f_s' = \sum_{s=1}^{n} \left[(\lambda a_{rs} - b_{rs}) + \sum_{\epsilon=1}^{k} \lambda t_\epsilon p_s^\epsilon q_r^\epsilon \right] f_s'$$
$$= \lambda t_1 q_r' (\lambda^{a_1}),$$

---

* This is the determinant in which all the u's appear, when Δ_k is split up into factors; and the second determinant contains all the v's.

† It may be well to remark explicitly that it has been found convenient to alter the notation used in my previous paper (thus, the present p's are the former κ's, &c.).

where f'_s has the same meaning as in the paper already quoted, so that

$$\sum_{s=1}^{n} (\lambda a_{rs} - b_{rs}) f'_s = 0.$$

Similarly, $$\sum_{r=1}^{n} (\lambda c_{rs} - b_{rs}) g'_r = \lambda t_1 p'_s (\lambda^{\beta_1});$$

and analogous results hold for each set of f's and g's.

Using these facts and bordering with the π's, κ's in order, we see that

$$\Delta_0(\lambda)/\Delta_1(\lambda) = -t_1 \lambda^{a_1+\beta_1+1},$$

$$\Delta_1(\lambda)/\Delta_2(\lambda) = -t_2 \lambda^{a_2+\beta_2+1},$$

&c.,

for it must be remembered that in none of these determinants can squares or higher powers of any t appear.* Now, in order that these powers of λ may be the invariant-factors, it is necessary and sufficient that the a's and β's should be arranged in descending order of magnitude.

The next step is to consider the linear functions of x and y which appear in the process of reduction; the function U is found from $\Delta_1(\lambda)$ by putting the linear functions $\frac{\partial C}{\partial y_1}, \ldots, \frac{\partial C}{\partial y_n}$ in the last row in place of $u'_1, \ldots, u'_n$. It is then easy to see that

$$U_1/\Delta_1(\lambda) = f'_1 \frac{\partial C}{\partial y_1} + \ldots + f'_n \frac{\partial C}{\partial y_n} + \text{terms with } t_1 \lambda^{a_1} \text{ as a factor.}$$

Now, owing to the definition of the f's, we see that the expression

$$f'_1 \frac{\partial A}{\partial y_1} + \ldots + f'_n \frac{\partial A}{\partial y_n}$$

* For t_1^2 would be multiplied by a determinant, such as

$$\begin{vmatrix} p_r q_g, & p_s q_g \\ p_r q_h, & p_s q_h \end{vmatrix},$$

which vanishes identically.

is only of degree $(a_1 - 1)$ in λ, not of degree a_1,* so that we may write

$$U_1/\Delta_1(\lambda) = X_1 + X_2\lambda + \ldots + X_a\lambda^{a-1} + \text{terms with } t_1\lambda^a \text{ as a factor}$$
$$(a = a_1).$$

Similarly, we shall have

$$V_1/\Delta_1(\lambda) = Y_1 + Y_2\lambda + \ldots + Y_\beta\lambda^{\beta-1} + \text{terms with } t_1\lambda^\beta \text{ as a factor}$$
$$(\beta = \beta_1).$$

Thus we shall have, say

$$U_1/\Delta_1(\lambda) = X_1 + X_2\lambda + \ldots + X_a\lambda^{a-1} + t_1\lambda^a(X_{a+1} + X_{a+2}\lambda + \ldots),$$
$$V_1/\Delta_1(\lambda) = Y_1 + Y_2\lambda + \ldots + Y_\beta\lambda^{\beta-1} + t_1\lambda^\beta(Y_{\beta+1} + Y_{\beta+2}\lambda + \ldots),$$

and then, according to the usual process given by Weierstrass and Stickelberger, we have to pick out the coefficients of $1/\lambda$, $1/\lambda^2$ in the expansion of $-U_1V_1/\Delta_0\Delta_1$; or the coefficients of $\lambda^{a+\beta}$, $\lambda^{a+\beta-1}$ in the product

$$\frac{1}{t_1}\left[X_1 + \ldots + X_a\lambda^{a-1} + t_1\lambda^a(X_{a+1} + X_{a+2}\lambda + \ldots)\right]$$
$$\times\left[Y_1 + \ldots + Y_\beta\lambda^{\beta-1} + t_1\lambda^\beta(Y_{\beta+1} + \ldots)\right].$$

Hence we find the two sets of terms

(in C)

$$X_1Y_{a+\beta+1} + X_2Y_{a+\beta} + \ldots + X_aY_{\beta+2} + t_1X_{a+1}Y_{\beta+1} + X_{a+2}Y_\beta + \ldots + X_{a+\beta+1}Y_1,$$

(in B) $\quad X_1Y_{a+\beta} + X_2Y_{a+\beta-1} + \ldots + X_aY_{\beta+1} + X_{a+1}Y_\beta + \ldots + X_{a+\beta}Y_1.$

It is easy to see that we find similar sets of terms for each pair of indices a and β; we can now remove the terms in C which are multiplied by $t_1, t_2, \ldots, t_k$, and so obtain the reduced form of A. The comparison of these terms shows that

$$X_{a+1} = q_1'x_1 + \ldots + q_n'x_n, \quad Y_{\beta+1} = p_1'y_1 + \ldots + p_n'y_n,$$

with similar results for each group of p's and q's.

Finally, it is to be noted that (as proved by Kronecker) the numbers a, β in addition to the ordinary invariant-factors form a *complete* set of invariants.

---

* For $\quad f_1'\left(\lambda\frac{\partial A}{\partial y_1} - \frac{\partial B}{\partial y_1}\right) + \ldots + f_n'\left(\lambda\frac{\partial A}{\partial y_n} - \frac{\partial B}{\partial y_n}\right) = 0,$

so that $\lambda\left[f_1'\frac{\partial A}{\partial y_1} + \ldots + f_n'\frac{\partial A}{\partial y_n}\right]$ is only of degree a_1 in λ.

As a consequence of the results that

$$\Delta_0(\lambda)/\Delta_1(\lambda) = -t_1\lambda^{\alpha_1+\beta+1},$$
&c.,

it should be observed that the π's, κ's do *not* give values of $\Delta_1(\lambda)$, $\Delta_2(\lambda)$, ..., which are *regular* for the other invariant-factors of $|\lambda C - B|$. These must be treated independently (as is usually the case in dealing with invariant-factors to different bases), and each of them will give a group of terms of the ordinary type found by Weierstrass and Stickelberger.

This general method of dealing with bilinear forms requires very little change for the problem of quadratic forms or symmetrical bilinears. In this respect it differs from Kronecker's last method,* which requires special modifications in dealing with a family of quadratics. Here we have only to note that $\alpha_r = \beta_r\ (r = 1, 2, \ldots, k)$, and that each q is equal to its corresponding p, and each κ to π. Then, clearly, each Y is the same function of the y's that the corresponding X is of the x's: and so, in the symmetrical bilinear family, we have sets of terms

$$X_1Y_{2a+1} + \ldots + X_aY_{a+2} + X_{a+2}Y_a + \ldots + X_{2a+1}Y_1,$$
$$X_1Y_{2a} + \ldots + X_aY_{a+1} + X_{a+1}Y_a + \ldots + X_{2a}Y_1;$$

or, in the quadratic forms,

$$2(X_1X_{2a+1} + X_2X_{2a} + \ldots + X_aX_{a+2}),$$
$$2(X_1X_{2a} + X_2X_{2a-1} + \ldots + X_aX_{a+1}).$$

3. *Reduction of a Single Alternate Form to a Canonical Shape by Congruent Substitutions.*

Here the number n of x's and of y's is even; otherwise Δ_0 would vanish. We make the assumption that $u_r^s = -v_r^s$, so that all Δ's are skew-symmetrical determinants, and hence $\Delta_1 = 0 = \Delta_3 = \Delta_5 = \ldots$, while Δ_0, Δ_2, Δ_4, ... are all perfect squares.†

---

* *Berliner Sitzungsberichte*, 1890, p. 1225, for bilinear forms; p. 1375, and 1891 pp. 9, 33, for quadratics; from which it will be seen that Kronecker's discussion of the quadratic case is much longer than that of the bilinear.

† Note, the Δ's, &c., are here as originally defined, *i.e.*, without any λ; *e.g.*,

$$\Delta_0 = |a_{rs}|.$$

Suppose then that k is odd; we can apply the preliminary lemma, and we shall have

$$(k, k) = 0 = (k+1, k+1),$$

for these are both skew-symmetrical determinants of odd order. Also, changing rows into columns and then changing the sign of every row, we see that

$$(k+1, k) = -(k, k+1) = E_k \text{ say},$$

and we have $\Delta_{k-1}\Delta_{k+1} = \begin{vmatrix} 0, & (k+1, k) \\ (k, k+1), & 0 \end{vmatrix} = E_k^2.$

Hence
$$\frac{W_{k-1}}{\Delta_{k-1}} = \frac{W_{k+1}}{\Delta_{k+1}} + \frac{1}{\Delta_{k-1}E_k}(U_k'V_k - U_kV_k').$$

Again, taking U_k, changing rows into columns, and changing the signs of every column but the last, we see that U_k and V_k are congruent functions of the ξ's and η's respectively. If, now, we write

$$\xi_r = \frac{\partial A}{\partial y_r}, \quad \eta_r = -\frac{\partial A}{\partial x_r},$$

we shall have that U_k and V_k are congruent functions of the x's and y's; and the same holds for U_k', V_k'.

Making this substitution for the ξ's and η's, we have that

$$W_0 = \Delta_0 A,$$

and so we have $A = \dfrac{W_0}{\Delta_0} = \dfrac{W_2}{\Delta_2} + \dfrac{1}{\Delta_0 E_1}(U_1'V_1 - U_1V_1'),$

$$\frac{W_2}{\Delta_2} = \frac{W_4}{\Delta_4} + \frac{1}{\Delta_2 E_3}(U_3'V_3 - U_3V_3'),$$

&c.

Hence A is the sum of $\frac{1}{2}n$ terms of the type just written, or

$$A = \frac{1}{\Delta_0 E_1}(U_1'V_1 - U_1V_1') + \frac{1}{\Delta_2 E_3}(U_3'V_3 - U_3V_3') + \dots$$
$$\dots + \frac{1}{\Delta_{n-2}E_{n-1}}(U_{n-1}'V_{n-1} - U_{n-1}V_{n-1}'),$$

which is the canonical type for an alternate form.* Just as a quadratic form can be reduced to a sum of squares in an infinity of ways, so this can be reduced in an infinity of ways.

Jordan† has given a special case of this, which is obtained by writing

$$u_a^a = 1, \quad u_r^a = 0 \quad (r = 1, 2, \ldots, a-1, a+1, \ldots, n).$$

Jordan takes (virtually) the substitution

$$X = -\frac{\partial A}{\partial y_1}, \quad Y = +\frac{\partial A}{\partial x_1}, \quad X' = -\frac{\partial A}{\partial y_2}, \quad Y' = +\frac{\partial A}{\partial x_2},$$

then (if $a_{12} \neq 0$) it is easy to show that the form

$$A - \frac{1}{a_{12}}(XY' - X'Y)$$

is independent of the four variables x_1, y_1, x_2, y_2; and, starting afresh from this form, we can remove four more variables, and so on until the whole form is reduced. It is not difficult to prove by induction that the coefficients of the substitution are all Pfaffians. Thus I find

$$X_1 = (12)\,x_2 + (13)\,x_3 + \ldots + (1n)\,x_n,$$

$$X_2 = (21)\,x_1 + (23)\,x_3 + \ldots + (2n)\,x_n,$$

$$X_3 = \frac{1}{(12)}\left[(1234)\,x_4 + \ldots + (123n)\,x_n\right],$$

$$X_4 = \frac{1}{(12)}\left[(1243)\,x_3 + \ldots + (124n)\,x_n\right],$$

$$X_5 = \frac{1}{(1234)}\left[(123456)\,x_6 + \ldots + (12345n)\,x_n\right],$$

$$X_6 = \frac{1}{(1234)}\left[(123465)\,x_5 + \ldots + (12346n)\,x_n\right],$$

&c.,

---

* If n should be odd, it can be readily proved that

$$W_1 = \Delta_1 A,$$

and then we find $\frac{1}{2}(n-1)$ terms of the type just given.

† *Liouville's Journal*, Vol. XIX. (2me sér.), 1874, p. 35.

and then the reduced form of A will be

$$\frac{1}{(12)}(X_1Y_2-X_2Y_1)+\frac{(12)}{(1234)}(X_3Y_4-X_4Y_3)$$
$$+\frac{(1234)}{(123456)}(X_5Y_6-X_6Y_5)+\ldots,$$

where the numbers in brackets denote Pfaffians in the ordinary way; for example,

$$(12)=a_{12},\quad (1234)=a_{12}a_{34}+a_{23}a_{14}+a_{31}a_{24},\ \&c.$$

Frobenius* has given a neat proof of Jordan's transformations; and E. von Weber† obtains a form equivalent to Jordan's. Muth‡ has published a short investigation showing how to modify a method of Kronecker's§ so as to obtain the reduced form of an alternate bilinear form by means of congruent substitutions; this investigation (like Frobenius's) is, in the first place, concerned only with forms whose coefficients are integers (in any assigned region of rationality).

E. von Weber also applies his results to reduce a family of alternate forms; I had worked out this reduction independently (using the transformation above), but shall omit the algebra here.

4. *Simultaneous Reduction of a Symmetric and an Alternate Form by Congruent Substitutions.*

Let A, B be respectively symmetric and alternate; then we must find first the invariant-factors of the determinant $|\lambda A-B|$. By definition of A, B, we have

$$A'=A,\quad B'=-B,$$

where accents refer to the conjugate bilinear forms. Hence

$$\lambda A-B=\lambda A'+B',$$

so that $$|\lambda A-B|=|\lambda A'+B'|=|\lambda A+B|,$$

for the determinant of a bilinear form is equal to that of its conjugate.

---

* *Crelle* (1879), Vol. LXXXVI., p. 146: the special consideration of alternate forms is in § 7 (p. 166).

† *Münchener Sitzungsberichte* (1898), p. 369.

‡ *Crelle* (1900), Vol. CXXII., p. 89.

§ *Crelle* (1891), Vol. CVII., p. 135.

Thus for every factor $(\lambda-c)^l$ of $|\lambda A-B|$ there will be another $(\lambda+c)^l$.

Further, if c be complex (and if the coefficients of A and B are *real*), there will be two corresponding conjugate complex roots c_0, $-c_0$ (where c_0 is the complex quantity conjugate to c); but, if $c_0=-c$, or if c be a pure imaginary, these two $(c_0, -c_0)$ are the same as the other two $(-c, +c)$. We shall prove now that the invariant-factors of $|\lambda A-B|$ occur (in general) in pairs.

Consider the value of Δ_k when the sign of λ is changed; first interchange rows and columns and then change the signs of the first n rows and the last k columns. Remembering that $a_{rs}=a_{sr}$, $b_{rs}=-b_{sr}$, we see that the new determinant only differs from Δ_k by having the u's and v's interchanged; but Stickelberger has proved (*St.*, § 5) that we may in general take $u_r^a=v_r^a$; and, making this assumption, we see that changing the sign of λ in Δ_k will only multiply Δ_k by $(-1)^{n-k}$.

So, writing $$\Delta_k=(\lambda-c)^l f(\lambda) \qquad [f(c)\neq 0],$$
we have $$(-1)^{n-k}\Delta_k=(-\lambda-c)^l f(-\lambda).$$

Thus Δ_k has factors of the type $(\lambda-c)^l$, $(\lambda+c)^l$ in pairs; and the invariant-factors (which are obtained from the quotients Δ_{k-1}/Δ_k) will occur in pairs of the type $(\lambda-c)^e$, $(\lambda+c)^e$.

This theorem is due to Kronecker [*Kr.*, p. 440 (p. 477)]. We shall now combine the terms in the reduced forms of A, B which correspond to the pair of invariant-factors $(\lambda-c)^e$, $(\lambda+c)^e$. We have to split $A(\lambda A-B)^{-1}A$ into partial fractions; thus we consider only the special fraction $-(U_kV_k)/(\Delta_{k-1}\Delta_k)$, and evaluate the fractions in $1/(\lambda-c)$, $1/(\lambda+c)$ which can be obtained from this. As explained before (p. 322), we only require the coefficients of $1/(\lambda-c)$, $1/(\lambda-c)^2$, $1/(\lambda+c)$, $1/(\lambda+c)^2$. To find them we write $\lambda=c+t$, $\lambda=-c+t'$, and expand first in powers of t, then in powers of t'.

We shall take for the present U_k to denote the value of U_k with $\lambda=c+t$; and U_k' its value with $\lambda=-c+t'$. The same notation will be used for Δ_k, V_k, and then we have

$$U_k'=\begin{vmatrix} (-c+t')a_{rs}-b_{rs}, & u_r^a, & u_r^k \\ u_s^a, & 0, & 0 \\ \dfrac{\partial A}{\partial y_s}, & 0, & 0 \end{vmatrix} \quad \begin{pmatrix} r, s=1, 2, \ldots, n \\ a=1, 2, \ldots, k-1 \end{pmatrix},$$

writing the determinant in the shortened form suggested by Frobenius and Nanson.

Change the signs of the first n rows and then of the last k columns in U'_k; finally, change rows into columns, then (remembering $a_{rs} = a_{sr}$, $b_{rs} = -b_{sr}$)

$$U'_k = (-1)^{n-k} \begin{vmatrix} (c-t')\, a_{rs} - b_{rs}, & u^{\bullet}_{r}, & \dfrac{\partial A}{\partial y_r} \\ u^{a}_{s}, & 0, & 0 \\ u^{k}_{s}, & 0, & 0 \end{vmatrix}.$$

Hence, if we change the x's to y's and write $t' = -t$ in U'_k, we should have

$$U'_k = (-1)^{n-k} V_k,$$

and, with a similar change,

$$V'_k = (-1)^{n-k} U_k.$$

By the same process we find

$$\Delta'_k = (-1)^{n-k} \Delta_k, \quad \Delta'_{k-1} = (-1)^{n-k+1} \Delta_{k-1},$$

where $t' = -t$.

If, then, we write

$$-\frac{U_k V_k}{\Delta_k \Delta_{k-1}} = \frac{1}{t^e} (X_1 + X_2 t + \dots)(Y_{e+1} + Y_{e+2} t + \dots),$$

we shall have

$$-\frac{U'_k V'_k}{\Delta'_k \Delta'_{k-1}} = \frac{-1}{(-t')^e} (Y_1 - Y_2 t' + Y_3 t'^2 - \dots)(X_{e+1} - X_{e+2} t' + \dots),$$

where Y_r is the same function of the y's that X_r is of the x's. Now the terms we require will come from the coefficients of $1/t$, $1/t^2$, $1/t'$, $1/t'^2$, which will be

$$(X_1 Y_{2e} + X_2 Y_{2e-1} + \dots + X_e Y_{e+1}),$$

$$(X_1 Y_{2e-1} + X_2 Y_{2e-2} + \dots + X_{e-1} Y_{e+1}),$$

$$(X_{2e} Y_1 + X_{2e-1} Y_2 + \dots + X_{e+1} Y_e),$$

$$-(X_{2e-1} Y_1 + X_{2e-2} Y_2 + \dots + X_{e+1} Y_{e-1}),$$

respectively. These will be parts of the coefficients of $1/(\lambda - c)$, $1/(\lambda - c)^2$, $1/(\lambda + c)$, $1/(\lambda + c)^2$ in the expansions of $A\,(\lambda A - B)^{-1} A$ in powers of $(\lambda - c)$, $(\lambda + c)$.

Finally, combining these terms as explained (p. 323), we find the corresponding parts of A, B in the forms

$$(A)\quad X_1Y_{2e}+X_{2e}Y_1+\dots+X_eY_{e+1}+X_{e+1}Y_e,$$

$$(B)\quad c\,(X_1Y_{2e}-X_{2e}Y_1)+\dots+c\,(X_eY_{e+1}-X_{e+1}Y_e)$$
$$+(X_1Y_{2e-1}-X_{2e-1}Y_1)+\dots+(X_{e-1}Y_{e+1}-X_{e+1}Y_{e-1}),$$

where the substitutions are *congruent*.

It may be observed that the substitutions will all be real; and, further, that by means of a similar process, interchanging the parts played by A, B, we can obtain

$$(A)\quad \frac{1}{c}(X_1Y_{2e}+X_{2e}Y_1)+\dots+\frac{1}{c}(X_eY_{e+1}+X_{e+1}Y_e)$$
$$+X_1Y_{2e-1}+X_{2e-1}Y_1+\dots+X_{e-1}Y_{e+1}+X_{e+1}Y_{e-1},$$

$$(B)\quad X_1Y_{2e}-X_{2e}Y_1+\dots+X_eY_{e+1}-X_{e+1}Y_e,$$

as typical terms in the reduced forms.

In case c is a complex quantity, we shall have an exactly similar pair of terms from using c_0 instead of c; if it is desired to restrict ourselves to *real* transformations, we can combine the corresponding terms after first dividing them into real and imaginary parts. If c be a pure imaginary, X_r and X_{e+r} will be conjugate imaginaries.

If c be zero and e be odd, we may still write $u_r^a = v_r^a$ (*St.*, § 5), and in this case there will be no invariant-factor paired off with this one. But we shall have

$$U_k = \begin{vmatrix} \lambda a_{rs}-b_{rs}, & u_r^a, & u_r^k \\ u_s^a, & 0, & 0 \\ \dfrac{\partial A}{\partial y_s}, & 0, & 0 \end{vmatrix},$$

and, treating this as before, we find

$$U_k = (-1)^{n-k}\begin{vmatrix} -\lambda a_{rs}-b_{rs}, & u^a, & \dfrac{\partial A}{\partial y_r} \\ u_s^a, & 0, & 0 \\ u_s^k, & 0, & 0 \end{vmatrix},$$

or
$$U_k = (-1)^{n-k}\,V_k,$$

where in V_k we replace λ by $(-\lambda)$ and the y's by x's.

Thus, if $c = 2p+1$, we have to pick out the coefficients of λ^{2p}, λ^{2p-1} in the product

$$(X_1+X_2\lambda+X_3\lambda^2+\ldots)(Y_1-Y_2\lambda+Y_3\lambda^2-\ldots),$$

where Y_r is the same function of the y's that X_r is of the x's. So we have typical terms

$$(A)\quad X_1Y_{2p+1}-X_2Y_{2p}+\ldots-X_{2p}Y_2+X_{2p+1}Y_1,$$

$$(B)\quad (X_{2p}Y_1-X_1Y_{2p})+\ldots\pm(X_{p+1}Y_p-X_pY_{p+1}).$$

Now return to the case of exception previously alluded to (p. 326), which occurs when $c=0$ and the index of an invariant-factor is *even;* then (as shown by Stickelberger) it is not permissible to write $u_r^a = v_r^a$, and, to avoid this difficulty, we use the preliminary lemma given above. In the first place, one of the consecutive invariant-factors is the same (*St.*, § 5). Suppose that the invariant-factors (to base λ) given by Δ_{k-1}/Δ_k and Δ_k/Δ_{k+1} are the same; let their common index be e, where e is even. By our lemma we have a part of $A(\lambda A-B)^{-1}A$,

$$-\frac{1}{\Delta_{k-1}^2\Delta_{k+1}}\Big[(k+1,\,k+1)\;U_kV_k+(k,\,k)\;U_k'V_k' \\ -(k+1,\,k)\;U_k'V_k-(k,\,k+1)\;U_kV_k'\Big],$$

where in all the determinants we write $(\lambda a_{rs}-b_{rs})$ instead of a_{rs}. We may assume now that $u_r^a = v_r^a$; this will make (k,k) and $(k+1,\,k+1)$ divisible by a higher power of λ than appears in *every* kth minor of $|\,\lambda A-B\,|$ (*St.*, § 5); let that which appears in every kth minor be λ^l, so that λ^{l+e} divides every $(k-1)$th minor and λ^{l-e} every $(k+1)$th. Thus, if

$$\Delta_{k-1} = \lambda^{l+e}(\delta_1+\delta_2\lambda+\delta_3\lambda^2+\ldots),$$

we have, by changing the sign of λ as above (p. 335),

$$\begin{aligned}(-\lambda)^{l+e}\big[\delta_1+\delta_2(-\lambda)+\delta_3(-\lambda)^2+\ldots\big] &= (-1)^{n-k+1}\Delta_{k-1} \\ &= (-1)^{n-k+1}\lambda^{l+e}\big[\delta_1+\delta_2\lambda+\delta_3\lambda^2+\ldots\big].\end{aligned}$$

But $\delta_1\neq 0$, or Δ_{k-1} would be divisible by a higher power of λ than λ^{l+e}, and so $(n-k+l)$ is odd (as e is even). Also, from the last equation, $\delta_2 = 0$, $\delta_4 = 0$, &c.; so we may write

$$\Delta_{k-1} = \lambda^{l+e}(\delta_1+\delta_3\lambda^2+\delta_5\lambda^4+\ldots).$$

In the same way we have

$$\Delta_{k+1} = \lambda^{l-e}(\epsilon_1+\epsilon_3\lambda^2+\epsilon_5\lambda^4+\dots).$$

Again, if we change the sign of λ in $(k, k+1)$, we get $(k+1, k)$ multiplied by $(-1)^{n-k}$, and so we write

$$(k, k+1) = \lambda^l(\gamma_1+\gamma_2\lambda+\gamma_3\lambda^2+\dots)$$
$$(k+1, k) = (-1)^{n-k+l}\lambda^l(\gamma_1-\gamma_2\lambda+\gamma_3\lambda^2-\dots)$$
$$= -\lambda^l(\gamma_1-\gamma_2\lambda+\gamma_3\lambda^2-\dots),$$

and, applying the same process to (k, k), we see that, if

$$(k, k) = \lambda^l(a_1+a_2\lambda+a_3\lambda^2+\dots),$$

then also $$(k, k) = -\lambda^l(a_1-a_2\lambda+a_3\lambda^2-\dots).$$

Hence $a_1 = 0$, $a_3 = 0$, ..., and so

$$(k, k) = \lambda^{l+1}(a_2+a_4\lambda^2+a_6\lambda^4+\dots).$$

Similarly, $$(k+1, k+1) = \lambda^{l+1}(\beta_2+\beta_4\lambda^2+\beta_6\lambda^4+\dots).$$

Using the same arguments, we readily see that, if we write

$$U_k = \lambda^l(\xi_1+\xi_2\lambda+\dots), \quad U'_k = \lambda^l(\xi'_1+\xi'_2\lambda+\dots),$$

then $$V_k = -\lambda^l(\eta_1-\eta_2\lambda+\dots), \quad V'_k = -\lambda^l(\eta'_1-\eta'_2\lambda+\dots),$$

where η_r, η'_r are the same functions of the y's as ξ_r, ξ'_r of the x's.

For brevity write

$$(a) = a_2+a_4\lambda^2+\dots, \text{ \&c.},$$
$$(\xi) = \xi_1+\xi_2\lambda+\dots, \text{ \&c.},$$
$$(\eta) = \eta_1-\eta_2\lambda+\dots, \text{ \&c.},$$
$$(\gamma) = \gamma_1+\gamma_2\lambda+\dots, \quad (\gamma') = \gamma_1-\gamma_2\lambda+\dots;$$

then we have our typical term in the expression for $A(\lambda A-B)^{-1}A$,

$$-\frac{1}{\lambda^e}\frac{1}{(\delta)^2(\epsilon)}\left[-\lambda(\beta)(\xi)(\eta)-\lambda(a)(\xi')(\eta')+(\gamma)(\xi)(\eta')-(\gamma')(\xi')(\eta)\right].$$

Now put $$(\Xi) = (\xi)\left[\frac{(\gamma)}{(\delta)^2(\epsilon)}\right]^{\frac{1}{2}}, \quad (\mathrm{H}) = (\eta)\left[\frac{(\gamma')}{(\delta)^2(\epsilon)}\right]^{\frac{1}{2}},$$

$$(\Xi') = (\xi')\left[\frac{(\gamma')}{(\delta)^2(\epsilon)}\right]^{\frac{1}{2}}, \quad (\mathrm{H}') = (\eta')\left[\frac{(\gamma)}{(\delta)^2(\epsilon)}\right]^{\frac{1}{2}},$$

$$(p) = \frac{(\beta)}{[(\gamma)(\gamma')]^{\frac{1}{2}}}, \quad (q) = \frac{(a)}{[(\gamma)(\gamma')]^{\frac{1}{2}}};$$

then (H), (H′) are derived from (Ξ), (Ξ′) respectively, by changing λ to $-\lambda$, and the x's to y's; further, (p), (q) are even functions of λ. With these abbreviations we see that our typical term becomes

$$\frac{1}{\lambda^e}\left[(\Xi')(\mathrm{H})-(\Xi)(\mathrm{H}')+\lambda\,(p)(\Xi)(\mathrm{H})+\lambda\,(q)(\Xi')(\mathrm{H}')\right].$$

It is now possible to determine (g), (h), (g'), (h') power-series in λ^2 so that the expression in square brackets is equal to

$$\left[(g')(\Xi')+\lambda\,(g)(\Xi)\right]\left[(h)(\mathrm{H})-\lambda\,(h')(\mathrm{H}')\right]$$
$$-\left[(h)(\Xi)+\lambda\,(h')(\Xi')\right]\left[(g')(\mathrm{H}')-\lambda\,(g)(\mathrm{H})\right],*$$

or to
$$(X)(Y')-(X')(Y),$$

where (Y) and (Y') are obtained from (X) and (X') respectively by changing the x's to y's, and λ to $-\lambda$.

Now, if we write

$$(X) = X_1+X_2\lambda+X_3\lambda^2+\ldots, \qquad (Y) = Y_1-Y_2\lambda+\ldots,$$
$$(X') = X_{e+1}-X_{e+2}\lambda+X_{e+3}\lambda^2-\ldots, \quad (Y') = Y_{e+1}+Y_{e+2}\lambda+\ldots,$$

it is clear that Y_r is the same function of the y's as X_r of the x's, and we find that the coefficients of $1/\lambda$, $1/\lambda^2$ in $\frac{1}{\lambda^e}\left[(X)(Y')-(X')(Y)\right]$ are respectively

$$X_1Y_{2e}+X_2Y_{2e-1}+\ldots+X_{2e-1}Y_2+X_{2e}Y_1,$$

$$(X_1Y_{2e-1}-X_{2e-1}Y_1)+(X_2Y_{2e-2}-X_{2e-2}Y_2)+\ldots+(X_{e-1}Y_{e+1}-X_{e+1}Y_{e-1}),$$

and these will be the reduced parts of A, B corresponding to the *pair* of invariant-factors λ^e, λ^e (e even).

In considering the infinite roots of $|\lambda A-B|=0$, it is easier to treat them as zero roots of $|\mu B-A|=0$. We use the same notation as before, but with $(\mu b_{rs}-a_{rs})$ in place of $(\lambda a_{rs}-b_{rs})$, and

---

* We have to consider an expression of the type $x'y-xy'+axy+bx'y'$, which is equal to $$[(x'+\beta x)(y-\alpha y')-(x+\alpha x')(y'-\beta y)]/(1+\alpha\beta),$$ provided that $$2\beta/a = -2\alpha/a = (1+\alpha\beta).$$ These equations will determine α, β by solving a single quadratic equation. In the special case required α, β, a, b will be power-series containing only *odd* powers of λ.

$\dfrac{\partial B}{\partial x_r}$, $\dfrac{\partial B}{\partial y_r}$ in place of $\dfrac{\partial A}{\partial x_r}$, $\dfrac{\partial A}{\partial y_r}$, and we are concerned with the invariant-factors to base μ only.

Such invariant-factors may occur singly with an *even* index (*St.*, § 5); but in pairs if the index be odd.* We shall consider first the case when the index is even, then we can treat one invariant-factor by itself; in this case (*St.*, § 5) we may still put $u_r^\circ = v_r^\circ$.

We readily see that changing the sign of μ in Δ_k only interchanges the rows and columns, and so does not affect the value of Δ_k; but the corresponding change in U_k gives $(-V_k)$, with the y's changed into x's, for $\dfrac{\partial B}{\partial x_r} = -\dfrac{\partial B}{\partial y_r}$ if the y's are changed into x's (as B is alternate). Thus we have (if $e = 2p$) to pick out the coefficients of μ^{2p-2}, μ^{2p-1} in the product

$$-(X_1+X_2\mu+X_3\mu^2+\dots)(Y_1-Y_2\mu+Y_3\mu^2-\dots),$$

where Y_r is the same function of the y's that X_r is of the x's. Hence we find the typical terms

$$(A)\quad -X_1Y_{2p-1}+X_2Y_{2p-2}-\dots+X_{2p-2}Y_2-X_{2p-1}Y_1,$$

$$(B)\quad (X_1Y_{2p}-X_{2p}Y_1)-(X_2Y_{2p-1}-X_{2p-1}Y_2)+\dots\pm(X_pY_{p+1}-X_{p+1}Y_p).$$

We have now to consider the case of a pair of invariant-factors of the type μ^e (e odd). The investigation of the corresponding reduced parts of A, B offers little difficulty after the reduction for the pair λ^e, λ^e (e even); we give the similar results without full explanations.

Changing the sign of μ in Δ_{k-1}, Δ_{k+1} will not affect their values, and so we have

$$\Delta_{k-1} = \mu^{l+e}(\delta_1+\delta_3\mu^2+\delta_5\mu^4+\dots),$$

$$\Delta_{k+1} = \mu^{l-e}(\epsilon_1+\epsilon_3\mu^2+\epsilon_5\mu^4+\dots),$$

where e is odd and l is odd [instead of $(n-k+l)$ being odd, as before]. Thus we may write

$$(k,\,k+1) = \mu^l(\gamma_1+\gamma_2\mu+\gamma_3\mu^2+\dots);$$

then $$(k+1,\,k) = -\mu^l(\gamma_1-\gamma_2\mu+\gamma_3\mu^2-\dots).$$

---

* This theorem (and the corresponding one relating to the invariant-factors of $|\lambda A - B|$ to base λ) are due in the first place to Kronecker [*Kr.*, p. 441 (p. 477)].

by changing the sign of μ. Also

$$(k, k) = \mu^{l+1}(\alpha_2+\alpha_4\mu^2+\dots),$$

$$(k+1, k+1) = \mu^{l+1}(\beta_2+\beta_4\mu^2+\dots).$$

We shall have also $U_k = \mu^l(\xi_1+\xi_2\mu+\xi_3\mu^2+\dots),$

$$V_k = -(-\mu)^l(\eta_1-\eta_2\mu+\eta_3\mu^2-\dots)$$

$$= \mu^l(\eta_1-\eta_2\mu+\eta_3\mu^2-\dots),$$

where η_r is the same function of the y's as ξ_r of the x's. Similar results hold for U'_k, V'_k.

Just as before, we can reduce the expression for

$$(-W_{k-1}/\Delta_{k-1}+W_{k+1}/\Delta_{k+1})$$

to the form $$\frac{1}{\mu^e}\left[(X)(Y')-(Y)(X')\right],$$

where $(X) = X_1+X_2\mu+X_3\mu^2+\dots$, $(Y) = Y_1-Y_2\mu+Y_3\mu^2-\dots$,

$(X') = X_{e+1}-X_{e+2}\mu+X_{e+3}\mu^2-\dots$, $(Y') = Y_{e+1}+Y_{e+2}\mu+Y_{e+3}\mu^2+\dots$,

Y_r being the same function of the y's that X_r is of the x's. Hence we find the typical parts

$$(A)\quad X_1Y_{2e-1}+X_2Y_{2e-2}+\dots+X_{2e-2}Y_2+X_{2e-1}Y_1,$$

$$(B)\quad (X_1Y_{2e}-X_{2e}Y_1)+\dots+(X_eY_{e+1}-X_{e+1}Y_e).$$

If $|\lambda A-B| \equiv 0$, we may apply the method explained above (§ 2, p. 326). Here we have $\beta_r = \alpha_r$, for, by interchanging the p's and q's and changing the sign of λ, we do not alter the value of $\Delta_k(\lambda)$ (except possibly in sign); hence the invariant-factors belonging to these terms will be of the type λ^{2a+1}, and so (as before, p. 336) we may take each q equal to the corresponding p. Then, by the same investigation as given for the case of invariant-factors to base λ, we find the typical groups of terms

$$C)\quad X_1Y_{2a+1}-X_2Y_{2a}+\dots+(-1)^a tX_{a+1}Y_{a+1}+\dots-X_{2a}Y_2+X_{2a+1}Y_1,$$

$$X_1Y_{2a}-X_2Y_{2a-1}+\dots+X_{2a-1}Y_2-X_{2a}Y_1.$$

Thus we have in the original forms

$$(A)\quad (X_1Y_{2a+1}+X_{2a+1}Y_1)-(X_2Y_{2a}+X_{2a}Y_2)+\ldots$$
$$\ldots+(-1)^{a-1}(X_aY_{a+2}+X_{a+2}Y_a),$$

$$(B)\quad (X_1Y_{2a}-X_{2a}Y_1)-(X_2Y_{2a-1}-X_{2a-1}Y_2)+\ldots$$
$$\ldots+(-1)^{a-1}(X_aY_{a+1}-X_{a+1}Y_a),$$

and there are similar sets of terms for each a. These a's are Kronecker's *Minimalgradzahlen*, and may be found as described in my former paper (*Proc. Lond. Math. Soc.*, pp. 87–92 and p. 111 above); the invariants used by Kronecker are the numbers $2a+1$, which are the numbers of variables in the sets of reduced terms.

We have now six types of reduced forms, corresponding to five types of invariant-factors and the singular case, when $|\lambda A-B|\equiv 0$. These will be found to be in agreement with Kronecker's results,* except in the arrangement of suffixes; there is also a superficial difference in the first and second classes (following Kronecker's order of arrangement, the sixth and first in the foregoing), due to the fact that Kronecker has reduced his results so as to give the neatest type for the bilinear form $(A+B)$. To indicate the real agreement it will be sufficient to consider two special cases—

(a) Corresponding to the case $|\lambda A-B|\equiv 0$, with a *Minimalgradzahl* 2, Kronecker gives the type

$$A+B = x_1y_2+x_2y_3+x_3y_4+x_4y_5,$$

and so $$A-B = A'+B' = x_2y_1+x_3y_2+x_4y_3+x_5y_4.$$

Making the substitutions

$$x_1+2x_3+x_5 = 2X_1,\qquad x_2+x_4 = 2X_2,$$
$$x_1\qquad -x_5 = 2X_3,\qquad -x_2+x_4 = 2X_4,$$
$$x_1-2x_3+x_5 = 2X_5,$$

with the congruent substitutions for the y's, we find

$$A = X_1Y_2+X_2Y_1-(X_3Y_4+X_4Y_3),$$
$$B = -X_2Y_3+X_3Y_2+X_4Y_5-X_5Y_4,$$

agreeing with our general result.

* See the paper *Kr.*, p. 440 (*Werke*, Bd. I., p. 475): the list is reproduced on p. 146 of Dr. Muth's *Theorie der Elementartheiler*, Leipzig, 1899.

(*b*) Again, Kronecker gives a form

$$A+B = x_1y_2+ax_2y_1+x_2y_3+ax_3y_2+x_3y_4+ax_4y_3,$$

and so $$A-B = x_2y_1+ax_1y_2+x_3y_2+ax_2y_3+x_4y_3+ax_3y_4.$$

If we put $a=(1+c)/(1-c)$ and then make the substitution

$$x_1 = -(X_3+2cX_4), \quad x_3 = \frac{1-c}{1+c}X_3,$$

$$x_2 = -\frac{1-c}{2c}X_1, \qquad x_4 = (1+c)X_2,$$

with the congruent substitutions for the y's, we have

$$A = X_1Y_4+X_4Y_1+X_2Y_3+X_3Y_2,$$

$$B = c(X_1Y_4-X_4Y_1)+c(X_2Y_3-X_3Y_2)+(X_1Y_3-X_3Y_1),$$

agreeing with what has been found before, corresponding to the invariant-factors $(\lambda-c)^2$, $(\lambda+c)^2$ of $|\lambda A-B|$.

5. *Application of the Reduction to Properties of Automorphic Substitutions.*

Herr Alfred Loewy * has proved certain propositions relating to automorphic substitutions of a real symmetric bilinear form (or of a quadratic form) of non-zero determinant. These are deduced (in § 9, p. 424, of the paper quoted in the footnote) from similar properties proved for conjugate imaginary substitutions which are automorphic for Hermite's forms (see below, p. 350). The following investigation is complete in itself.

Suppose that S is a real symmetric bilinear form and P a real substitution automorphic for S; then, in Frobenius's symbolical form,

$$P'SP = S.$$

Frobenius has proved† that we can find P_0 similar (*ähnlich*)‡ to P and S_0 congruent with S, so that $P_0'S_0P_0 = S_0$. Further, P_0, S_0 will be each separable (*zerlegbar*) into two parts such that $P_0=P_1+P_2$, $S_0=S_1+S_2$, where the variables in P_1, S_1 are the same and do not

---

* *Nova Acta Leop.-Carol. Akad.*, 1898, Vol. LXXI., p. 379; an abstract appeared in *Math. Annalen*, Vol. L., p. 557.

† *Crelle*, 1878, Vol. LXXXIV., p. 1; this proposition is given in § 10.

‡ Two forms P, P_0 are called *ähnlich* by Frobenius, if a third form Q can be found such that $P_0 = QPQ^{-1}$.

appear in P_2, S_2, and *vice versa*; finally, $P'_1 S_1 P_1 = S_1$, $P'_2 S_2 P_2 = S_2$. The invariant-factors of $| P_1 - \lambda E_1 |$ are all those of $| P_0 - \lambda E |$ (or of $| P - \lambda E |$) which have the base $(\lambda + 1)$; and the other invariant-factors of $| P - \lambda E |$ appear as those of $| P_2 - \lambda E_2 |$. Here the unit-form E (*Einheitsform*) has been separated into E_1, E_2 in the same way as P_0, S_0. The process followed in finding P_1, S_1, P_2, S_2 from P, S shows that these are all real and obtained by real transformations. Then we have

$$P_1 = -(S_1 + T_1)^{-1}(S_1 - T_1), \quad P_2 = +(S_2 + T_2)^{-1}(S_2 - T_2),$$

where T_1, T_2 are real alternate forms, each containing only the appropriate set of variables.

It follows that $| P_1 - \lambda E_1 |$ has the same invariant-factors as $| \mu_1 S_1 - T_1 |$ $[\mu_1 = (1+\lambda)/(1-\lambda)]$, and $| P_2 - \lambda E_2 |$ the same as $| \mu_2 S_2 - T_2 |$ $[\mu_2 = (1-\lambda)/(1+\lambda)]$.

Now, by hypothesis, $| P_1 - \lambda E_1 |$ has only invariant-factors of the type $(1+\lambda)^e$; so that $| \mu_1 S_1 - T_1 |$ has only those of the type μ_1^e. Thus, if e be even, we have a pair of equal invariant-factors, and the corresponding part of S_1 is (p. 339)

$$x_1 y_{2e} + x_2 y_{2e-1} + \ldots + x_{2e} y_1,$$

and the *signature** of this is zero; while, if e be odd, we have a single invariant-factor giving (see p. 337)

$$x_1 y_e + x_2 y_{e-1} + \ldots + x_e y_1,$$

and the absolute value of the signature is unity. It follows that the signature of S_1 is not greater (in absolute value) than the number of *odd* indices of invariant-factors $(1+\lambda)^e$ of $| P_0 - \lambda E |$.

Now take S_2; we have here five cases—

(i.) The pair of invariant-factors $(\mu_2 - c)^e$, $(\mu_2 + c)^e$ ($c \neq 0$ and *real*), corresponding to $(\lambda - b)^e$, $(\lambda - 1/b)^e$ ($b^2 \neq 1$, and b real).

The corresponding part of S_2 will be

$$x_1 y_{2e} + x_2 y_{2e-1} + \ldots + x_{2e} y_1,$$

which has signature zero.

---

* Frobenius defines the *signature* as the number of positive squares, less the number of negative squares, when the form is reduced to squares by *real* transformations.

(ii.) μ_2^e, μ_2^e (e even) corresponding to $(\lambda-1)^e$, $(\lambda-1)^e$. Here the reduced part of S_2 is the same as in (i.).

(iii.) μ_2^e (e odd) corresponding to $(\lambda-1)^e$. Here the reduced part of S_2 is

$$x_1 y_e + x_2 y_{e-1} + \ldots + x_e y_1,$$

with the absolute value of the signature equal to unity.

(iv.) $(\mu_2-c)^e$, $(\mu_2+c)^e$, $(\mu_2-c_0)^e$, $(\mu_2+c_0)^e$ (c imaginary, with c_0 for its conjugate, $c+c_0 \neq 0$) corresponding to $(\lambda-b)^e$, $(\lambda-1/b)^e$, $(\lambda-b_0)^e$ $(\lambda-1/b_0)^e$. We find for S_2 the part

$$(x_1 y_{2e} + \ldots + x_{2e} y_1) + \text{the conjugate imaginary},$$

and we can readily reduce this to a *real* form, whose signature is seen to be zero.

(v.) $(\mu_2-c)^e$, $(\mu_2+c)^e$ (c pure imaginary), corresponding to $(\lambda-b)^e$, $(\lambda-1/b)$ (mod $b=1$). This gives for S_2

$$x_1 y_{2e} + x_2 y_{2e-1} + \ldots + x_{2e} y_1,$$

and, from the method used in making the reduction in the last section, x_{e+r} and x_r are conjugate imaginaries. It is now easy to see that the signature of this part of S_2 is zero if e be even, or 2 if e be odd. It follows that the absolute value of the signature of S_2 is not greater than the number of odd indices of invariant-factors of $|P_2-\lambda E|$ of the type $(\lambda-b)^e$ (mod $b=1$).

Hence, finally, as the absolute value of the signature of $S_0 = S_1 + S_2$ is not greater than the sum of the absolute values of the signatures of S_1 and S_2, it follows that the absolute value for S_0 is not greater than the number of odd indices of invariant-factors of $|P_0-\lambda E|$ of the type $(\lambda-b)^e$ (mod $b=1$, including $b=\pm 1$). But, since S_0 is derived from S by a *real* substitution, their signatures are equal; and, since P, P_0 are similar, the invariant-factors of $|P-\lambda E|$, $|P_0-\lambda E|$ are the same. Hence the theorem follows:

If P be a real substitution which is automorphic for the real quadratic form S, the signature of S is not greater in absolute value than the number of odd indices of invariant-factors of $|P-\lambda E|$, only those of the type $(\lambda-b)^e$ (mod $b \mp 1$) being counted.

This theorem is equivalent to the inequality in § 9 of Herr Loewy's paper. Herr Loewy asks me to state that this form of the theorem was familiar to him when his paper was published; but that, for the applications he had in view (§§ 10–12), the form of the theorem given in his paper was found more suitable; this form of the theorem can be deduced immediately from the above by introducing the *characteristic* (as defined by Loewy)* of S. A similar remark applies to the theorem stated on p. 163 above.

6. *Simultaneous Reduction of two Hermite's Forms to a Canonical Shape.*

A *Hermite's form* is a bilinear form in which the coefficients a_{rs}, a_{sr} are conjugate imaginaries; and the variables x_r, y_r are also conjugate imaginaries, but, for purposes of symbolical calculation, it is often convenient to leave this out of reckoning, just as we frequently use symmetrical bilinear forms instead of quadratic forms.

Symbolically, a Hermite's form (A) is defined by the condition

$$A' = A_0$$

where the suffix 0 indicates that we should take the conjugate imaginary of each *coefficient* (the effect of the suffix is not to be extended to the variables).

If A, B are Hermite's forms to be reduced to canonical shapes, we are restricted to conjugate imaginary substitutions on the x's and the y's; we have thus, in Frobenius's notation, to determine a form S which will give

$$S_0'AS, \quad S_0'BS$$

as reduced forms.

We shall see that we can modify Stickelberger's results (*St.*, § 1) so as to reduce A, B by substitutions of the type considered. We show, first, that the invariant-factors of $|\lambda A - B|$ occur in pairs of the form $(\lambda - c)^e$, $(\lambda - c_0)^e$, where c, c_0 are conjugate imaginaries.

For, according to Stickelberger (*St.*, § 5), we may always choose our u's and v's so that u_r^s, v_r^s are conjugate complex quantities; in fact, they may be real and equal except in two special cases. With this assumption, take the conjugate imaginary of Δ_k (without changing λ) and change rows into columns; we then have Δ_k once more. Hence

$$\Delta_k = (\Delta_k)_0.$$

Thus, if $\qquad \Delta_k = (\lambda - c)\, f(\lambda) \qquad [f(c) \neq 0]$,

we have $\qquad \Delta_k = (\lambda - c_0)^e f_0(\lambda) \qquad [f_0(c_0) \neq 0]$.

---

* The *characteristic* of a quadratic form is the number of positive or negative squares in the reduced form; the *smaller* of these numbers being taken.

It follows that factors of Δ_k of the type $(\lambda-c)^l$, $(\lambda-c_0)^l$ always appear in pairs, and thus, since the invariant-factors are given by the series of quotients Δ_{k-1}/Δ_k, the invariant-factors also appear in pairs of the type $(\lambda-c)^e$, $(\lambda-c_0)^e$. If c is real, or $c = c_0$, this result will no longer hold.

Next we combine the parts of the fraction $-(U_k V_k)/(\Delta_k \Delta_{k-1})$ which are obtained from the invariant-factors $(\lambda-c)^e$, $(\lambda-c_0)^e$. Let us write U_k for its value when $\lambda = c+t$, U_k' for its value when $\lambda = c_0 + t'$; then, taking the conjugate imaginary of U_k' (without altering the x's or t'), we find, changing rows into columns,

$$(U_k')_0 = \begin{vmatrix} (c+t')\,a_{rs} - b_{rs}, & v_r^a, & \dfrac{\partial A_0}{\partial y_r} \\ u_s^a, & 0, & 0 \\ u_s^k, & 0, & 0 \end{vmatrix} \quad \begin{pmatrix} r, s = 1, 2, \ldots, n \\ a = 1, 2, \ldots, k-1 \end{pmatrix}.$$

Now $\dfrac{\partial A_0}{\partial y_r}$ is the same function of the x's as $\dfrac{\partial A}{\partial x_r}$ is of the y's, by definition of A. Hence, if t' be changed to t in $(U_k')_0$, we obtain the same function of the x's as V_k is of the y's. Similar results hold for $(V_k')_0$ and U_k. Hence we may write

$$-\frac{U_k V_k}{\Delta_k \Delta_{k-1}} = \frac{1}{t^e}\,(X_1 + X_2 t + \ldots)(Y_{e+1} + Y_{e+2} t + \ldots)$$

and

$$-\frac{U_k' V_k'}{\Delta_k' \Delta_{k-1}'} = \frac{1}{t'^e}\,(Y_1 + Y_2 t' + \ldots)(X_{e+1} + X_{e+2} t' + \ldots),$$

where the X's are derived from the x's and the Y's from the y's by conjugate imaginary substitutions, by virtue of what has been proved connecting U_k and V_k', U_k' and V_k, Δ_k and Δ_k'.

Combining the coefficients of $1/t$, $1/t^2$, $1/t'$, $1/t'^2$ (as explained previously), we find the reduced parts

$$(A) \quad X_1 Y_{2e} + X_2 Y_{2e-1} + \ldots + X_{2e} Y_1,$$

$$(B) \quad c\,(X_1 Y_{2e} + \ldots + X_e Y_{e+1}) + c_0\,(X_{e+1} Y_e + \ldots + X_{2e} Y_1)$$
$$+ X_1 Y_{2e-1} + X_2 Y_{2e-2} + \ldots + X_{2e-1} Y_1.$$

If $c = c_0$, we do not have a pair of invariant-factors in general; but, writing $\lambda = c + t = c_0 + t$, we find that

$$(U_k)_0 = \begin{vmatrix} (c+t)\,a_{rs} - b_{rs}, & v^a, & \dfrac{\partial A_0}{\partial y_r} \\ u^a, & 0, & 0 \\ u^k_s, & 0, & 0 \end{vmatrix} \quad \begin{pmatrix} r, s = 1, 2, \ldots, n \\ a = 1, 2, \ldots, k-1 \end{pmatrix},$$

which is the same function of the x's as V is of the y's $\left(\text{by virtue of the relation between } \dfrac{\partial A_0}{\partial y_r} \text{ and } \dfrac{\partial A}{\partial x_r}\right)$.

Hence, if we write

$$U_k = t^l\,(\xi_1 + \xi_2 t + \ldots),$$

we have

$$V_k = t^l\,(\eta_1 + \eta_2 t + \ldots),$$

where the ξ's are derived from the x's and the η's from the y's by conjugate imaginary substitutions. If we write

$$\Delta_k = t^l\,(\delta_1 + \delta_2 t + \delta_3 t^2 + \ldots),$$

we shall have $(\delta_1 + \delta_2 t + \delta_3 t^2 + \ldots)_0 = \delta_1 + \delta_2 t + \delta_3 t^2 + \ldots$,

i.e., $\delta_1, \delta_2, \delta_3, \ldots$ are all real.

So write

$$-\Delta_k \Delta_{k-1} = t^{2l+e}\,(a_1 + a_2 t + \ldots)^2,$$

and then $a_1, a_2, \ldots$ will be all real.* Thus

$$-\frac{U_k V_k}{\Delta_k \Delta_{k-1}} = \frac{1}{t^e}\left(\frac{\xi_1 + \xi_2 t + \ldots}{a_1 + a_2 t + \ldots}\right)\left(\frac{\eta_1 + \eta_2 t + \ldots}{a_1 + a_2 t + \ldots}\right).$$

If we write now

$$X_1 + X_2 t + \ldots = (\xi_1 + \xi_2 t + \ldots)/(a_1 + a_2 t + \ldots),$$

$$Y_1 + Y_2 t + \ldots = (\eta_1 + \eta_2 t + \ldots)/(a_1 + a_2 t + \ldots),$$

the substitutions for the Y's and X's are still conjugate imaginaries. Accordingly the reduced parts are

(A) $\quad X_1 Y_e + X_2 Y_{e-1} + \ldots + X_e Y_1,$

(B) $\quad c\,(X_1 Y_e + X_2 Y_{e-1} + \ldots + X_e Y_1) + X_1 Y_{e-1} + X_2 Y_{e-2} + \ldots + X_{e-1} Y_1.$

---

* They might be pure imaginaries, but we can avoid this by changing the signs of the resulting terms in A, B; this change is equivalent to removing the factor i from each a.

The case $c = 0$ offers no special feature; and the case of infinite roots is treated, as usual, by taking them as zero roots of $| \mu B - A | = 0$. We shall thus find reduced parts (for $c = \infty$)

$$(A) \quad X_1 Y_{e-1} + X_2 Y_{e-2} + \ldots + X_{e-1} Y_1,$$

$$(B) \quad X_1 Y_e + X_2 Y_{e-1} + \ldots + X_e Y_1.$$

It follows that the problem of reducing two Hermite's forms by conjugate imaginary substitutions has been completely solved, except in the singular case when $| \lambda A - B | \equiv 0$.

In the singular case, following out the method given above (§ 2, p. 326), it is easy to see that reduced forms of the types

$$(A) \quad X_1 Y_{2a+1} + X_2 Y_{2a} + \ldots + X_a Y_{a+2} + X_{a+2} Y_a + X_{a+3} Y_{a-1} + \ldots + X_{2a+1} Y_1,$$

$$(B) \quad X_1 Y_{2a} + \ldots + X_a Y_{a+1} + X_{a+1} Y_a + \ldots + X_{2a} Y_1$$

will appear, where a is a *Minimalgradzahl.*

Collecting all our results, we have the following theorem:—

If A, B are two Hermite's forms, and σ is the absolute value of the signature of B, then

$$\sigma \leqq \text{the number of odd } h\text{'s} + \text{the number of even } l\text{'s},$$

where h is the index of any invariant factor of $| \lambda A - B |$ corresponding to a real root (not zero) and l is the index of an invariant-factor to base λ. If $| A | = 0$, we are to include in the h's the indices of invariant-factors corresponding to infinite roots of $| \lambda A - B |$, such indices being determined as those corresponding to zero roots of $| A - \mu B | = 0$ (*cf. Proc. Lond. Math. Soc.*, pp. 158–163 above).

This result holds still if $| \lambda A - B | \equiv 0$. In this case, if the k-th minors of $| \lambda A - B |$ are the first which do not all vanish identically, the number of invariant-factors to base λ is $(d' - k)$; and the number corresponding to $\lambda = \infty$ is $(d - k)$, where d, d' are the defects of A, B respectively (i.e., $n - d$, $n - d'$ are the ranks of A, B).

This theorem is slightly extended from one given by Loewy (*Crelle*, Vol. CXXII., 1900, p. 69); the changes are due to the inclusion of the possibilities (i.) $| A | = 0$, (ii.) $| \lambda A - B | \equiv 0$. Loewy's proof depends on reducing the Hermite's forms to two real quadratics; but it is interesting to see the connexion with our former reductions.

7. *Simultaneous Reduction of a Hermite's Form and an associated Hermite's Form; with an Application.*

We first define an *associated Hermite's form* (*beigeordnete Hermite'sche Form*); B will be such a form if $B' = -B_0$. Bilinear forms of this type have been introduced by Herr A. Loewy in investigating automorphic substitutions for Hermite's forms.*

If A be a Hermite's form, and it is desired to reduce A, B by conjugate imaginary substitutions, we have only to observe that, if $B = iC$, we have $B_0 = -iC_0$, and so $C' = C_0$. Hence C is a Hermite's form, and the problem is solved at once by means of the investigation in the last section. We can, of course, give an independent investigation (which was the method I followed originally), but this is rather longer, and the results do not reduce to so neat a form without extra labour.

The problem of § 4 ought to be deducible as a case of the above; for a real symmetric form is a special case of a Hermite's form, and a real alternate form of an associated Hermite's form. If such a connexion could be pointed out, it would shorten the investigation of § 4 considerably.

The application is to find an alternative proof of the fundamental inequality of § 5 in Loewy's paper. Loewy shows (§ 4) that, if S be an automorphic substitution of a Hermite's form (A), so that

$$S_0' A S = A,$$

then S can be reduced to the form

$$S = e^{i\phi}(A+B)^{-1}(A-B),$$

where $B_0' = -B$ (or B is an associated Hermite's form), and ϕ is some real angle.

In order to examine the characteristic equation of S we consider the form $(rE-S)$; now

$$\begin{aligned} rE-S &= (A+B)^{-1}\left[r(A+B)-e^{i\phi}(A-B)\right] \\ &= (A+iC)^{-1}\left[r(A+iC)-e^{i\phi}(A-iC)\right] \qquad (B=iC) \\ &= i\,(r+e^{i\phi})(A+iC)^{-1}(C-sA), \end{aligned}$$

* *Nova Acta Leop.-Carol. Akad.*, Vol. LXXI., 1898, p. 379, § 4; and *Math. Annalen*, Vol. L., p. 557.

where $$s = i(r-e^{i\phi})/(r+e^{i\phi}).$$

Hence, to investigate the invariant-factors of $|rE-S|$, we have only to find those of $|sA-C|$, where C is another (arbitrary) Hermite's form; corresponding to an invariant-factor $(s-c)^e$ there will be one $(r-b)^e$, where b, c are connected by

$$c = i(b-e^{i\phi})/(b+e^{i\phi})$$

or $$b = e^{i\phi}(1-ic)/(1+ic).$$

It follows that, if b' corresponds to c_0, we have

$$b' = e^{i\phi}(1-ic_0)/(1+ic_0).$$

Hence, taking the conjugate imaginary,

$$b'_0 = e^{-i\phi}(1+ic)/(1-ic)$$

or $$bb'_0 = 1,$$

i.e., $$b'_0 = 1/b \quad \text{or} \quad b' = 1/b_0.$$

But, if c be real, $b' = b$, since $c_0 = c$, and thus $bb_0 = 1$ or mod $b = 1$.

We conclude that invariant-factors of the type $(r-b)^e$, $(r-1/b_0)^e$ occur in pairs, unless mod $b = 1$. Corresponding to the pair of invariant-factors $(s-c)^e$, $(s-c_0)^e$, we found a part of A of the type

$$x_1 y_{2e} + x_1 y_{2e-1} + \ldots + x_{2e} y_1.$$

Taking the pair of terms $x_1 y_{2e} + x_{2e} y_1$, we can write them in the form

$$\tfrac{1}{2}\left[(x_1+x_{2e})(y_1+y_{2e}) - (x_1-x_{2e})(y_1-y_{2e})\right],$$

and similarly for every other pair.

It follows that the *characteristic** of this part of A is e; and the *signature** is zero.

Next, if c be real, we have a part of A, corresponding to $(r-b)^e$, when mod $b = 1$,

$$x_1 y_e + x_2 y_{e-1} + \ldots + x_e y_1,$$

---

* Loewy (§ 5) defines the *characteristic* (q') of a Hermite's form G as the smaller of the two numbers q, $(n-q)$, when G is put in the form

$$G = \sum_{a=1}^{q} x_a y_a - \sum_{a=q+1}^{n} x_a y_a$$

(the variables being conjugate imaginaries). Frobenius defines the *signature* (σ) of G as $(2q-n)$; so that $2q' = n - \text{mod}\ \sigma$.

for which $q' = \frac{1}{2}e$ or $\frac{1}{2}(e-1)$ and mod $\sigma = 0$ or 1, according as e is even or odd. Now the characteristics of a sum of sets of terms is not less than the sum of the characteristics of the sets separately; while the absolute value of the signature is not greater than the sum of the absolute values of the signatures of the separate sets. *Hence, combining our results, we have that*

$$q \geqq s+\Sigma E\left(\tfrac{1}{2}h\right),$$

$$|\sigma| \leqq p,$$

where 2s is the sum of the indices of all the invariant-factors of $|rE-S|$ *which vanish for values of r whose absolute values are not unity;** *h represents the index of any invariant-factor which vanishes for a value of r whose absolute value is unity,* $E\left(\frac{1}{2}h\right)$ *is the greatest integer contained in* $\frac{1}{2}h$. *Further, p is the number of the odd h's.*

The above is Loewy's fundamental inequality (§ 5); it is, of course, assumed that (A), the determinant of A, does not vanish; or we should have to consider some further possibilities (as on p. 349). This proof was originally sketched out in a letter from me to Herr Loewy (April, 1900); but the proof of the reduced form of A which I gave there was insufficient. Herr Loewy is publishing a short note † in which he proves his inequality by combining my suggestion with the results of a recent paper of his own (*Crelle*, Vol. CXXII., 1900, p. 53).

The theorem in § 7 of Loewy's paper becomes almost intuitive by this method of investigation. This theorem is to the effect that, if we are given a set of invariant-factors satisfying the above inequality and a Hermite's form A, then we can find a substitution which (with its conjugate imaginary) is automorphic for A, and whose characteristic equation possesses the assigned invariant-factors.

---

* These invariant-factors occur in pairs with equal indices; so that the sum of their indices is an even number 2s.

† *Göttinger Nachrichten*, June 30th, 1900.