ON A GENERAL CONVERGENCE THEOREM, AND THE THEORY OF THE REPRESENTATION OF A FUNCTION BY SERIES OF NORMAL FUNCTIONS

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THE theory of integral equations, as recently developed by Hilbert and by Schmidt, has resulted in a certain unification of the theory of the series of normal functions which represent prescribed functions in an interval. By this means the validity of such representation has, however, in the first instance, been established only for the case of a function which, together with its first and second differential coefficients, is continuous in the whole interval of representation, and which satisfies at the ends of the interval the same conditions that are imposed upon the normal functions themselves. An extension of the theory to the case of functions of a less restricted type has recently been given by Kneser.* A method of development of the theory of series of normal functions, on foundations laid by Schwarz and Poincaré has been given in detailed investigations by Stekloff and others, but involves a restriction upon the type of the functions represented by the series, of a similar character to that in the theory of integral equations.

It seems desirable to obtain sufficient conditions for the convergence of the series at a particular point, and for the uniform convergence of the series in any interval contained in the whole interval of representation, comparable in generality with the known sufficient conditions applicable in the case of Fourier's series. In the present communication a fundamental convergence theorem is established, which, when applied to the case of series of Sturm-Liouville functions,† suffices to shew that the question whether the series corresponding to a given function converges, or not, at a particular point, depends only upon the nature of the function in

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† The convergence of the series of Sturm-Liouville functions has been investigated for the case of a function of limited total fluctuation by Kneser, *Math. Annalen*, Vols. LVIII., LX. The case of the series representing analytical functions has been treated by A. C. Dixon, by the method of residues, *Proceedings*, Ser. 2, Vol. 3.

an arbitrarily small neighbourhood of the point, whilst the nature of the function throughout the whole interval of representation is restricted only by the postulation that it shall possess a Lebesgue integral in that interval; the function being therefore not necessarily limited in the interval. The convergence theorem is further employed to shew that, subject to the same condition as regards the nature of the function, the question whether the series converges uniformly, or not, in an interval in which the function is continuous, and which is contained in the given interval of representation, depends only on the nature of the function in an interval which encloses the interval of continuity in its interior, exceeding it in length only by an arbitrarily small amount.

It is further shewn that a sufficient restriction upon the nature of the function in an arbitrarily small neighbourhood of a particular point, to ensure the convergence of the series, is that the function shall be of limited total fluctuation (*à variation bornée*) in that neighbourhood. It is shewn that a similar restriction is sufficient in the case of uniform convergence in an interval.

When the end-points of the interval of representation are singular points of the linear differential equation satisfied by the normal functions, as, for example, in the case of Legendre's or Bessel's functions, the fundamental theorem is not applicable to the whole interval of representation. In this case, neighbourhoods of the end-points must be excluded and the theorem applied to the remainder of the interval; the parts of the series depending on the excluded neighbourhoods of the end-points requiring separate consideration. As an instance of such series, the case of the series of Legendre's functions is treated in detail.

A few other applications of the fundamental convergence theorem are given, to the proof of the validity, under very general conditions, of known modes of representation of functions by means of definite integrals and by series.

A General Convergence Theorem.

1. The following convergence theorem will be first established :—

Let $f(x')$ be a function which has a Lebesgue integral in the interval (α, β) of the variable x', whether the function be limited or unlimited in that interval. Let $F(x', x, n)$ be a function defined for all values of x' in (α, β), and for values of x belonging to a certain set of points G contained in (α, β), and for positive values of n. Let $F(x', x, n)$ satisfy the following conditions :—(1) *that* $|F(x', x, n)|$ *does not exceed a definite* *positive number $\bar{F}$, for all values of x' and x such that $|x'-x|$ is not less than a fixed positive number μ ($< b-a$), where x' belongs to (α, β), and x to G, and for all values of n;* (2) *that $\int_{\alpha_1}^{\beta_1} F(x', x, n)\, dx'$ exists as a Lebesgue integral for all values of α_1, β_1 such that $\alpha \leqslant \alpha_1 < \beta_1 \leqslant \beta$, for each value of x belonging to G but not interior to the interval $(\alpha_1-\mu, \beta_1+\mu)$, and that it is less in absolute value, for each value of n, than a positive number A_n, independent of α_1, β_1 and x;* (3) *that*

$$\lim_{n=\infty} A_n = 0.$$

Then $\int_{\alpha}^{x-\mu} f(x')\, F(x', x, n)\, dx'$ converges, as n is indefinitely increased, uniformly to zero, for all values of x belonging to G, and in the interval $(\alpha+\mu, \beta)$; also $\int_{x+\mu}^{\beta} f(x')\, F(x', x, n)\, dx'$ converges uniformly to zero as n is indefinitely increased, for all values of x belonging to G and in the interval $(\alpha, \beta-\mu)$. The positive number n may be either a variable capable of having all positive values, or it may be restricted to have the values in a sequence with no upper limit, as, for example, the sequence of positive integers.

In particular, the set G may consist of the whole interval (α, β), in which case the integrals converge uniformly to zero as n is increased indefinitely for all values of x in the intervals $(\alpha+\mu, \beta)$, $(\alpha, \beta-\mu)$ respectively; or G may consist of the points of an interval $(\alpha+\lambda, \beta-\lambda)$ contained in (α, β), in which case the integrals converge uniformly in the intervals $(\alpha+\lambda, \beta-\lambda)$, if $\lambda \geqslant \mu$, or in the intervals $(\alpha+\mu, \beta)$, $(\alpha, \beta-\mu)$ respectively, if $\lambda \leqslant \mu$.

In proving this theorem, it will be sufficient to consider the first of the two integrals only. Let it be first assumed that $f(x')$ is limited in the given interval (α, β); and let U, L denote its upper and lower limits respectively in that interval.

We may divide the interval (L, U) into portions

$$(c_0, c_1),\ (c_1, c_2),\ \ldots,\ (c_{p-1}, c_p),$$

where $c_0 = L$, $c_p = U$, and such that $c_q - c_{q-1}$ is less than an arbitrarily chosen positive number η, for all the values $1, 2, 3, \ldots, p$, of q.

Let that set of points in (α, β) for which $c_q \leqslant f(x') < c_{q+1}$ be denoted by E_q; and for any fixed value of x ($\geqslant \mu$) and belonging to G, let e_q be that part of E_q which is in the interval $(\alpha, x-\mu)$.

Let a function $f_1(x')$ be defined by the following rule:—For those values of x' for which $c_q \leqslant f(x') < c_{q+1}$, let $f_1(x') = c_q$, for each value of q; and for $f(x') = c_p$, let $f_1(x') = c_p$.

We have now

$$\left| \int_a^{x-\mu} f(x') F(x', x, n)\, dx' - \int_a^{x-\mu} f_1(x') F(x', x, n) dx' \right| < \eta \bar{F} (\beta - a);$$

where x is any point of the set G in the interval $(a+\mu, \beta)$, and for all values of n. Also we have

$$\int_a^{x-\mu} f_1(x') F(x', x, n)\, dx' = \sum_{q=0}^{q=p} c_q \int_{(e_q)} F(x', x, n)\, dx'.$$

Let the set of points E_q be enclosed in the interiors of a set of non-overlapping intervals H_q, such that $m(H_q) - m(E_q) = \zeta$; where $m(H_q)$, $m(E_q)$ denote the measures of H_q and of E_q, and ζ is an arbitrarily chosen positive number sufficiently small. A finite, or infinite, set H_q of intervals can always be so chosen that this condition is satisfied. If h_q denote that part of H_q which is in the interval $(a, x-\mu)$, it can be seen that

$$m(h_q) - m(e_q) \leqslant \zeta.$$

For, if possible, let $$m(h_q) - m(e_q) = \zeta + \gamma,$$

where γ is a positive number. Let the set e_q be enclosed in the interiors of non-overlapping intervals of a set l_q, all in the interval $(a, x-\mu)$, such that $m(l_q) < m(e_q) + \gamma$; and let $\bar{H}_q$ denote that set of non-overlapping intervals which consists of the set l_q together with that part of H_q which is not in the interval $(a, x-\mu)$. Observing that $m(l_q) < m(h_q) - \zeta$, we have then

$$m(\bar{H}_q) = m(H_q) - m(h_q) + m(l_q) < m(H_q) - \zeta < m(E_q);$$

and this is impossible, since E_q cannot be enclosed in intervals of a set of smaller measure than $m(E_q)$. Since therefore no such positive number γ can exist, we have $m(h_q) - m(e_q) \leqslant \zeta$. We have now

$$\left| \int_{(e_q)} F(x', x, n)\, dx' - \int_{(h_q)} F(x', x, n) dx' \right| < \zeta \bar{F}.$$

Let the intervals of the set H_q have lengths $\gamma_1, \gamma_2, \gamma_3, \ldots$ in descending order of magnitude. We may choose r_q so that

$$m(H_q) - (\gamma_1 + \gamma_2 + \ldots + \gamma_{r_q}) < \zeta.$$

Of the intervals $\gamma_1, \gamma_2, \ldots, \gamma_{r_q}, \ldots$, let $\gamma_{s_1}, \gamma_{s_2}, \ldots, \gamma_{s_t}, \ldots$ fall wholly or partly in the interval $(a, x-\mu)$; one of these intervals may be only partially in $(a, x-\mu)$. Let s_t be the greatest of the numbers $s_1, s_2, \ldots, s_t, \ldots$ which does not exceed r_q; we have then

$$\gamma_{s_{t+1}} + \gamma_{s_{t+2}} + \ldots < \zeta, \quad \text{and} \quad m(h_q) - (\gamma_{s_1} + \gamma_{s_2} + \ldots + \gamma_{s_t}) < \zeta,$$

as may be seen by applying the same argument that has been employed above to shew that

$$m(h_q) - m(e_q) \leqslant \zeta.$$

Here, in case the point $x-\mu$ is interior to an interval γ_{s_r}, we take only that part of γ_{s_r} which is in the interval $(a, x-\mu)$.

We have now $m(h_q)-m(D_q) < \zeta$, where D_q is the finite set of intervals $\gamma_{s_1}, \gamma_{s_2}, \ldots, \gamma_{s_t}$; the number t of intervals of this set D_q does not exceed the number r_q, which is independent of x.

We have now

$$\left| \int_{(h_q)} F(x', x, n)\, dx' - \int_{(D_q)} F(x', x, n)\, dx' \right| < \zeta \bar{F}.$$

Also
$$\left| \int_{(D_q)} F(x', x, n)\, dx' \right| < tA_n < r_q A_n.$$

Combining the inequalities which have been shewn to hold, we find that

$$\left| \int_a^{x-\mu} f(x')\, F(x', x, n)\, dx' \right| < \eta \bar{F}(\beta - a) + \sum_{q=0}^{q=p} c_q\, (2\zeta \bar{F} + r_q A_n).$$

Now, let ϵ be an arbitrarily fixed positive number; we can then fix η so that $\eta \bar{F}(\beta - a) < \frac{1}{3}\epsilon$; then the numbers c_q for $q = 0, 1, 2, \ldots, p$ can be fixed. We can then choose ζ so that

$$2\zeta \bar{F} \sum_{q=0}^{q=p} c_q < \tfrac{1}{3}\epsilon.$$

The numbers r_q depend only on ζ and q, being independent of x, and thus $\sum_{q=0}^{q=p} r_q c_q$ is fixed; we can choose n_1 so that

$$A_n \sum_{q=0}^{q=p} r_q c_q < \tfrac{1}{3}\epsilon,$$

provided
$$n \geqslant n_1.$$

We now have
$$\left| \int_a^{x-\mu} f(x')\, F(x', x, n)\, dx' \right| < \epsilon, \quad \text{for } n \geqslant n_1,$$

and for all values of x belonging to G, and in the interval $(a+\mu, \beta)$. The uniform convergence of the integral to zero has accordingly been established.

Next, let $f(x')$ be no longer limited in (a, β). A positive number N can be so determined that

$$\int |f(x')|\, dx' < \tfrac{1}{2}\epsilon / \bar{F},$$

when the integral is taken over that set of points K_N in (a, β) for each of which $|f(x')| > N$. If k_N be that part of K_N which lies in $(a, x-\mu)$, for any fixed value of x in $(a+\mu, \beta)$, we have

$$\int_{k_N} |f(x')|\, dx' < \tfrac{1}{2}\epsilon/\bar{F}.$$

Let the function $f_2(x')$ be defined by the rule that $f_2(x') = f(x')$, when $|f(x')| \leqslant N$, and $f_2(x') = 0$, when $|f(x')| > N$. Thus $f_2(x')$ vanishes at all points of K_N, and it is a limited summable function. We have now

$$\int_a^{x-\mu} f(x')\, F(x', x, n)\, dx' = \int_{(k_N)} f(x')\, F(x', x, n)\, dx' + \int_a^{x-\mu} f_2(x')\, F(x', x, n)\, dx'.$$

A value n_1 of n can be so determined that

$$\left| \int_a^{x-\mu} f_2(x')\, F(x', x, n)\, dx' \right| < \tfrac{1}{2}\epsilon, \quad \text{for } n \geqslant n_1,$$

and for all values of x belonging to G, and in the interval $(a+\mu, \beta)$. Also

$$\left| \int_{k_N} f(x')\, F(x', x, n)\, dx' \right| < \tfrac{1}{2}\epsilon,$$

for all the values of x and of n. Therefore we have

$$\left| \int_a^{x-\mu} f(x')\, F(x', x, n)\, dx' \right| < \epsilon, \quad \text{for } n \geqslant n_1,$$

and for all the values of x. The theorem has now been completely established. A special case of this theorem, in a somewhat different form, was given in my paper* "On the Uniform Convergence of Fourier's Series," and was there applied to the theory of Dirichlet's integral.

2. It may be remarked that the above proof is applicable to establish the following somewhat more general theorem:—

If the functions $f(x')$, $F(x', x, n)$ satisfy the conditions before stated, $\int_{a_1}^{\beta_1} f(x')\, F(x', x, n)\, dx'$ *converges to zero, as n is indefinitely increased, uniformly for all values of a_1, β_1 and x, which are such that $a \leqslant a_1 < \beta_1 \leqslant \beta$, and such that x belongs to the set G, and is not interior to the interval $(a_1-\mu, \beta_1+\mu)$.*

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* *Proceedings*, Ser. 2, Vol. 5, p. 275.

Let us consider the special case in which $F(x', x, n)$ is independent of x, say $F(x', x, n) = \phi(x', n)$. We then obtain the following theorem:—

If $f(x')$ have a Lebesgue integral in the interval (α, β); and $\phi(x', n)$ be such that $|\phi(x', n)|$ have a definite upper limit for all values of x' in (α, β), and for all the values of n; and if further $\int_{\alpha_1}^{\beta_1} \phi(x', n)\, dx$ exist and be numerically less than A_n, for all values of α_1, β_1, such that $\alpha \leqslant \alpha_1 < \beta_1 \leqslant \beta$, where A_n is independent of α_1, β_1, and where $\lim_{n=\infty} A_n = 0$, then $\int_{\alpha_1}^{\beta_1} f(x')\, \phi(x', n)\, dx'$ converges to zero, as n is indefinitely increased, uniformly for all values of α_1 and β_1. The number n may be either capable of having all positive values, or may be restricted to have the values in a sequence, for example, in the sequence of positive integers.

A special case of this theorem is that, if $f(x')$ have a Lebesgue integral in the interval $(-\pi, \pi)$, then $\int_{\alpha}^{\beta} f(x') \cos nx'\, dx'$, $\int_{\alpha}^{\beta} f(x') \sin nx'\, dx'$ converge to zero as n is indefinitely increased, and uniformly for all values of α and β, such that $-\pi \leqslant \alpha < \beta \leqslant \pi$.

The theorems may be generalized so as to apply to the case of a function of any number of variables. As is clear from the theory of Lebesgue integration, the proof of the fundamental theorem is applicable, without any essential modification, to this more general case. It will be sufficient to state the main theorem for the case of a function of three variables, as follows:—

Let $f(x', y', z')$ be a limited, or unlimited, function defined for all points in the space V bounded by a closed surface S, and having a Lebesgue integral through V. Let $F(x', y', z', x, y, z, n)$ be a function defined for all values of (x', y', z') in V, and for all values of (x, y, z) corresponding to the points of a given set G contained in V; and for positive values of n.

Let $F(x', y', z', x, y, z, n)$ satisfy the following conditions:—(1) that $|F(x', y', z', x, y, z, n)|$ does not exceed a definite number $\bar{F}$, for all positions of the points $(x', y', z')(x, y, z)$, such that

$$(x'-x)^2+(y'-y)^2+(z'-z)^2 \geqslant \mu^2,$$

where μ is a fixed positive number, and (x', y', z') belong to V, and (x, y, z) to G; (2) $\int_{(V_1)} F(x', y', z', x, y, z, n)(dx'\, dy'\, dz')$ exists as a Lebesgue integral for every volume V_1 not exterior to V, and bounded by a surface S_1 not exterior to S, and for all values of (x, y, z) corresponding to points *of G such that a sphere with centre (x, y, z) and radius μ has no volume in common with V_1, and that the integral is in absolute value less than A_n, a number independent of V_1 and of x; (3) that $\lim_{n=\infty} A_n = 0$. Then $\int_{(V_1)} f(x', y', z') F(x', y', z', x, y, z, n)(dx' dy' dz')$ converges to zero, as n is indefinitely increased, uniformly for all points (x, y, z) belonging to G and of which the minimum distance from points of V_1 is not less than μ. The convergence is also uniform for all such volumes V_1, under the conditions stated.*

In particular, if the integral be taken through the whole volume V with the exception of a sphere of centre (x, y, z) and radius μ (or of the portion of such sphere which is in V), then the convergence is uniform for all points (x, y, z) belonging to G.

It is clear that the statement might be made more general by replacing the volumes V, V_1 by any bounded and measurable sets of points. If H denote the measurable set of points for which the function $f(x', y', z')$ is defined, and in which it has a Lebesgue integral; the set G for which $F(x', y', z', x, y, z, n)$ is defined and satisfies the conditions of the theorem, being contained in H, then the integral of

$$f(x', y', z') F(x', y', z', x, y, z, n),$$

taken through a measurable set H_1 contained in H, converges to zero as n is indefinitely increased, subject to the conditions of the theorem, uniformly for all points (x, y, z) belonging to G, and of which the minimum distance from the points of H_1 is $\geqslant \mu$. The convergence is uniform for all such sets H_1. The original statement of the theorems will be, however, sufficient for the purpose of the applications to be made below.

3. The theorems of §§ 1, 2 may be extended to cases in which the given domain of the function is unbounded, provided an additional convergency condition is satisfied. It will be sufficient to give the extension of that case of the theorem in which the set G consists of all the points of the interval (α, β).

Let us assume that $f(x')$ has a Lebesgue integral in every finite interval contained in the unlimited interval $(-\infty, \infty)$. Let it be assumed also that $| F(x', x, n) | < \bar{F}$, for all values of x', x such that $| x'-x | \geqslant \mu$, and for all values of n. Further, let it be assumed that, if K be any arbitrarily chosen positive number, then, if $\beta - \alpha \leqslant K$,

$$\left| \int_\alpha^\beta F(x', x, n)\, dx' \right| < A_n,$$

where A_n depends only on n and K; and for all values of x not interior to the interval $(\alpha-\mu, \beta+\mu)$. Let it also be assumed that for each value of K, $\lim_{n=\infty} A_n = 0$.

Let x be confined to an arbitrarily chosen interval (α_1, β_1). If, then, corresponding to each arbitrarily chosen positive number ϵ, a number $\xi \leqslant \alpha_1-\mu$ can be determined, such that

$$\left| \int_{\xi'}^{\xi} f(x') F(x', x, n)\, dx \right| < \epsilon,$$

for all values of $\xi' < \xi$, and for all values of n; and, if further, a number $\eta \geqslant \beta_1+\mu$ can be so determined that

$$\left| \int_{\eta}^{\eta'} f(x') F(x', x, n)\, dx' \right| < \epsilon,$$

for all values of $\eta' > \eta$, the numbers ξ, η being independent of n and of x, then the integrals

$$\int_{-\infty}^{\xi} f(x') F(x', x, n)\, dx', \quad \int_{\eta}^{\infty} f(x') F(x', x, n)\, dx'$$

exist as
$$\lim_{\xi'=-\infty} \int_{\xi'}^{\xi} f(x') F(x', x, n)\, dx',$$

and
$$\lim_{\eta'=\infty} \int_{\eta}^{\eta'} f(x') F(x', x, n)\, dx',$$

respectively, and neither of them numerically exceeds ϵ. We suppose these conditions to be satisfied for every interval (α_1, β_1) of x.

We have then

$$\int_{-\infty}^{x-\mu} f(x') F(x', x, n)\, dx' = \int_{-\infty}^{\xi} f(x') F(x', x, n)\, dx' + \int_{\xi}^{x-\mu} f(x') F(x', x, n)\, dx'.$$

The variable x being confined to the interval (α_1, β_1), ξ can be so chosen that, for all such values of x, and for all values of n, the first integral on the right-hand side is numerically not greater than ϵ. Moreover, the second integral is, for all sufficiently large values of n, and for all values of x in (α_1, β_1), numerically less than ϵ, in accordance with the theorem of § 1. It has therefore been shewn that

$$\int_{-\infty}^{x-\mu} f(x') F(x', x, n)\, dx'$$

converges uniformly to zero for all values of x in the interval (α_1, β_1). The integral $\int_{x+\mu}^{\infty} f(x') F(x', x, n)\, dx'$ can be similarly shewn to converge uniformly to zero, for all values of x in (α_1, β_1). In particular, if

$$\int_{-\infty}^{0} |f(x')|\, dx', \quad \int_{0}^{\infty} |f(x')|\, dx'$$

both exist as $\displaystyle \lim_{h=\infty} \int_{-h}^{0} |f(x')|\, dx', \quad \lim_{h=\infty} \int_{0}^{h} |f(x')|\, dx',$

the additional convergency conditions are satisfied. For

$$\left| \int_{\xi'}^{\xi} f(x')\, F(x', x, n)\, dx' \right| \leqslant \bar{F} \int_{\xi'}^{\xi} |f(x')|\, dx',$$

and $$\left| \int_{\eta}^{\eta'} f(x')\, F(x', x, n)\, dx' \right| \leqslant \bar{F} \int_{\eta}^{\eta'} |f(x')|\, dx';$$

hence ξ, η can be so chosen that for all values of $\xi' < \xi$, and for all values of $\eta' > \eta$, the expressions on the right-hand sides of these inequalities are each $< \epsilon$. The following theorem has thus been established :—

Let $f(x')$ possess a Lebesgue integral in every finite interval, and let $|F(x, x', n)|$ have a finite upper limit for all values of x and x' such that $|x - x'| \geqslant \mu$, and for all the values of n. Further, let it be assumed that the integral of $F(x, x', n)$ in any interval (a, β) whatever, such that $\beta - a \leqslant K$, when K is an arbitrarily chosen positive number, is numerically less than a number A_n dependent only on n and K, for all values of x not interior to the interval $(a-\mu, \beta+\mu)$, and that $\lim_{n=\infty} A_n = 0$, for each value of K. Then, if $\int_{-\infty}^{\infty} |f(x')|\, dx'$ have a definite value as the double limit

$$\lim_{h=\infty,\, k=\infty} \int_{-k}^{h} |f(x')|\, dx'$$

of the Lebesgue integral, the integrals

$$\int_{-\infty}^{x-\mu} f(x')\, F(x', x, n)\, dx', \quad \int_{x+\mu}^{\infty} f(x')\, F(x', x, n)\, dx',$$

both converge to zero as n is indefinitely increased, uniformly for all values of x in any finite interval.

It is clear that a similar theorem may be stated for the case of a function of three variables, or of any number of variables.

It should be observed that for special forms of the function $F(x', x, n)$, the condition that $\int_{-\infty}^{\infty} |f(x')|\, dx'$ exists may be replaced by a less stringent condition depending on the nature of the function $F(x', x, n)$.

THE CONVERGENCE OF A DEFINITE INTEGRAL.

4. Let $F(x', x, n)$ be defined for all values of x' in (α, β), and for all values of x in a set G, which may, in particular, consist of all points in (α, β), or of all points in an interval $(\alpha+\lambda, \beta-\lambda)$, and for positive values of n. It is also assumed that $f(x')$, $F(x', x, n)$ satisfy the conditions of the theorem of § 1, for every sufficiently small positive value of μ.

We propose to consider the limiting value of the integral

$$\int_\alpha^\beta f(x')\, F(x', x, n)\, dx'$$

as n is indefinitely increased, the integral being assumed to exist for all points x in G. The integral is equivalent to the sum

$$\int_\alpha^{x-\mu} f(x')\, F(x', x, n)\, dx' + \int_{x+\mu}^{\beta} f(x')\, F(x', x, n)\, dx' + \int_{x-\mu}^{x+\mu} f(x')\, F(x', x, n)\, dx',$$

where the first integral is omitted if $x \leqslant \alpha+\mu$, and the second is omitted if $x+\mu \geqslant \beta$. When $x < \alpha+\mu$, the lower limit in the third integral is replaced by α; and when $x > \beta-\mu$, the upper limit in the third integral is replaced by β. For any fixed value of μ, the first and second integrals converge uniformly to zero for all values of x in G, as n is indefinitely increased.

We therefore consider the integral

$$\int_{x-\mu}^{x+\mu} f(x')\, F(x', x, n)\, dx',$$

which is equivalent to

$$\int_0^\mu f(x+t)\, F(x+t, x, n)\, dt + \int_0^\mu f(x-t)\, F(x-t, x, n)\, dt.$$

Let us assume that, at a particular point x, the two limits $f(x+0)$, $f(x-0)$ have definite finite values; then

$$\int_0^\mu f(x+t)\, F(x+t, x, n)\, dt$$
$$= f(x+0) \int_0^\mu F(x+t, x, n)\, dt + \int_0^\mu \{f(x+t) - f(x+0)\}\, F(x+t, x, n)\, dt,$$

and $\int_0^\mu f(x-t)\, F(x-t, x, n)\, dt$

$$= f(x-0) \int_0^\mu F(x-t, x, n)\, dt + \int_0^\mu \{f(x-t) - f(x-0)\}\, F(x-t, x, n)\, dt.$$

If, for a fixed x, belonging to G, the positive number μ can be chosen so small that

$$\int_0^{\mu} \{f(x-t)-f(x+0)\}\, F(x+t, x, n)\, dt,$$

and

$$\int_0^{\mu} \{f(x-t)-f(x-0)\}\, F(x-t, x, n)\, dt,$$

are, for all values of n, each numerically less than an arbitrarily chosen positive number ϵ, and if the two limits

$$\lim_{n=\infty} \int_0^{\mu} F(x+t, x, n)\, dt, \qquad \lim_{n=\infty} \int_0^{\mu} F(x-t, x, n)\, dt$$

have definite values P, Q independent of μ, we see that

$$\lim_{n=\infty} \int_a^{\beta} f(x')\, F(x', x, n)\, dx' = Pf(x+0)+Qf(x-0),$$

x being a fixed point in the interior of the interval (a, β).

It has therefore been shewn that *it is sufficient for the convergence of* $\int_a^{\beta} f(x')\, F(x', x, n)\, dx'$ *for a fixed value of* x, *in the interior of the interval* (a, β), *to the value* $Pf(x+0)+Qf(x-0)$, *that*

$$\lim_{n=\infty} \int_0^{\mu} F(x+t, x, n)\, dt \quad \textit{and} \quad \lim_{n=\infty} \int_0^{\mu} F(x-t, x, n)\, dt$$

should have the values P, Q *independent of* μ, *and that*

$$\int_0^{\mu} \{f(x+t)-f(x+0)\}\, F(x+t, x, n)\, dt,$$

$$\int_0^{\mu} \{f(x-t)-f(x-0)\}\, F(x-t, x, n)\, dt$$

should both be numerically less than an arbitrarily chosen positive number ϵ, *for a sufficiently small value of* μ, *and for all values of* n. *It is assumed that the conditions of the fundamental convergence theorem are satisfied for a set* G *to which* x *belongs.*

Let us next assume that the function $f(x)$ is such that, for a particular point x in G, a neighbourhood can be found such that the function $f(x')$ is of limited total fluctuation (*à variation bornée*) in that neighbourhood.

We may then replace the function $f(x')$ by $f_1(x')-f_2(x')$, where $f_1(x')$, $f_2(x')$ are monotone in the neighbourhood of the point x.

We have then, by applying the second mean value theorem, for a sufficiently small value of μ,

$$\int_0^\mu \{f_1(x+t)-f_1(x+0)\} F(x+t, x, n)\, dt$$
$$= \{f_1(x+\mu)-f_1(x+0)\} \int_{\mu_1}^\mu F(x+t, x, n)\, dt,$$

where $0 \leqslant \mu_1 < \mu$. A similar equation holds for the function $f_2(x)$.

Let us now assume that the function F is such that $\int_{\mu'}^{\mu} F(x+t, x, n)\, dt$ is numerically less than a fixed positive number, for all values of μ' such that $0 \leqslant \mu' < \mu$, and for all values of n. The number μ may be so chosen that $f_1(x+\mu)-f_1(x+0)$, $f_2(x+\mu)-f_2(x+0)$ are each less than an arbitrarily chosen positive number. It follows that μ can be so chosen that

$$\int_0^\mu \{f(x+t)-f(x+0)\} F(x+t, x, n)\, dt$$

is, for all values of n, less than an arbitrarily chosen positive number. It is clear that μ may be so chosen that, subject to a similar condition, a similar property belongs to

$$\int_0^\mu \{f(x-t)-f(x-0)\} F(x-t, x, n)\, dt.$$

It has therefore been shewn that, *for a point x in G, the conditions contained in the last theorem that*

$$\int_0^\mu \{f(x\pm t)-f(x+0)\} F(x\pm t, x, n)\, dt,$$

should both be numerically less than an arbitrarily chosen positive number ϵ, for a sufficiently small value of μ, and for all values of n, are satisfied if a neighbourhood of x exists so small that in that neighbourhood $f(x')$ is of limited total fluctuation, provided also μ can be so chosen that the integrals $\int_{\mu_1}^\mu F(x\pm t, x, n)\, dt$ are both numerically less than some fixed positive number for all values of μ_1 such that $0 \leqslant \mu_1 < \mu$, and for all values of n. When the other conditions of the theorem are also satisfied, the integral converges to the value $Pf(x+0)+Qf(x-0)$.

If x coincides with the end-point a of the interval (a, β), that point being assumed to belong to G, $\int_a^\beta f(x') F(x', a, n)\, dx'$ converges to the value $f(a+0) \lim_{n=\infty} \int_0^\mu F(a+t, a, n)\, dx'$, provided this expression have a definite meaning, and provided also that μ can be so chosen that

$$\int_0^{\mu} \{f(a+t)-f(a+0)\}\, F(x', a, n)\, dx'$$

is numerically less than an arbitrarily chosen positive number for all values of n. This last condition is satisfied, in particular, if a neighbourhood on the right of the point a exists in which $f(x)$ has limited total fluctuation, and if also μ can be so chosen that $\int_{\mu_1}^{\mu} F(a+t, a, n)\, dt$ is numerically less than some fixed positive number, for all values of μ_1 in the interval $(0, \mu)$, and for all values of n. A similar statement may be made for the case $x = \beta$.

5. Having found sufficient conditions for the convergence of the integral $\int_a^{\beta} f(x')\, F(x', x, n)\, dx'$ at any point x of G, at which $f(x)$ has definite functional limits, we proceed to find sufficient conditions that the convergence of the integral to its limit may be uniform in an interval (a_1, β_1) contained in the interior of (a, β), and in which the function $f(x')$ is continuous. It will be assumed that all points of (a_1, β_1) belong to G.

It is sufficient for such uniform convergence that the two integrals

$$\int_0^{\mu} \{f(x\pm t)-f(x)\}\, F(x\pm t, x, n)\, dt$$

should converge to zero, as μ is indefinitely diminished, uniformly for all values of x in (a_1, β_1), it being assumed also that $\lim_{n=\infty} \int_{-\mu}^{\mu} F(x+t, x, n)\, dt$ exists at each point of (a_1, β_1), independently of the value of μ, and that the convergence to the limit is uniform in (a_1, β_1).

This clearly follows from the discussion in § 4.

If we assume that (a_1, β_1) is contained in the interior of an interval (a_2, β_2) in which $f(x)$ has limited total fluctuation, the function as before being supposed continuous in (a_1, β_1), we see, from the proof of the second theorem in § 4, that *the convergence will be uniform in (a_1, β_1), provided μ can be chosen so small that the integrals $\int_{\mu_1}^{\mu} F(x\pm t, x, n)\, dt$ are both numerically less than some fixed positive number, for all values of μ_1 such that $0 \leqslant \mu_1 < \mu$, and for all values of x in (a_1, β_1), and for all values of n; it being assumed that $\lim_{n=\infty} \int_0^{\mu} F(x\pm t, x, n)\, dt$ exists at each point of (a_1, β_1), and so that the convergence to the limit is uniform in that interval.*

Since (a_1, β_1) is contained in the interior of an interval (a_2, β_2) in which the total fluctuation of the function $f(x)$ is limited, μ can be so chosen that the interval $(x-\mu, x+\mu)$ is, for each value of x in (α_1, β_1), an interval in which the function has limited total fluctuation.

It has thus been shewn that the uniform convergence of the integral in an interval (α_1, β_1) contained in the interior of (α, β) depends only on the nature of the function $f(x)$ in an interval $(\alpha_1-\epsilon, \beta_1+\epsilon)$ containing (α_1, β_1), where ϵ is arbitrarily small, and not on its nature in the remainder of the interval (α, β); subject, of course, to the condition that the function has a Lebesgue integral in the whole interval (α, β). In particular, the convergence of the integral at a particular point x, depends only on the nature of the function in an arbitrarily small neighbourhood of x. These results are known for the particular case of the convergence of Fourier's series. The result in the case of convergence at a point is due to Riemann.

6. In case the function $F(x', x, n)$ is never negative, the criteria for the convergence of the integral $\int_\alpha^\beta f(x')F(x',x,n)\,dx'$ admit of simplification.

At any point x at which $f(x+0)$, $f(x-0)$ exist and are finite, μ can be chosen so small that $|f(x+t)-f(x+0)|$, $|f(x-t)-f(x-0)|$ are both less than an arbitrarily chosen positive number η, for $0 < t \leqslant \mu$. It follows that, for a properly chosen value of μ,

$$\left| \int_0^\mu \{f(x\pm t)-f(x\pm 0)\}\, F(x\pm t, x, n)\,dt \right| < \eta \int_0^\mu F(x\pm t, x, n)\,dt,$$

and the expression on the right-hand side is arbitrarily small if $\int_0^\mu F(x\pm t, x, n)\,dt$ is less than some fixed finite number for all values of n. We thus obtain the following theorem :—

When $F(x', x, n)$ is never negative, it is sufficient for the convergence of $\int_\alpha^\beta f(x')\,F(x', x, n)\,dx'$, for a fixed value of x in the interior of (α, β), to the value $Pf(x+0)+Qf(x-0)$, that

$$\int_0^\mu F(x+t, x, n)\,dt \quad \textit{and} \quad \int_0^\mu F(x-t, x, n)\,dt$$

should be less than fixed positive numbers for all values of n, and for a sufficiently small value of μ, and that they should have definite limits P, Q independent of μ, when n is indefinitely increased.

If $f(x)$ is continuous in the interval (α_1, β_1), it follows, from the well known property of uniform continuity, that a value of μ can be determined such that $|f(x\pm t)-f(x\pm 0)| < \eta$, for $0 < t \leqslant \mu$, and for all values of x in (α_1, β_1). If, then,

$$\int_0^\mu F(x+t,\, x,\, n)\, dt \quad \text{and} \quad \int_0^\mu F(x-t,\, x,\, n)\, dt$$

are both less than fixed positive numbers for all values of n, and for all values of x in (α_1, β_1), provided μ be sufficiently small, then the convergence of the integral is uniform in (α_1, β_1). We thus obtain the following theorem :—

When $F(x', x, n)$ is never negative, it is sufficient for the uniform convergence of $\int_\alpha^\beta f(x')\, F(x', x, n)\, dx'$, to $f(x)$ for all values of x in the interval (α_1, β_1) interior to (α, β), where $f(x)$ is continuous in (α_1, β_1), that

$$\int_{-\mu}^{\mu} F(x+t,\, x,\, n)\, dt$$

should be less than a fixed positive number for all values of n, and for all values of x in (α_1, β_1), and for a sufficiently small value of μ ; and also that it have a definite limit, independent of μ, for all values of x in (α_1, β_1) when n is indefinitely increased, the convergence to the limit being uniform in (α_1, β_1). It is assumed that G contains (α_1, β_1).

APPLICATIONS OF THE THEORY.

7. As a first application of the preceding theory, let

$$F(x',\, x,\, n) = \frac{[1-(x'-x)^2]^n}{\int_0^1 (1-t^2)^n\, dt},$$

where $\qquad 0 \leqslant x \leqslant 1, \quad \text{and} \quad 0 \leqslant x' \leqslant 1\,;$

n denoting a positive integer. We take G to consist of the interval (0, 1).

If $|x'-x| \geqslant \mu$, we have

$$|F(x',\, x,\, n)| \leqslant \frac{(1-\mu^2)^n}{\int_0^1 (1-t^2)^n\, dt} < \frac{\int_0^\mu (1-t^2)^n\, dt}{\mu \int_0^1 (1-t^2)^n\, dt} < \frac{1}{\mu}.$$

Also
$$\int_{\alpha_1}^{\beta_1} F(x',\, x,\, n)\, dx' = \frac{\int_{\alpha_1}^{\beta_1} [1-(x'-x)^2]^n\, dx'}{\int_0^1 (1-t^2)^n\, dt};$$

and provided x does not lie in the interior of the interval $(\alpha_1-\mu,\ \beta_1+\mu)$, this is less than $\dfrac{(\beta_1-\alpha_1)(1-\mu^2)^n}{\int_0^1 (1-t^2)^n\,dt}$ or than $\dfrac{(1-\mu^2)^n}{\frac{1}{\sqrt{n}}\left(1-\frac{1}{n}\right)^n}$, since

$$\int_0^1 (1-t^2)^n\,dt > \int_0^{1/\sqrt{n}} (1-t^2)^n\,dt > \frac{1}{\sqrt{n}}\left(1-\frac{1}{n}\right)^n.$$

We may thus take $\qquad A_n = (1-\mu^2)^n n^{\frac{1}{2}} \left(1-\dfrac{1}{n}\right)^{-n},$

and then $\qquad \lim\limits_{n=\infty} A_n = e \lim\limits_{n=\infty} \dfrac{n^{\frac{1}{2}}}{(1+\lambda)^n},$

where $\qquad 1+\lambda = \dfrac{1}{1-\mu^2}.$

Hence $\lim\limits_{n=\infty} A_n = 0$; and therefore the conditions of the theorem of § 1 are satisfied for each positive value of μ.

Applying the criteria of § 6, we have

$$\int_0^{\mu} F(x\pm t,\ x,\ n)\,dt = \frac{\int_0^{\mu} (1-t^2)^n\,dt}{\int_0^1 (1-t^2)^n\,dt} < 1;$$

and the integral may be expressed in the form

$$1-\frac{\int_{\mu}^1 (1-t^2)^n\,dt}{\int_0^1 (1-t^2)^n\,dt},$$

and it has been shewn above that the limit of this is 1, when n is indefinitely increased, as may be seen by putting $\alpha_1=\mu$, $\beta_1=1$, $x'-x=\mu$. Therefore

$$\lim_{n=\infty} \int_0^{\mu} F(x\pm t,\ x,\ n)\,dt = 1.$$

We now see that $\qquad \lim\limits_{n=\infty} \dfrac{\int_0^1 [1-(x'-x)^2]^n f(x')\,dx'}{2\int_0^1 (1-t^2)^n\,dt}$

converges to the limit $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at any point x in the interior of (0, 1), at which $f(x+0)$, $f(x-0)$ exist, the function $f(x)$ being restricted only by the postulation that it has a Lebesgue integral in the interval

(0, 1). At the points 0, 1 the integral converges to the values $\frac{1}{2}f(1+0)$, $\frac{1}{2}f(1-0)$, provided these limits exist.

Moreover, we see from the second theorem of § 6 that the convergence to the limit $f(x)$ is uniform in any interval (a, b) in the interior of (0, 1), provided $f(x)$ be continuous in (a, b).

This last result has been established by Landau* for the case in which $f(x)$ is continuous in the whole interval (0, 1). He has applied it to prove Weierstrass' fundamental theorem, that if $f(x)$ be continuous in (a, b), a rational integral function $G(x)$ can be determined such that $|f(x)-G(x)|$ is less than a prescribed positive number, for all values of x in the interval (a, b). The proof of this is immediate; for we have only to choose a value of n sufficiently large, to make the rational integral function of x,

$$\frac{\int_0^1 [1-(x'-x)^2]^n f(x')\,dx'}{2\int_0^1 (1-t^2)^n\,dt}$$

differ from $f(x)$ by less than a prescribed positive number, for all points x such that $a \leqslant x \leqslant b$.

This method of proving Weierstrass' theorem may be extended to the case of a function of any number of variables. It will be sufficient to consider the case of three variables.

$$\text{Let } F(x', y', z', x, y, z, n) = \frac{\{1-(x'-x)^2-(y'-y)^2-(z'-z)^2\}^n}{8\pi\int_0^1 (1-t^2)^n\,dt};$$

and let the function $f(x', y', z')$ have a Lebesgue integral in the sphere

$$x'^2+y'^2+z'^2 = 1.$$

As before $$F(x', y', z', x, y, z, n) < \frac{1}{8\pi\mu},$$

provided $$(x'-x)^2+(y'-y)^2+(z'-z)^2 \geqslant \mu^2.$$

Also $\int F(x', y', z', x, y, z, n)\,(dx'\,dy'\,dz')$ taken through any volume in the given sphere which has no part in common with the sphere of radius μ,

* See his paper "Ueber die Approximation einer stetigen Funktion durch eine ganze rationale Funktion," *Rendiconti del circ. mat. di Palermo*, Vol. xxv., p. 337.

and centre at (x, y, z), is less than $\dfrac{(1-\mu^2)^n}{6\int_0^1 (1-t^2)^n\,dt}$, or than

$$\frac{\sqrt{n}}{6}(1-\mu^2)^n\left(1-\frac{1}{n}\right)^{-n},$$

which converges to zero as n is indefinitely increased. Therefore the conditions of the theorem in § 2 are satisfied.

To shew that

$$\frac{\int f(x', y', z')\,[1-(x'-x)^2-(y'-y)^2-(z'-z)^2]^n\,(dx'\,dy'\,dz')}{8\pi\int_0^1 (1-t^2)^n\,dt},$$

where the integral in the numerator is taken through the volume

$$x'^2+y'^2+z'^2 = 1,$$

converges to $f(x, y, z)$ uniformly in any volume contained in the interior of the sphere, provided the function is continuous through that volume, we have only to consider the above integral taken through the sphere

$$(x'-x)^2+(y'-y)^2+(z'-z)^2 = \mu^2.$$

If $\quad x' = x+t\sin\theta\cos\phi, \quad y' = y+t\sin\theta\sin\phi, \quad z' = z+t\cos\theta,$

the integral reduces to

$$\frac{\int_0^\mu \phi(x, y, z, t)(1-t^2)^n\,dt}{2\int_0^1 (1-t^2)^n\,dt},$$

where $\phi(x, y, z, t)$ is continuous with respect to (x, y, z) and to t. As before, this integral converges to $\phi(x, y, z, 0)$ or $f(x, y, z)$ uniformly in the given volume through which the function is continuous. Weierstrass' theorem is deduced immediately, as in the case of a function of one variable.

8. Let us consider the limit

$$\lim_{k=0}\frac{1}{k\sqrt{\pi}}\int_{-\infty}^{\infty} f(x')\,e^{-(x'-x)^2/k^2}\,dx',$$

employed by Weierstrass himself, to prove his fundamental theorem.

We assume that $f(x)$ is a function which has a Lebesgue integral in every finite interval, and is such that $\int_{-\infty}^{\infty} |f(x')|\,dx'$ exists as the double limit for $\alpha = -\infty$, $\beta = \infty$, of the Lebesgue integral $\int_{\alpha}^{\beta} |f(x')|\,dx'$.

Writing $k = 1/n$, $F(x', x, n) = \dfrac{n}{\sqrt{\pi}}\, e^{-n^2(x'-x)^2}$,

we see that, if $|x'-x| \geqslant \mu$, then

$$F(x', x, n) \leqslant \frac{n}{\sqrt{\pi}}\, e^{-n^2\mu^2} \leqslant \frac{1}{\mu\pi^{\frac{1}{2}}2^{\frac{1}{2}}}\, e^{-\frac{1}{2}}.$$

Also $$\int_{a_1}^{b_1} \frac{n}{\sqrt{\pi}}\, e^{-n^2(x'-x)^2}\,dx'$$

in any interval (a_1, b_1) such that x is not interior to the interval $(a_1-\mu, b_1+\mu)$, is less than $\dfrac{n}{\sqrt{\pi}}(b_1-a_1)\,e^{-n^2\mu^2}$ or than $\dfrac{n}{\sqrt{\pi}}\,e^{-n^2\mu^2}K$, where $b_1-a_1 \leqslant K$; and this converges to zero as n is indefinitely increased. It has thus been shewn that the conditions of the theorem in § 3 are satisfied.

Again,

$$\int_{-\mu}^{\mu} F(x+t, x, n)\,dt = \int_{-\mu}^{\mu} \frac{n}{\sqrt{\pi}}\, e^{-n^2t^2}\,dt < \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\,dt < 1\,;$$

and $$\lim_{n=\infty} \int_0^{\pm\mu} \frac{n}{\sqrt{\pi}}\, e^{-n^2t^2}\,dt = \lim_{n=\infty} \frac{1}{\sqrt{\pi}} \int_0^{\pm n\mu} e^{-t^2}\,dt = \tfrac{1}{2}.$$

Therefore the conditions of the theorems of § 6 are satisfied.

It has therefore been shewn that, *if $f(x)$ have a Lebesgue integral in every finite interval, and if $\int_{-\infty}^{\infty} |f(x')|\,dx'$ exists, then*

$$\frac{1}{k\sqrt{\pi}} \int_{-\infty}^{\infty} f(x')\, e^{-(x'-x)^2/k^2}\,dx'$$

converges, for $k = 0$, to the value $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at any point x at which $f(x+0)$, $f(x-0)$ exist. Moreover, the convergence to the value $f(x)$ is uniform in any finite interval in which $f(x)$ is continuous.

It is easy to extend the theorem to the case of the limit

$$\lim_{k=0} \left(\frac{1}{k\sqrt{\pi}}\right)^p \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots f(x'_1, x'_2, \ldots, x'_p)\, e^{-[(x'_1-x_1)^2+(x'_2-x_2)^2+\ldots+(x'_p-x_p^2)]/k^2}\, dx'_1\,dx'_2 \ldots dx'_p\,;$$

and from this Weierstrass' theorem for continuous functions of p variables can be immediately deduced.

[*Added May 26th*, 1908.]

The condition contained in the statement of the above result, that $\int_{-\infty}^{\infty} |f(x')|\,dx'$ should exist, may be replaced by a much less stringent condition.* Referring to § 3, we see that it is only necessary that when x is confined to an interval (α_1, β_1), ξ and η can be so determined that

$$\int_{\xi'}^{\xi} f(x')\frac{n}{\sqrt{\pi}}e^{-n^2(x'-x)^2}\,dx' \quad \text{and} \quad \int_{\eta}^{\eta'} f(x')\frac{n}{\sqrt{\pi}}e^{-n^2(x'-x)^2}\,dx'$$

should both be numerically $< \epsilon$, for all values of $\xi' < \xi$, and $\eta' > \eta$, and for all values of n. We have now

$$\left|\int_{\eta}^{\eta'} f(x')\frac{n}{\sqrt{\pi}}e^{-n^2(x'-x)^2}\,dx'\right| < \int_{\eta}^{\eta'} |f(x')|\frac{n}{\sqrt{\pi}}e^{-n^2(x'-\beta_1)^2}\,dx'$$

$$< \frac{1}{\sqrt{\pi}}\int_{n\eta}^{n\eta'}\left|f\left(\frac{z'}{n}\right)\right|e^{-(z'-n\beta_1)^2}\,dz'.$$

Now, let it be assumed that, for all values of x greater than some fixed value, the condition $|f(x)| < x^p e^{qx}$ is satisfied, where p and q are fixed positive numbers. Let η be so chosen that $f(z'/n)$ satisfies this condition for $z'/n > \eta$; we have then

$$\left|\int_{\eta}^{\eta'} f(x')\frac{n}{\sqrt{\pi}}e^{-n^2(x'-x)^2}\,dx'\right| < \frac{1}{\sqrt{\pi}}\int_{n\eta}^{n\eta'}\left(\frac{z'}{n}\right)^p e^{q(z'/n)}e^{-(z'-n\beta_1)^2}\,dz'$$

$$< \frac{1}{\sqrt{\pi}}\int_{n(\eta-\beta_1)}^{n(\eta'-\beta_1)}\left(\beta_1+\frac{u}{n}\right)^p e^{q\beta_1+qu/n}e^{-u^2}\,du$$

$$< \frac{1}{\sqrt{\pi}}\int_{\eta-\beta_1}^{\infty}(\beta_1+u)^p e^{q\beta_1+qu}e^{-u^2}\,du.$$

Since the integral $\int_{\eta-\beta_1}^{\infty}(\beta_1+u)^p e^{qu-u^2}\,du$ exists, as is well known, η can be so chosen that

$$\frac{1}{\sqrt{\pi}}\int_{\eta-\beta_1}^{\infty}(\beta_1+u)^p e^{q\beta_1+qu-u^2}\,du < \epsilon.$$

Similarly, it can be shewn that if, for sufficiently large negative values of x, the condition $|f(x)| < |x|^p e^{q|x|}$ is satisfied, then

$$\int_{\xi'}^{\xi} f(x')\frac{n}{\sqrt{\pi}}e^{-n^2(x'-x)^2}\,dx'$$

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* That this is the case was suggested to me by Mr. Bromwich.

is numerically $<\epsilon$, for all values of $\xi' < \xi$, and of n, if ξ be properly chosen. We have thus established the following more general theorem :—

If $f(x)$ have a Lebesgue integral in every finite interval, and be such that for $|x| > a$, the condition $|f(x)| < |x|^p e^{q|x|}$ is satisfied, when a, p, q are fixed positive numbers, then $\frac{1}{k\sqrt{\pi}}\int_{-\infty}^{\infty} f(x')\, e^{-(x'-x)^2/k^2}\, dx'$ converges for $k=0$ to the value $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at any point x at which $f(x+0)$, $f(x-0)$ exist. Moreover, the convergence to the value $f(x)$ is uniform in any finite interval in which $f(x)$ is continuous.

9. Let $s_n(x)$ denote the sum

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\, dx' + \frac{1}{\pi}\sum_{n=1}^{r=n-1}\int_{-\pi}^{\pi} f(x')\cos r(x'-x)\, dx'$$

of the first $2n+1$ terms of Fourier's series. Denoting by $S_n(x)$ the arithmetic mean $(s_1+s_2+\ldots+s_n)/n$, formed in accordance with Césaro's method, it is easily found that

$$S_n(x) = \frac{1}{2n\pi}\int_{-\pi}^{\pi} f(x')\left\{\frac{\sin\frac{1}{2}n(x'-x)}{\sin\frac{1}{2}(x'-x)}\right\}^2 dx'.$$

To evaluate $\lim_{n=\infty} S_n(x)$, let

$$F(x', x, n) = \frac{1}{2n\pi}\left\{\frac{\sin\frac{1}{2}n(x'-x)}{\sin\frac{1}{2}(x'-x)}\right\}^2.$$

As $|x'-x|$ approaches the extreme value 2π, $F(x', x, n)$ approaches the value $n/2\pi$. Consequently, the conditions of the theorem of § 1 would not be satisfied if G were taken to include the whole interval $(-\pi, \pi)$. The conditions are, however, satisfied if we take the set G of values of x to consist of the points of the interval $(-\pi+\lambda,\ \pi-\lambda)$, where λ is a fixed positive number as small as we please. In that case $F(x', x, n)$ is less than the greater of the numbers $\frac{1}{2n\pi}\operatorname{cosec}^2\frac{1}{2}\mu$, and $\frac{1}{2n\pi}\operatorname{cosec}^2\frac{1}{2}\lambda$, which is $\frac{1}{2n\pi}\operatorname{cosec}^2\frac{1}{2}\mu$, if we choose μ to be $<\lambda$. The number $\bar{F}$ is then $\frac{1}{2\pi}\operatorname{cosec}^2\frac{1}{2}\mu$.

Also

$$\int_{a_1}^{\beta_1} F(x', x, n)\, dx' < \frac{1}{2n\pi}\operatorname{cosec}^2\tfrac{1}{2}\mu\int_{a_1}^{\beta_1}\sin^2\tfrac{1}{2}n(x'-x)\, dx < \frac{1}{n}\operatorname{cosec}^2\tfrac{1}{2}\mu\,;$$

and the limit of this value is zero, when n is indefinitely increased.

Therefore the conditions of the theorem of § 1 are satisfied for every value of μ, such that $0 < \mu \leqslant \lambda$, when G consists of the interval $(-\pi+\lambda, \pi-\lambda)$.

We have also

$$\int_0^\mu F(x\pm t, x, n)\, dt = \frac{1}{2n\pi}\int_0^\mu \left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^2 dt < \frac{1}{2n\pi}\int_0^\pi \left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^2 dt < \tfrac{1}{2}.$$

Also $$\lim_{n=\infty}\int_0^\mu F(x\pm t, x, n)\, dt = \lim_{n=\infty}\frac{1}{2n\pi}\int_0^\pi \left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^2 dt = \tfrac{1}{2};$$

and therefore the conditions of the theorems in § 6 are satisfied.

It has therefore been shewn that, *if $f(x')$ have a Lebesgue integral in the interval $(-\pi, \pi)$, the function $S_n(x)$, formed in accordance with Cesàro's method of arithmetic means, converges to the value*

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}$$

at any interior point x at which the functional limits exist. Moreover, the convergence of $S_n(x)$ to the value $f(x)$ is uniform in any interval (a, b) interior to $(-\pi, \pi)$ in which the function is continuous.*

To find $\lim\limits_{n=\infty} S_n(-\pi)$, we have

$$S_n(-\pi) = \frac{1}{2n\pi}\left\{\int_{-\pi}^{-\pi+\mu} + \int_{\pi-\mu}^{\pi} + \int_{-\pi+\mu}^{\pi-\mu}\right\} f(x')\left\{\frac{\sin \frac{1}{2}n(x'+\pi)}{\sin \frac{1}{2}(x'+\pi)}\right\}^2 dx'$$

$$= \frac{1}{2n\pi}\int_0^\mu f(-\pi+t)\left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^2 dt + \frac{1}{2n\pi}\int_0^\mu f(\pi-t)\left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^2 dt$$

$$+\frac{1}{2n\pi}\int_{-\pi+\mu}^{\pi-\mu} f(x')\left\{\frac{\sin \frac{1}{2}n(x'+\pi)}{\sin \frac{1}{2}(x'+\pi)}\right\}^2 dt.$$

The limit of the third of these integrals has been shewn to be zero. If

* I take the opportunity of correcting an error which occurs in this connection in my treatise "On the Theory of Functions of a Real Variable, and on Fourier's Series." It is erroneously stated, on p. 712, that the convergence of $S_n(x)$ to the value $\lim\limits_{h=0}\{f(x+h)+f(x-h)\}$ is uniform in any interval in which $f(x)$ is limited, and in which the limit everywhere exists. The source of the error is at the top of p. 711, where the incorrect statement is made, that in any interval (a, b) in which $f(x)$ is limited, and in which $\lim\limits_{h=0}\{f(x+h)+f(x-h)\}$ has everywhere definite values, n may be so chosen that the upper limit of $|F(z)|$ in the interval $\left(0, \frac{\pi}{2n+1}\right)$ is less than ϵ. A sequence of continuous functions $\{S_n(x)\}$ which converges uniformly in any interval must, as is well known, have a continuous limit.

the two limits $f(-\pi+0)$, $f(\pi-0)$ both exist and are finite, we see that

$$\lim S_n(-\pi) = \tfrac{1}{2}\{f(-\pi+0)+f(\pi-0)\}.$$

Clearly $S_n(\pi)$ has the same limit.

10. The preceding theory may also be employed to establish the validity of Fourier's representation of a function in an unlimited interval by means of a single integral, under very general conditions.

Let us consider the integral

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f(x')\,\frac{\sin u(x'-x)}{x'-x}\,dx',$$

where u is here written instead of n. It will be assumed that $f(x')$ has a Lebesgue integral in every finite interval. It is unnecessary to assume that $\int_{-\infty}^{\infty} |f(x')|\,dx'$ exists as the double limit of the Lebesgue integral $\int_k^h |f(x')|\,dx'$. It will, in fact, be sufficient to make the more general assumption that $\int_a^{\infty} \left|\frac{f(x')}{x'}\right| dx'$, $\int_{-\infty}^{-a} \left|\frac{f(x')}{x'}\right| dx'$, where a is a positive number, both exist as the limits of Lebesgue integrals. Assuming that x is confined to a finite interval (a_1, β_1), we have

$$\left|\frac{1}{\pi}\int_{\eta}^{\eta'} \frac{f(x')}{x'-x}\sin u\,(x'-x)\,dx'\right| < \frac{1}{\pi}\int_{\eta}^{\eta'}\left|\frac{f(x')}{x'-x}\right| dx'$$

$$< \frac{1}{\pi}\,(1+\zeta)\int_{\eta}^{\eta'}\left|\frac{f(x')}{x'}\right| dx',$$

where η is chosen so great that $\frac{x'}{x'-\beta_1} < 1+\zeta$ for $x' \geqslant \eta$, and ζ denotes an arbitrarily chosen positive number.

It now follows that η may be so chosen that, for all values of u, and provided x is confined to the interval (a_1, β_1), $\frac{1}{\pi}\int_{\eta}^{\eta'} f(x')\,\frac{\sin u\,(x'-x)}{x'-x}\,dx'$ is numerically less than ϵ, for all values of $\eta' > \eta$. It may similarly be shewn that ξ may be so chosen that $\int_{\xi'}^{\xi} f(x')\,\frac{\sin u\,(x'-x)}{x'-x}\,dx'$ is numerically less than ϵ, for all values of $\xi' < \xi$.

We have now to shew that the conditions of the theorem in § 4 are satisfied. Writing

$$F(x', x, u) = \frac{1}{\pi}\,\frac{\sin u(x'-x)}{x'-x},$$

we have $$|F(x', x, u)| \leqslant \frac{1}{\mu\pi}, \quad \text{for} \quad |x'-x| \geqslant \mu.$$

Also we have

$$\int_a^\beta F(x', x, u)\, dx' = \frac{1}{\pi(a-x)} \int_a^\gamma \sin u\,(x'-x)\,dx' + \frac{1}{\pi(\beta-x)} \int_\gamma^\beta \sin u\,(x'-x)\,dx',$$

provided x is exterior to the interval (a, β), where $a \leqslant \gamma \leqslant \beta$. It follows that

$$\left| \int_a^\beta F(x', x, u)\, dx' \right| < \frac{4}{\mu\pi u},$$

if x is not interior to the interval $(a-\mu, \beta+\mu)$; and $4/\mu\pi u$ converges to zero as u is indefinitely increased. It now follows that

$$\frac{1}{\pi} \int_{-\infty}^{x-\mu} f(x') \frac{\sin u\,(x'-x)}{x'-x}\, dx' \quad \text{and} \quad \frac{1}{\pi} \int_{x+\mu}^{\infty} f(x') \frac{\sin u\,(x'-x)}{x'-x}\, dx'$$

converge to zero as u is indefinitely increased, uniformly for all values of x in any finite interval.

We have now to consider the convergence of

$$\frac{1}{\pi} \int_{x-\mu}^{x+\mu} f(x') \frac{\sin u\,(x'-x)}{x'-x}\, dx'.$$

Let x be confined to an interval (a_1, β_1), and let the criteria provided in §§ 4, 5 be applied. We have then

$$\frac{1}{\pi} \int_0^\mu \frac{\sin ut}{t}\, dt = \frac{1}{\pi} \int_0^{\mu u} \frac{\sin t}{t}\, dt;$$

and this has the limit $\frac{1}{2}$, when u is indefinitely increased.

Also

$$\frac{1}{\pi} \int_{\mu_1}^{\mu} \frac{\sin ut}{t}\, dt = \frac{1}{\pi} \int_{\mu_1 u}^{\mu u} \frac{\sin t}{t}\, dt = \frac{1}{\pi} \int_0^{\mu u} \frac{\sin t}{t}\, dt - \frac{1}{\pi} \int_0^{\mu_1 u} \frac{\sin t}{t}\, dt;$$

and both the last integrals are well known to be numerically less than fixed numbers independent of μ, μ_1 and u.

The following theorem has now been established :—

If $f(x')$ have a Lebesgue integral in every finite interval, and if $\int_a^\infty \left|\frac{f(x')}{x'}\right| dx'$, $\int_{-\infty}^{-a} \left|\frac{f(x')}{x'}\right| dx'$, where a is positive, exist as the limits of Lebesgue integrals $\Big($*this condition being satisfied in particular if $\int_{-}^{\infty} |f(x')|\, dx'$ exists*$\Big)$, *then $\frac{1}{\pi} \int_{-\infty}^{\infty} f(x') \frac{\sin u\,(x'-x)}{x'-x}\, dx'$ converges for*

$u = \infty$, to the value $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at a point x for which a neighbourhood exists in which $f(x')$ is of limited total fluctuation. Moreover, the convergence to the value $f(x)$ is uniform in any finite interval in which $f(x)$ is continuous, and which is in the interior of an interval in which $f(x)$ has limited total fluctuation.

From § 4, we see that a sufficient condition of convergence of the integral at a point x is that

$$\int_0^{\mu} \frac{f(x+t)-f(x+0)}{t} \sin ut\,dt, \quad \text{and} \quad \int_0^{\mu} \frac{f(x-t)-f(x-0)}{t} \sin ut\,dt,$$

should converge to zero as u is indefinitely increased. This condition is certainly satisfied if $|f(x+t)-f(x+0)| \leqslant At^{1-a}$, for all sufficiently small values of t, when A, $1-a$ are fixed positive numbers, and if $|f(x-t)-f(x-0)|$ satisfies a similar condition. The conditions are satisfied at a point of continuity of $f(x)$ at which the four derivatives are limited, and generally provided any of the known sufficient criteria for the convergence of Fourier's series at a point are satisfied.

The preceding theory may also be applied to the case of Poisson's integral which occurs in the theory of Fourier's series.

SERIES OF STURM-LIOUVILLE NORMAL FUNCTIONS.

11. The differential equation

$$\frac{d}{dx}\left(k\frac{dV}{dx}\right)+(gr-l)\,V = 0 \tag{1}$$

occurs in the theory of the conduction of heat in a heterogeneous bar, and in connection with other problems of mathematical physics.

Those solutions of this equation for an interval (a, b) of the variable x which satisfy the boundary conditions

$$\frac{dV}{dx}-hV = 0, \text{ for } x = a, \quad \text{and} \quad \frac{dV}{dx}+HV = 0, \text{ for } x = b, \tag{2}$$

where h, H are positive constants, were studied by Liouville and by Sturm in a series of memoirs published in the first two volumes of *Liouville's Journal.*

Special cases of the boundary equations are obtained by letting one or both of the constants h, H have the value zero. Other special cases are obtained by supposing h or H, or both of them to be infinite, in which case the corresponding boundary condition is $V = 0$.

It is assumed that g, k, l are functions of x which are positive, and do not vanish in the interval (a, b); r is a parameter. It will be further assumed that g, k have everywhere finite differential coefficients, and that l and $(gk)^{-\frac{1}{4}}$ have limited total fluctuation in (a, b). If we transform the equation (1) by means of the substitutions

$$z = \int_a^x \left(\frac{g}{k}\right)^{\frac{1}{2}} dx, \quad \theta = (gk)^{-\frac{1}{4}}, \quad V = \theta U, \quad r = \rho^2,$$

the differential equation (1) becomes

$$\frac{d^2U}{dz^2} + (\rho^2 - l_1)\, U = 0, \tag{3}$$

where
$$l_1 = \frac{1}{\theta (gk)^{\frac{1}{2}}} \left\{ l \left(\frac{k}{g}\right)^{\frac{1}{2}} \theta - \frac{d(gk)^{\frac{1}{2}}}{dz} \frac{d\theta}{dz} - (gk)^{\frac{1}{2}} \frac{d^2\theta}{dz^2} \right\}.$$

The boundary equations (2) become

$$\frac{dU}{dz} - h'U = 0, \text{ for } z = 0, \quad \text{and} \quad \frac{dU}{dz} + H'U = 0, \text{ for } z = \pi, \tag{4}$$

where it is assumed that
$$\int_a^b \left(\frac{g}{k}\right)^{\frac{1}{2}} dx = \pi,$$

an equation which is always satisfied if a slight formal change in the variable x be made. The constants h', H' are real, but no longer necessarily positive. We shall suppose that neither h' nor H' is infinite.

Writing the equation (3) in the form

$$\frac{d^2U}{dz^2} + \rho^2 U = l_1 U,$$

we have, as an equation satisfied by a solution of this equation,

$$U = A \cos \rho z + B \sin \rho z + \frac{1}{\left(\frac{d}{dz}\right)^2 + \rho^2} l_1 U$$

$$= A \cos \rho z + B \sin \rho z + \frac{1}{2\iota\rho} \left\{ \frac{1}{\frac{d}{dz} - \iota\rho} - \frac{1}{\frac{d}{dz} + \iota\rho} \right\} l_1 U$$

$$= A \cos \rho z + B \sin \rho z + \frac{1}{\rho} \int_0^z l_1' U' \sin \rho (z - \zeta)\, d\zeta,$$

where l_1', U' are what l_1 and U become when ζ is substituted in them for z. If this value of U be substituted in the first of the boundary conditions (4), we find that $B\rho - Ah' = 0$, in order that the condition may be satisfied.

We have therefore, if we assume that $U = 1$, when $z = 0$,

$$U = \cos \rho z + \frac{h'}{\rho} \sin \rho z + \frac{1}{\rho} \int_0^z l_1' U' \sin \rho (z-\zeta) \, d\zeta,$$

which was obtained in a different manner by Liouville.*

Let $\bar{U}$ be the upper limit of $|U|$ in the interval $(0, \pi)$; then the absolute value of the expression on the right-hand side does not exceed

$$\left(1+\frac{h'^2}{\rho^2}\right)^{\frac{1}{2}} + \frac{\bar{U}}{\rho} \int_0^\pi |l_1'| \, d\zeta.$$

From the continuity of U, we have therefore

$$\bar{U} \leqslant \left(1+\frac{h'^2}{\rho^2}\right)^{\frac{1}{2}} + \frac{\bar{U}}{\rho} \int_0^\pi |l_1'| \, d\zeta.$$

If ρ is sufficiently large to be not less than $\int_0^\pi |l_1'| \, d\zeta$, we see that

$$\bar{U} \leqslant \left(1+\frac{h'^2}{\rho^2}\right)^{\frac{1}{2}} \left\{1 - \frac{1}{\rho} \int_0^\pi |l_1'| \, d\zeta\right\}^{-1}.$$

It follows that, for all positive values of ρ, greater than a fixed positive number, the values of $\bar{U}$, for all such values of ρ, do not exceed a fixed positive number.

If the value of U be substituted in the second of the boundary equations (4), we obtain an equation for the determination of the values of ρ. We find, on substitution, that ρ must satisfy the equation

$$\tan \pi\rho = \frac{P}{\rho - P'},$$

where
$$P = h' + H' + \int_0^\pi l_1' U' \left(\cos \rho\zeta - \frac{H'}{\rho} \sin \rho\zeta\right) d\zeta,$$

$$P' = \frac{H'h'}{\rho} + \int_0^\pi l_1' U' \left(\frac{H' \cos \rho\zeta}{\rho} + \sin \rho\zeta\right) d\zeta.$$

Since $|U'|$ cannot exceed some fixed number, independent of ζ and ρ, we see that $|P|$, $|P'|$ cannot exceed fixed numbers independent of ρ.

The roots of the equation for ρ have been discussed by Liouville, who shewed that they are of the form $n + a_n/n$, where $|a_n|$ is less than some number independent of n. It is necessary, however, for our purpose to obtain a somewhat more exact expression for these roots.

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* *Liouville's Journal*, Vol. II., p. 24.

12. In the following investigation of the forms of U, and of the roots of the equation

$$\tan \pi\rho = \frac{P}{\rho - P'}$$

the notation $a(z, n)$ will be used to denote a function of z and n which for all values of these variables does not exceed in absolute magnitude some fixed positive number independent of z and n. The notation will be used for a variety of such functions, in order to render unnecessary the introduction of a number of fresh symbols. In the same manner $a(n)$ will be used for a function of n which is such that $|a(n)|$ does not exceed some number independent of n.

The equation

$$U = \cos \rho z + \frac{h'}{\rho} \sin \rho z + \frac{1}{\rho} \int_0^z l_1' U' \sin \rho (z - \zeta)\, d\zeta$$

is of the form
$$U = \cos \rho z + \frac{a(\rho, z)}{\rho}.$$

Substituting the corresponding value of U' under the integral, we have

$$\begin{aligned} U = {} & \cos \rho z \left[1 - \frac{1}{\rho} \int_0^z l_1' \sin \rho\zeta \left\{ \cos \rho\zeta + \frac{a(\rho, \zeta)}{\rho} \right\} d\zeta \right] \\ & + \sin \rho z \left[\frac{h'}{\rho} + \frac{1}{\rho} \int_0^z l_1' \cos \rho\zeta \left\{ \cos \rho\zeta + \frac{a(\rho, \zeta)}{\rho} \right\} d\zeta \right]. \end{aligned}$$

Now $\int_0^z l_1' \sin \rho\zeta \cos \rho\zeta\, d\zeta$, or $\frac{1}{2} \int_0^z l_1' \sin 2\rho\zeta\, d\zeta$ is of the form $\frac{a(\rho, z)}{\rho}$; it being assumed that l_1' is of limited total fluctuation. Again, $\int_0^z l_1' \cos^2 \rho\zeta\, d\zeta$ is of the form $a(z) + \frac{a(\rho, z)}{\rho}$; and hence we have, as the form of U,

$$U = \cos \rho z \left\{ 1 + \frac{a(\rho, z)}{\rho^2} \right\} + \sin \rho z \left\{ \frac{a(z)}{\rho} + \frac{a(\rho, z)}{\rho^2} \right\};$$

where $|a(\rho, z)|$, $|a(z)|$ are less than fixed numbers independent of ρ and z, and of z respectively.

The numbers P, P' are of the forms $h' + H' + h'' + a/\rho$, a/ρ respectively, where the numbers a are in each case less in absolute magnitude than fixed numbers independent of ρ; and h'' denotes $\frac{1}{2} \int_0^\pi l_1'\, d\zeta$. Consequently, the equation for the determination of ρ is of the form

$$\tan \pi\rho = \frac{h'+H'+h''+\frac{a}{\rho}}{\rho-\frac{a}{\rho}};$$

therefore, for sufficiently large values of ρ,

$$\pi\rho = n\pi + \frac{h'+H'+h''}{\rho}+\frac{a}{\rho^2}, \quad \text{or} \quad \rho = n+\frac{h'+H'+h''}{n\pi}+\frac{a}{n^2}.$$

It then follows that, for all values of n, which represents one of the positive integers,

$$\rho_n = n+\frac{c}{n}+\frac{a}{n^2},$$

where c is the constant $\frac{h'+H'+h''}{\pi}$, and a denotes some number of which the absolute value is less than a fixed number independent of n. All the positive roots of the equation for the determination of ρ are given in this form; it is clear that the notation employed enables us to use what is primarily an asymptotic expression, available for large values of n, to represent all the roots.

We shall now employ the expression for ρ_n to express the function U_n, which corresponds to the value ρ_n of ρ, in terms of n and z. We have

$$\cos \rho_n z = \cos nz \left\{1+\frac{a(z, n)}{n^2}\right\} - \sin nz \left\{\frac{cz}{n}+\frac{a(z, n)}{n^2}\right\},$$

$$\sin \rho_n z = \sin nz \left\{1+\frac{a(z, n)}{n^2}\right\} + \cos nz \left\{\frac{cz}{n}+\frac{a(z, n)}{n^2}\right\}.$$

Substituting these values in the expression

$$\cos \rho_n z \left\{1+\frac{a(\rho, z)}{\rho_n^2}\right\} + \sin \rho_n z \left\{\frac{a(z)}{\rho_n}+\frac{a(\rho, z)}{\rho_n^2}\right\},$$

or in the equivalent expression

$$\cos \rho_n z \left\{1+\frac{a(\rho, z)}{n^2}\right\} + \sin \rho_n z \left\{\frac{a(z)}{n}+\frac{a(\rho, z)}{n^2}\right\},$$

we find that

$$U_n(z) = \cos nz \left\{1+\frac{a(z, n)}{n^2}\right\} + \sin nz \left\{\frac{a(z)}{n}+\frac{a(z, n)}{n^2}\right\}$$

which is the required expression for $U_n(z)$.

It is easily seen that $\int_0^\pi \{U_n(z)\}^2 dz$ is of the form $\frac{\pi}{2}+\frac{a(n)}{n^2}$; for

$$\int_0^\pi a(z)\sin 2nz\,dz = \frac{1}{n}\Big[-a(z)\cos 2nz\Big]_0^\pi + \frac{1}{n}\int_0^\pi a'(z)\cos 2nz\,dz,$$

and $a'(z)$ is a limited function, and thus $\int_0^\pi a(z)\sin 2nz\,dz$ is of the form $\frac{a(n)}{n}$.

We take now, as the normal function $\phi_n(z)$,

$$\phi_n(z) = U_n(z)\left[\int_0^\pi \{U_n(z)\}^2 dz\right]^{-\frac{1}{2}};$$

and thus $\int_0^\pi \{\phi_n(z)\}^2 dz = 1,\quad \int_0^\pi \phi_n(z)\,\phi_{n'}(z)\,dz = 0,\quad$ for $\quad n \neq n'$.

It follows, from the above forms for $U_n(z)$, $\phi_n(z)$, that

$$\phi_n(z) = \sqrt{\frac{2}{\pi}}\cos nz\left\{1+\frac{a(z,n)}{n^2}\right\} + \sin nz\left\{\frac{a(z)}{n}+\frac{a(z,n)}{n^2}\right\}.$$

It is necessary for our purposes to find a corresponding expression for $\phi_n'(z)$. We have

$$\frac{dU_n}{dz} = -\rho_n\sin\rho_n z + h'\cos\rho_n z + \int_0^z l_1' U'\cos\rho_n(z-\xi)\,d\xi$$

$$= -\left[n+\frac{a(z,n)}{n}\right]\left[\sin nz\left\{1+\frac{a(z,n)}{n^2}\right\} + \cos nz\left\{\frac{cz}{n}+\frac{a(z,n)}{n^2}\right\}\right]$$

$$+\left[h'+\frac{a(z,n)}{n}\right]\left[\cos nz\left\{1+\frac{a(z,n)}{n^2}\right\} - \sin nz\left\{\frac{cz}{n}+\frac{a(z,n)}{n^2}\right\}\right]$$

$$= -\left\{n+\frac{a(z,n)}{n}\right\}\sin nz + \left\{h'-cz+\frac{a(z,n)}{n}\right\}\cos nz.$$

On multiplying $\frac{dU_n}{dz}$ by $\left[\int_0^\pi \{U_n(z)\}^2 dz\right]^{-\frac{1}{2}}$, or $\sqrt{\frac{2}{\pi}}\left\{1+\frac{a(n)}{n^2}\right\}$, we find that

$$\frac{d\phi_n(z)}{dz} = -\sqrt{\frac{2}{\pi}}\left\{n+\frac{a(z,n)}{n}\right\}\sin nz + \sqrt{\frac{2}{\pi}}\left\{h'-cz+\frac{a(z,n)}{n}\right\}\cos nz,$$

the required form for $\phi_n'(z)$.

12. If we write $\lambda_n = \rho_n^2$, the positive numbers $\lambda_1, \lambda_2, \ldots$ are the characteristic values of ρ^2 (*Eigenwerthe*) for the equation (3), with the given boundary conditions, in accordance with the nomenclature of

the theory of integral equations. We may assume that the smallest characteristic number λ_1 is > 0; for if λ_1 were equal to zero, by changing the value of l_1 so that $\rho^2 - l_1$ remained unaltered, we should make λ_1 greater than zero.

Let $\mathfrak{f}_1(z)$, $\mathfrak{f}_2(z)$ be solutions of the equation

$$\frac{d^2u}{dz^2} - l_1 u = 0,$$

which, together with their first two differential coefficients, are continuous in the interval $(0, \pi)$; and such that $\mathfrak{f}_1'(z) - h'\mathfrak{f}_1(z) = 0$, for $z = 0$, and $\mathfrak{f}_2'(z) + H'\mathfrak{f}_2(z) = 0$, for $z = \pi$. The two functions satisfy the relation

$$\mathfrak{f}_1(z)\,\mathfrak{f}_2'(z) - \mathfrak{f}_2(z)\,\mathfrak{f}_1'(z) = -1,$$

if the arbitrary constant factors in $\mathfrak{f}_1(z)$, $\mathfrak{f}_2(z)$ are properly chosen. A function* $K(z, z')$ is defined for the whole interval, by the conditions

$$K(z, z') = \mathfrak{f}_2(z')\,\mathfrak{f}_1(z), \quad \text{for } z \leqslant z',$$

and
$$K(z, z') = \mathfrak{f}_1(z')\,\mathfrak{f}_2(z), \quad \text{for } z \geqslant z'.$$

This function is continuous in the interval $(0, \pi)$ of z, and symmetrical with respect to z and z'; it is the "nucleus" (Kern) for the system of normal functions. Writing

$$\frac{d}{dz} K(z, z') = K'(z, z'),$$

we have $K'(z'-0, z') = \mathfrak{f}_2(z')\,\mathfrak{f}_1'(z')$, and $K'(z'+0, z') = \mathfrak{f}_1(z')\,\mathfrak{f}_2'(z')$;

and therefore
$$K'(z'+0, z') - K'(z'-0, z') = -1.$$

The function $K'(z, z')$ is continuous for all values of z not equal to z'. The function $K(z, z')$ clearly satisfies the differential equation

$$\frac{d^2K(z, z')}{dz^2} - l_1 K(z, z') = 0,$$

for every value of z except $z = z'$.

It is known from the theory of integral equations that if the series $\sum\limits_{n=1}^{\infty} \dfrac{\phi_n(z)\,\phi_n(z')}{\lambda_n}$ is uniformly convergent in the interval $(0, \pi)$, then it represents the function $K(z, z')$.

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* See Kneser's "Integralgleichungen und Darstellung willkürlicher Functionen," *Math. Annalen*, Vol. LXIII., p. 483.

13. Let the sum $\phi_1(z)\,\phi_1(z')+\phi_2(z)\,\phi_2(z')+\ldots+\phi_n(z)\,\phi_n(z')$ be denoted by $F(z', z, n)$. In order to apply the theorem of § 1 to the case of this function, we suppose the set G to consist of all the points of the interval $(0, \pi)$ of the variable z.

We have first to verify that $|F(z', z, n)|$ is less than a fixed number, for all values of n, and for all the values of z, z' such that $|z-z'| \geqslant \mu$.

We have, employing the expression for $\phi_n(z)$ found in § 11,

$$F(z', z, n) = \sum_{r=1}^{r=n}\left[\sqrt{\frac{2}{\pi}}\cos rz\left\{1+\frac{a(z, r)}{r^2}\right\}+\sin rz\left\{\frac{a(z)}{r}+\frac{a(z, r)}{r^2}\right\}\right]$$

$$\left[\sqrt{\frac{2}{\pi}}\cos rz'\left\{1+\frac{a(z', r)}{r^2}\right\}+\sin rz'\left\{\frac{a(z')}{r}+\frac{a(z, r)}{r^2}\right\}\right].$$

We have to consider the sums of the various terms in this product. The series

$$\left|\frac{2}{\pi}\sum_{r=1}^{r=n}\cos rz\cos rz'\right| = \frac{1}{\pi}\left|\sum_{r=1}^{r=n}\{\cos r(z-z')+\cos r(z+z')\}\right|$$

$$=\frac{1}{2\pi}\left|\left\{\frac{\sin(2n+1)\frac{z-z'}{2}}{\sin\frac{z-z'}{2}}+\frac{\sin(2n+1)\frac{z+z'}{2}}{\sin\frac{z+z'}{2}}-2\right\}\right|$$

$$<\frac{1}{2\pi}(2\operatorname{cosec}\tfrac{1}{2}\mu-2),$$

provided $$|z-z'| \geqslant \mu.$$

The expression

$$\sqrt{\frac{2}{\pi}}\,a(z')\sum_{r=1}^{n}\frac{\cos rz\sin rz'}{r}\quad\text{or}\quad\frac{1}{\sqrt{2\pi}}\,a(z')\sum_{r=1}^{n}\frac{\sin r(z'+z)+\sin r(z'-z)}{r}$$

can be shewn to be numerically less than a fixed number, for all values of z, z' such that $|z-z'| \geqslant \mu$, and for all values of n. For it is known* that the sum $\sum_1^n \frac{\sin rx}{r}$ is given by

$$s_n(x) = s(x)+\int_0^{(n+\frac{1}{2})x}\frac{\sin z}{z}\,dz+\frac{\theta A}{n+\frac{1}{2}},$$

when $s(x)$ is the sum of the convergent series $\sum_1^\infty \frac{\sin rx}{r}$, A is a positive

* See *Theory of Functions of a Real Variable*, p. 649

number independent of x and n, and θ is such that $-1 < \theta < 1$; provided x is in an interval $(0, b)$, where $b < 2\pi$. It thus appears that $| s_n(x) |$ is less than some fixed positive number, provided x is in an interval $(-b, b)$, where $b < 2\pi$. The point $x = 0$ is a point of non-uniform convergence of the series $\sum\limits_1 \frac{\sin rx}{r}$, but the measure of non-uniform convergence is finite, the peaks of the approximation curves representing $s_n(x)$ being all of limited height. It follows that

$$\sum_{r=1}^{r=n} \frac{\sin r(z'+z)}{r}, \quad \sum_{r=1}^{r=n} \frac{\sin r(z'-z)}{r}, \quad \text{for} \quad | z-z' | \geqslant \mu,$$

are both less than fixed numbers independent of z, z', and n. Remembering that $| a(z') |$ is limited, the required result at once follows.

Similarly $\sqrt{\frac{2}{\pi}}\, a(z) \sum\limits_{r=1}^{n} \frac{\cos rz' \sin rz}{r}$ is numerically less than a fixed number independent of a, z and z', for $| z-z' | \geqslant \mu$.

The other terms such as

$$\sum_{r=1}^{r=n} a(z)\, a(z') \frac{\sin rz \sin rz'}{r^2}, \quad \sqrt{\frac{2}{\pi}} \sum_{r=1}^{r=n} \frac{\cos rz \sin rz'\, a(z, r)}{r^2}$$

are absolutely and uniformly convergent, as n is indefinitely increased. It therefore follows that $| F(z', z, n) |$ is less than a fixed number, for all values of n, and for all values of z, z' such that $|z-z'| \geqslant \mu$, and in the interval $(0, \pi)$.

We have next to consider

$$\sum_{r=1}^{n} \int_a^\beta \phi_r(z)\, \phi_r(z')\, dz', \quad \text{or} \quad \int_a^\beta F(z', z, n)\, dz'.$$

This may be written in the form

$$\sum_{r=1}^{n} \frac{\phi_r(z)}{\lambda_r} \int_a^\beta \left\{ l_1' \phi_r(z') - \frac{d^2\phi_r(z')}{dz'^2} \right\} dz',$$

by substituting for $\phi_r(z')$ its value as expressed by the differential equation given in § 12, which it satisfies. This is equivalent to

$$\int_a^\beta l_1' \sum_{r=1}^{r=n} \frac{\phi_r(z)\, \phi_r(z')}{\lambda_r}\, dz' - \sum_{r=1}^{r=n} \frac{\phi_r(z)\, \phi_r'(\beta)}{\lambda_r} + \sum_{r=1}^{r=n} \frac{\phi_r(z)\, \phi_r'(a)}{\lambda_r}.$$

Since $$\frac{1}{\lambda_r} = \frac{1}{\rho_r^2} = \frac{1}{r^2}\left\{ 1 + \frac{a(r)}{r^2} \right\},$$

we see that $\sum_{r=1}^{r=n} \frac{\phi_r(z)\,\phi_r(z')}{\lambda_r}$ takes the form

$$\sum_{r=1}^{n} \frac{1}{r^2} \left\{1+\frac{a(r)}{r^2}\right\} \left[\sqrt{\frac{2}{\pi}} \cos rz \left\{1+\frac{a(z,r)}{r^2}\right\} + \sin rz \left\{\frac{a(z)}{r}+\frac{a(z,r)}{r^2}\right\}\right]$$
$$\left[\sqrt{\frac{2}{\pi}} \cos rz' \left\{1+\frac{a(z',r)}{r^2}\right\} + \sin rz' \left\{\frac{a(z)}{r}+\frac{a(z,r)}{r^2}\right\}\right].$$

The various terms $\frac{2}{\pi} \sum_{r=1}^{r=n} \frac{\cos rz \cos rz'}{r^2}$, ... converge uniformly, as n is indefinitely increased; therefore the series $\sum_{r=1}^{n} \frac{\phi_r(z)\,\phi_r(z')}{\lambda_r}$ converges uniformly for all values of z and z' in $(0, \pi)$, and the limit of the sum is consequently $K(z, z')$.

The series $\sum_{r=1}^{r=n} \frac{\phi_r(z)\,\phi_r'(z')}{\lambda_r}$ is, on substituting the forms obtained in § 11, for $\phi_r(z)$, $\phi_r(z')$ and λ_r, of the form

$$\sum_{r=1}^{n} \frac{1}{r^2} \left\{1+\frac{a(r)}{r^2}\right\} \left[\sqrt{\frac{2}{\pi}} \cos rz \left\{1+\frac{a(z,r)}{r^2}\right\} + \sin rz \left\{\frac{a(z)}{r}+\frac{a(z,r)}{r^2}\right\}\right]$$
$$\left[-\sqrt{\frac{2}{\pi}} \sin rz' \left\{r+\frac{a(z,r)}{r}\right\} + \cos rz \left\{a(z)+\frac{a(z,r)}{r}\right\}\right].$$

The portion $-\left(\frac{2}{\pi}\right) \sum_{r=1}^{n} \frac{\cos rz \sin rz'}{r}$ converges uniformly for all values of z, z', such that $|z-z'| \geqslant \mu$, and the remainder of the series converges uniformly for all values of z, z'. Consequently, the series $\sum_{r=1}^{r=n} \frac{\phi_r(z)\,\phi_r'(\beta)}{\lambda_r}$ and the series $\sum_{r=1}^{r=n} \frac{\phi_r(z)\,\phi_r'(a)}{\lambda_r}$ converge uniformly for all values of z not interior to the interval $(a-\mu, \beta+\mu)$. By a known theorem, the limiting sums of these series are therefore $\frac{dK(z,\beta)}{d\beta}$, $\frac{dK(z,a)}{da}$ respectively.

It has now been shewn that, when x is not interior to the interval $(a-\mu, \beta+\mu)$, the sum

$$\sum_{r=1}^{n} \int_a^\beta \phi_r(z)\,\phi_r(z')\,dz' \quad \text{or} \quad \int_a^\beta F(z', z, n)\,dz'$$

converges uniformly to the value

$$\int_a^\beta l_1' K(z, z')\,dz' - \frac{dK(z,\beta)}{d\beta} + \frac{dK(z,a)}{da},$$

which is equal to $\int_a^\beta \frac{d^2K(z,z')}{dz'^2}\,dz' - \frac{dK(z,\beta)}{d\beta} + \frac{dK(z,a)}{da}$;

and is therefore zero. It thus appears that $\left|\int_a^\beta F(z', z, n)\,dz'\right|$ is less than some fixed number independent of a and β, for all values of z in $(0, \pi)$ which are not interior to the interval $(a-\mu, \beta+\mu)$. For, corresponding to an assigned ϵ, a value n_1 of n can be determined such that

$$\left|\int_a^\beta F(z', z, n)\,dz'\right| < (\beta-a)\,\epsilon+2\epsilon < (\pi+2)\,\epsilon, \quad \text{for} \quad n \geqslant n_1.$$

Therefore the conditions of the theorem in § 1 are satisfied for every value of μ such that $0<\mu<\pi$; the set G consisting of the whole interval $(0, \pi)$.

It follows that, for each value of μ,

$$\int_0^{z-\mu} f(x')\,F(x', x, n)\,dx' \quad \text{and} \quad \int_{z+\mu}^{\pi} f(x')\,F(x', x, n)\,dx'$$

converge to zero, as n is indefinitely increased, uniformly for all values of z in the interval $(0, \pi)$.

14. It will now be shewn that the function

$$F(z', z, n) \equiv \sum_{r=1}^{n} \phi_r(z)\,\phi_r(z')$$

satisfies the conditions for the validity of the theorems in §§ 4, 5.

We have, as in § 13,

$$\int_z^{z+\mu} \sum_1^n \phi_r(z)\,\phi_r(z')\,dz'$$
$$= \int_z^{z+\mu} l_1 \sum_{r=1}^{n} \frac{\phi_r(z)\,\phi_r(z')}{\lambda_r}dz' - \sum_{r=1}^{n} \frac{\phi_r(z)\,\phi_r'(z+\mu)}{\lambda_r} + \sum_{r=1}^{n} \frac{\phi_r(z)\,\phi_r'(z)}{\lambda_r}.$$

The series $\sum_{r=1}^{n} \frac{\phi_r(z)\,\phi_r'(z)}{\lambda_r}$ consists of parts which converge uniformly for all values of z and z', together with the part $\sum_{r=1}^{n}\left(-\frac{2}{\pi}\right)\frac{\cos rz \sin rz}{r}$, which is equivalent to $-\frac{1}{\pi}\sum_{r=1}^{n}\frac{\sin 2rz}{r}$, and this converges uniformly for all values of z in the interval $(\epsilon, \pi-\epsilon)$ of z, where ϵ is an arbitrarily small positive number. The function to which this sum converges in the interval $(\epsilon, \pi-\epsilon)$ is consequently $\frac{1}{2}\frac{dK(z, z)}{dz}$. Also

$$\lim_{n=\infty}\int_z^{z+\mu} l_1 \sum_{r=1}^{n} \frac{\phi_r(z)\,\phi_r(z')}{\lambda_r}dz' \quad \text{or} \quad \int_z^{z+\mu}\frac{d^2}{dz'^2}K(z, z')\,dz'$$

is equal to $$K'(z, z+\mu)-K'(z, z+0),$$

the convergence to this value being uniform, since $\sum_{r=1}^{n} \frac{\phi_r(z)\,\phi_r(z')}{\lambda_r}$ converges uniformly to $K(z, z')$.

We have therefore

$$\lim_{n=\infty} \int_z^{z+\mu} F(z', z, n)\, dz' = \tfrac{1}{2} \frac{dK(z, z)}{dz} - K'(z, z+0),$$

provided z is in the interval $(\epsilon, \pi-\epsilon)$, and the convergence is uniform in this interval. Referring to the notation of § 12, we have

$$\frac{d}{dz} K(z, z) = \mathfrak{f}_1'(z)\,\mathfrak{f}_2(z) + \mathfrak{f}_2'(z)\,\mathfrak{f}_1(z) = K'(z, z+0) + K'(z, z-0);$$

also it has been shewn that

$$K'(z, z+0) - K'(z, z-0) = -1.$$

Therefore $\int_z^{z+\mu} F(z', z, n)\, dz'$ converges to the limit $\frac{1}{2}$, as n is indefinitely increased, uniformly in the interval $(\epsilon, \pi-\epsilon)$.

In a precisely similar manner it may be shewn that $\int_{z-\mu}^{z} F(z', z, n)\, dz'$ converges to $\frac{1}{2}$, uniformly in the interval $(\epsilon, \pi-\epsilon)$.

At the point $z = 0$, we have

$$\lim_{n=\infty} \int_0^{\mu} F(z', 0, n)\, dz'$$
$$= \frac{d}{d\mu} K(0, \mu) - \left[\frac{d}{d\mu} K(0, \mu)\right]_{\mu=0} - \sum_{r=1}^{n} \frac{\phi_r(0)\,\phi_r'(\mu)}{\lambda_r} + \sum_{r=1}^{n} \frac{\phi_r(0)\,\phi_r'(0)}{\lambda_r}.$$

The series $\sum_{r=1}^{n} \frac{\phi_r(0)\,\phi_r'(z')}{\lambda_r}$ converges uniformly in the interval (μ_1, μ_2) of z', where $0 < \mu_1 < \mu < \mu_2$, since the series $\sum_{r=1}^{n} \frac{\sin rz'}{r}$ is uniformly convergent in that interval; therefore the series $\sum_{r=1}^{n} \frac{\phi_r(0)\,\phi_r'(\mu)}{\lambda_r}$ converges to the value $\frac{d}{d\mu} K(0, \mu)$. Again,

$$\sum_{r=1}^{n} \frac{\phi_r(0)\,\phi_r'(0)}{\lambda_r} = h' \sum_{r=1}^{n} \frac{\phi_r(0)\,\phi_r(0)}{\lambda_r} = h'K(0, 0).$$

Therefore
$$\begin{aligned} \lim_{n=\infty} \int_0^{\mu} F(z', 0, n)\, dz' &= -\,\mathfrak{f}_1(0)\,\mathfrak{f}_2'(0) + h'K(0, 0) \\ &= -\,\mathfrak{f}_1(0)\,\mathfrak{f}_2'(0) + h'\mathfrak{f}_1(0)\,\mathfrak{f}_2(0) \\ &= -\,\mathfrak{f}_1(0)\,\mathfrak{f}_2'(0) + \mathfrak{f}_1'(0)\,\mathfrak{f}_2(0) \\ &= 1. \end{aligned}$$

It may be shewn, in a similar manner, that

$$\lim_{n=\infty} \int_{\pi-\mu}^{\pi} F(z', \pi, n)\, dz' = 1.$$

We have next to shew that $\int_{\mu_1}^{\mu} F(z \pm t, z, n)\, dt$ are numerically less than a fixed positive number, for all values of μ_1, such that $0 \leqslant \mu_1 < \mu$, and for all values of n; z being any point in the interval $(0, \pi)$.

The value of the integral is

$$\sum_{r=1}^{n} \left[\sqrt{\frac{2}{\pi}} \cos rz + \sin rz \frac{a(z)}{r} + \frac{a(z, r)}{r^2}\right] \left[\sqrt{\frac{2}{\pi}}\; \frac{\sin rz'}{r} + \frac{a(z, r)}{r^2}\right]_{z'=z+\mu_1}^{z'=z+\mu},$$

for, as before, $\int a(z) \sin rz\, dz$ may be integrated by parts, and the result has the factor $1/r$. Of this, the part

$$\frac{2}{\pi} \Sigma \frac{\cos rz \sin r(z+\mu_1)}{r} \quad \text{or} \quad \frac{1}{\pi} \Sigma \frac{\sin r\mu_1 + \sin(2z+\mu_1)}{r}$$

is numerically less than a fixed positive number, for all values of z and all values of μ_1. The remainder consists of series which are uniformly convergent, and therefore the required result holds.

It has now been verified that the conditions of validity of the theorems of §§ 4, 5 are satisfied.

15. We are now in a position to state the following general theorem, which has been established by the foregoing investigation.

Let $f(z)$ be a function, limited or unlimited, which has a Lebesgue integral in the interval $(0, \pi)$. If the normal functions which satisfy the differential equation

$$\frac{d^2U}{dz^2} + (\rho^2 - l_1)\, U = 0,$$

where l_1 has limited total fluctuation in the interval $(0, \pi)$, and where ρ has values such that the boundary conditions

$$\frac{dU}{dz} - h'U = 0, \textit{ for } z = 0, \quad \textit{and} \quad \frac{dU}{dz} + H'U = 0, \textit{ for } z = \pi,$$

are satisfied, be denoted by $\phi_n(z)$, then the series

$$\sum_{r=1}^{\infty} \phi_r(z) \int_0^{\pi} \phi_r(z') f(z')\, dz'$$

converges to the value $\frac{1}{2}\{f(z+0)+f(z-0)\}$ *at any interior point* z *of the interval* $(0, \pi)$, *at which* $f(z+0), f(z-0)$ *exist and are finite, if a neighbourhood of the point* z *exists in which the function* $f(z)$ *is of limited total fluctuation. In any interval in which* $f(z)$ *is continuous, and which is contained in the interior of an interval in which the function has limited total fluctuation, the convergence of the series to the value* $f(z)$ *is uniform. At the points* $z=0$, $z=\pi$, *the series converges to the values* $f(0+0)$, $f(\pi-0)$, *if the function is of limited total fluctuation in neighbourhoods of these points.*

In order to pass back to the functions which satisfy the equation

$$\frac{d}{dx}\left(k\frac{dV}{dx}\right)+(gr-l)\,V=0, \tag{1}$$

with the boundary conditions

$$\frac{dV}{dx}-hV=0, \text{ for } x=a, \qquad \frac{dV}{dx}+HV=0, \text{ for } x=b, \tag{2}$$

we write
$$\phi_n(z)=(gk)^{\frac{1}{4}}\,V_n(x)\,;$$

then, since
$$dz=\left(\frac{g}{k}\right)^{\frac{1}{2}}dx,$$

we have
$$\int_a^b g\,V_n(x)\,V_{n'}(x)\,dx=0, \text{ for } n\neq n',$$

and
$$\int_a^b g\,\{V_n(x)\}^2\,dx=1.$$

Writing $\chi(x)$ for $f(z)$, the series becomes

$$\sum_{n=1}^{\infty}(gk)^{\frac{1}{4}}\,V_n(x)\int_a^b g'\,V_n(x')\,\frac{\chi(x')}{(g'k')^{\frac{1}{4}}}\,dx'.$$

If we now write $F(x)$ for $\chi(x)(gk)^{-\frac{1}{4}}$, and remember the assumption that g and k are such that $(gk)^{-\frac{1}{4}}$ has limited total fluctuation, and is continuous in (a, b), we obtain the following theorem :—

Let $F(x)$ *be a limited or unlimited function which has a Lebesgue integral in* (a, b). *Let* $V_n(x)$ *be the function which satisfies the equation* (1), *and is such that*

$$\int_a^b g\,\{V_n(x)\}^2\,dx=1,$$

and corresponds to the value r_n of r, found so that the boundary conditions (2) *are satisfied. Then, it being assumed that $(gk)^{-\frac{1}{4}}$ has limited total fluctuation in (a, b), the series*

$$\Sigma V_n(x) \int_a^b g' V_n(x') F(x') dx'$$

converges to the value $\frac{1}{2}\{F(x+0)+F(x-0)\}$ at any interior point of (a, b) at which the functional limits have definite finite values, and which is such that the function has limited total fluctuation in some neighbourhood of the point. In any interval in which $F(x)$ is continuous, and which is contained in the interior of another interval in which it has limited total fluctuation, the convergence of the series to the value $F(x)$ is uniform. The series converges to the values $F(a+0)$, $F(b-0)$ at the points $x = a$, $x = b$, if there exist neighbourhoods of these points in which the function has limited total fluctuation.

We have not considered the cases in which h or H is infinite, or in which both are infinite. The investigation in that case is of a precisely similar character, the details being slightly different on account of the somewhat different form of the functions $\phi_n(z)$.

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16. In the preceding investigation, the differential equation (1) has no singular points in the interval (a, b). As an example of a case in which there are singular points at the ends of the interval, the case of the series

$$\Sigma \frac{2n+1}{2} P_n(x) \int_{-1}^1 f(x') P_n(x') dx'$$

will be here considered. The normal functions $\sqrt{\frac{2n+1}{2}} P_n(x)$ satisfy Legendre's equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_n(x)}{dx}\right]+n(n+1)P_n(x) = 0.$$

Let
$$F(x', x, n) = \sum_{r=0}^{r=n} \frac{2r+1}{2} P_r(x) P_r(x');$$

then, by a well known formula of summation, we have the expression

$$F(x', x, n) = \frac{n+1}{2} \frac{P_{n+1}(x) P_n(x') - P_n(x) P_{n+1}(x')}{x-x'}.$$

We shall now verify that the conditions of the theorem of § 1 are satisfied for the interval $(-1+\epsilon, 1-\epsilon)$ of x, where ϵ is an arbitrarily chosen positive number. We cannot, in this case, apply the theorem to the interval $(-1, +1)$.

It is known that in the interval $(-1+\epsilon, 1-\epsilon)$, the value of $P_n(x)$ is given by

$$\left(\frac{2}{n\pi\sin\theta}\right)^{\frac{1}{2}}\left[\cos\left\{(n+\tfrac{1}{2})\theta-\frac{\pi}{4}\right\}+\frac{a(n,\theta)}{n}\right],$$

for every value of $n\ (>0)$; where $x=\cos\theta$, and $a(n,\theta)$ represents a function which is in absolute value less than some fixed number, for all values of $n\ (>0)$, and for all values of x in the interval $(-1+\epsilon, 1-\epsilon)$.

This value of $P_n(x)$ is clearly of the form $\dfrac{a(n, x)}{n^{\frac{1}{2}}}$.

If $|x-x'| \geqslant \mu$, we have

$$|F(x', x, n)| < \frac{n+1}{2\mu}\frac{1}{\sqrt{n(n+1)}}|a(n, x, x')| < \frac{1}{\mu}|a(n, x, x')|,$$

provided x, x' are in the interval $(-1+\epsilon, 1-\epsilon)$. Hence, in this interval, $|F(x', x, n)|$ is less than a fixed number, for all values of n, x', and x.

Again,

$$\int_{\alpha_1}^{\beta_1} F(x', x, n)\,dx' = \frac{n+1}{2}\left\{P_{n+1}(x)\int_{\alpha_1}^{\beta_1}\frac{P_n(x')}{x-x'}dx' - P_n(x)\int_{\alpha_1}^{\beta_1}\frac{P_{n+1}(x')}{x-x'}dx'\right\}$$

$$=\frac{n+1}{2}\left[P_{n+1}(x)\left\{\frac{1}{x-\alpha_1}\int_{\alpha_1}^{k_1}P_n(x')\,dx'+\frac{1}{x-\beta_1}\int_{k_1}^{\beta_1}P_n(x')\,dx'\right\}\right.$$

$$\left.-P_n(x)\left\{\frac{1}{x-\alpha_1}\int_{\alpha_1}^{k_2}P_{n+1}(x')\,dx'+\frac{1}{x-\beta_1}\int_{k_2}^{\beta_1}P_{n+1}(x')\,dx'\right\}\right],$$

where k_1 and k_2 are numbers such that $\alpha_1 \leqslant k_1 \leqslant \beta_1$, $\alpha_1 \leqslant k_2 \leqslant \beta_2$; and x is not interior to the interval $(\alpha_1-\mu, \beta_1+\mu)$.

If we employ the known formula

$$(2n+1)P_n(x') = \frac{dP_{n+1}(x')}{dx'} - \frac{dP_{n-1}(x')}{dx'},$$

we then find that

$$\int_{a_1}^{\beta_1} F(x', x, n)\,dx'$$

$$= \frac{n+1}{2}\Bigg[\frac{P_{n+1}(x)}{2n+1}\left\{\frac{1}{x-a_1}[P_{n+1}(k_1)-P_{n+1}(a_1)-P_{n-1}(k_1)+P_{n-1}(a_1)]\right.$$
$$\left.+\frac{1}{x-\beta_1}[P_{n+1}(\beta_1)-P_{n+1}(k_1)-P_{n-1}(\beta_1)+P_{n-1}(k_1)]\right\}$$
$$-\frac{P_n(x)}{2n+3}\left\{\frac{1}{x-a_1}[P_{n+2}(k_2)-P_{n+2}(a_1)-P_n(k_2)+P_n(a_1)]\right.$$
$$\left.+\frac{1}{x-\beta_1}[P_{n+2}(\beta_1)-P_{n+2}(k_2)-P_n(\beta_1)+P_n(k_2)]\right\}\Bigg].$$

Hence we have, by using the form $\frac{a(n, x)}{n^{\frac{1}{2}}}$ for $P_n(x)$,

$$\left|\int_{a_1}^{\beta_1} F(x', x, n)\,dx'\right| < \frac{1}{\mu n}\,|\,a(n)\,|,$$

where (a_1, β_1) is any interval in the interval $(-1+\epsilon, 1-\epsilon)$, and is not interior to the interval $(a_1-\mu, \beta_1+\mu)$. Thus $\left|\int_{a_1}^{\beta_1} F(x', x, n)\,dx'\right|$ is less than a fixed number independent of a_1, β_1; and this number converges to zero as n is indefinitely increased. It has therefore been shewn that the conditions of validity of the general convergence theorem of § 1 are satisfied for every value of $\mu > 0$.

17. The limit of $\int_{-1}^{-1+\epsilon} f(x')\,F(x', x, n)\,dx'$ will now be investigated.

It will be assumed that x is such that $x+1-\epsilon \geqslant \mu$. We have then

$$\int_{-1}^{-1+\epsilon} f(x')\,F(x', x, n)\,dx'$$
$$= \frac{n+1}{2}\,\frac{1}{x+1}\int_{-1}^{-1+\epsilon_1} f(x')\,[P_{n+1}(x)\,P_n(x')-P_n(x)\,P_{n+1}(x')]\,dx'$$
$$+\frac{n+1}{2}\,\frac{1}{x+1-\epsilon_1}\int_{-1+\epsilon_1}^{-1+\epsilon} f(x')\,[P_{n+1}(x)\,P_n(x')-P_n(x)\,P_{n+1}(x')]\,dx',$$

where ϵ_1 is a number such that $0 \leqslant \epsilon_1 \leqslant \epsilon$.

Let us now assume that $f(x')$ is monotone and limited in the interval

$(-1, -1+\epsilon)$; we have then

$$\int_{-1}^{-1+\epsilon_1} f(x')\left[P_{n+1}(x)\,P_n(x')-P_n(x)\,P_{n+1}(x')\right]dx'$$

$$=f(-1+0)\int_{-1}^{-1+\epsilon_2}\left[P_{n+1}(x)\,P_n(x')-P_n(x)\,P_{n+1}(x')\right]dx'$$

$$+f(-1+\epsilon_1-0)\int_{-1+\epsilon_2}^{-1+\epsilon_1}\left[P_{n+1}(x)\,P_n(x')-P_n(x)\,P_{n+1}(x')\right]dx',$$

where ϵ_2 is such that $0 \leqslant \epsilon_2 \leqslant \epsilon_1$. Now

$$(n+1)\int_{-1}^{-1+\epsilon_2} P_{n+1}(x)\,P_n(x')\,dx$$

$$=\frac{n+1}{2n+1}P_{n+1}(x)\left[P_{n+1}(-1+\epsilon_2)-P_{n-1}(-1+\epsilon_2)\right],$$

and the expression on the right-hand side is numerically less than $2\,|\,P_{n+1}(x)\,|$, which converges to zero as n is indefinitely increased, uniformly for all values of x in the interval $(-1+\epsilon+\mu,\ 1-\epsilon-\mu)$. It may similarly be shewn that

$$(n+1)\int_{-1}^{-1+\epsilon_2} P_n(x)\,P_{n+1}(x')\,dx'$$

has the same property. A precisely similar proof establishes also that

$$(n+1)\int_{-1+\epsilon_2}^{-1+\epsilon_1}\left[P_{n+1}(x)\,P_n(x')-P_n(x)\,P_{n+1}(x')\right]dx'$$

has the same property; therefore

$$\frac{n+1}{x+1}\int_{-1}^{-1+\epsilon_1} f(x')\left[P_{n+1}(x)\,P_n(x')-P_n(x)\,P_{n+1}(x')\right]dx'$$

converges to zero, uniformly for all values of x in the interval $(-1+\epsilon+\mu,\ 1-\epsilon-\mu)$.

Similarly also, it may be shewn that the other part of the expression for $\int_{-1}^{-1+\epsilon} f(x')\,F(x', x, n)\,dx'$ converges uniformly to zero. Since a function with limited total fluctuation is the difference of two monotone functions, it can now be seen that, if $f(x')$ is of limited total fluctuation in the interval $(-1, -1+\epsilon)$, then $\int_{-1}^{-1+\epsilon} f(x')\,F(x', x, n)\,dx'$ converges to zero, as n is indefinitely increased, uniformly for all values of x in the interval

$(-1+\epsilon+\mu,\ 1-\epsilon-\mu)$. Also, if $f(x')$ is of limited total fluctuation in the interval $(1-\epsilon,\ 1)$, a precisely similar proof establishes that

$$\int_{1-\epsilon}^{1} f(x')\, F(x',\ x,\ n)\, dx'$$

converges uniformly to zero, for all values of x in the same interval as before.

18. To prove that the conditions of the theorems in §§ 4, 5 are satisfied, we have

$$\int_x^1 \sum_{r=0}^{n} \frac{2r+1}{2} P_r(x)\, P_r(x')\, dx' = \tfrac{1}{2} \sum_{r=1}^{n} P_r(x) \left[P_{r-1}(x) - P_{r+1}(x)\right] + \tfrac{1}{2}(1-x)$$

$$= \tfrac{1}{2}\left[1 - P_n(x)\, P_{n+1}(x)\right].$$

Therefore

$$\int_x^{x+\mu} F(x',\ x,\ n)\, dx'$$

$$= \tfrac{1}{2}\left[1 - P_n(x)\, P_{n+1}(x)\right] - \int_{x+\mu}^{1-\epsilon} F(x',\ x,\ n)\, dx' - \int_{1-\epsilon}^{1} F(x',\ x,\ n)\, dx'.$$

It has been shewn in § 17 that $\int_{1-\epsilon}^{1} F(x',\ x,\ n)\, dx'$ converges uniformly to zero, as n is indefinitely increased, for all values of x in the interval $(-1+\epsilon+\mu,\ 1-\epsilon-\mu)$. It has been shewn in § 16, that $\int_{x+\mu}^{1-\epsilon} F(x',\ x,\ n)\, dx'$ converges uniformly to zero, for all values of x in the interval $(-1+\epsilon,\ 1-\epsilon)$, the conditions of the fundamental convergence theorem being satisfied. Also $P_n(x)\, P_{n+1}(x)$ converges uniformly to zero, for all values of x in the interval $(-1+\epsilon,\ 1-\epsilon)$. It therefore follows that $\int_x^{x+\mu} F(x',\ x,\ n)\, dx'$ converges to the value $\frac{1}{2}$, uniformly for all values of x in the interval $(-1+\epsilon+\mu,\ 1-\epsilon-\mu)$ of x. Similarly it can be shewn that $\int_{x-\mu}^{x} F(x',\ x,\ n)\, dx'$ converges to the value $\frac{1}{2}$, uniformly for all values of x in the same interval. We have next to shew that

$$\left| \int_{x+\mu_1}^{x+\mu} F(x',\ x,\ n)\, dx' \right|$$

is less than some fixed finite number for all values of μ_1 such that

$0 \leqslant \mu_1 \leqslant \mu$, for all values of n, and for all values of x in the interval $(-1+\epsilon, 1-\epsilon)$, the number μ being taken to be $< \epsilon$.

The integral

$$\int_{x+\mu_1}^{x+\mu} F(x', x, n)\, dx' \quad \text{or} \quad \sum_{r=0}^{n} \frac{2r+1}{2} P_r(x) \int_{x+\mu_1}^{x+\mu} P_r(x')\, dx'$$

is equivalent to

$$\tfrac{1}{2} \sum_{r=1}^{n} P_r(x)[P_{r+1}(x+\mu) - P_{r-1}(x+\mu) - P_{r+1}(x+\mu_1) + P_{r-1}(x+\mu_1)] + \tfrac{1}{2}(\mu - \mu_1).$$

Writing $x = \cos\theta$, $x+\mu = \cos\theta'$, $x+\mu_1 = \cos\theta''$, and substituting for $P_r(x)$ the value

$$\left(\frac{2}{r\pi \sin\theta}\right)^{\frac{1}{2}} \left[\cos\left\{(r+\tfrac{1}{2})\theta - \frac{\pi}{4}\right\} + \frac{a(r, \theta)}{r^2}\right],$$

with the corresponding values of $P_{r+1}(\cos\theta')$, $P_{r-1}(\cos\theta')$, $P_{r+1}(\cos\theta'')$ and $P_{r-1}(\cos\theta'')$, we obtain an aggregate of terms of which the first is

$$\Sigma \tfrac{1}{2} \left(\frac{2}{r\pi \sin\theta}\right)^{\frac{1}{2}} \left(\frac{2}{(r+1)\pi \sin\theta'}\right)^{\frac{1}{2}} \left[\cos\left\{(r+\tfrac{1}{2})\theta - \frac{\pi}{4}\right\} + \frac{a(r, \theta)}{r}\right]$$
$$\times \left[\cos\left\{(r+\tfrac{3}{2})\theta' - \frac{\pi}{4}\right\} + \frac{a(r, \theta)}{r+1}\right].$$

This consists partly of series with $\dfrac{1}{r^{\frac{3}{2}}(r+1)^{\frac{1}{2}}}$ or $\dfrac{1}{r^{\frac{1}{2}}(r+1)^{\frac{3}{2}}}$ as factors of the general term, and which converge uniformly, and partly of the series

$$\Sigma \left(\frac{1}{r(r+1)\pi^2 \sin\theta \sin\theta'}\right)^{\frac{1}{2}} \cos\left\{(r+\tfrac{1}{2})\theta - \frac{\pi}{4}\right\} \cos\left\{(r+\tfrac{3}{2})\theta' - \frac{\pi}{4}\right\},$$

and this is expressible as the sum of four series which, apart from factors independent of r which are less than fixed numbers, are of the forms

$$\sum_1^n \frac{\cos r(\theta+\theta')}{\sqrt{r(r+1)}}, \quad \sum_1^n \frac{\sin r(\theta+\theta')}{\sqrt{r(r+1)}}, \quad \sum_1^n \frac{\cos r(\theta-\theta')}{\sqrt{r(r+1)}}, \quad \sum_1^n \frac{\sin r(\theta-\theta')}{\sqrt{r(r+1)}}.$$

It is known* that these series all converge uniformly, for all values of θ and θ' such that $\theta+\theta'$ and $|\theta-\theta'|$ are in an interval interior to the interval $(0, 2\pi)$, and this condition is satisfied if x is in the interval $(-1+\epsilon, 1-\epsilon)$, and if $\mu < \epsilon$.

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* See *Theory of Functions of a Real Variable*, p. 729.

The part of the expression which depends on μ_1 is of the form

$$\tfrac{1}{2}\Sigma P_r(x)\left[\sqrt{\frac{2}{(r-1)\,\pi\sin\theta''}}\left\{\cos\left((r-\tfrac{1}{2})\,\theta''-\frac{\pi}{4}\right)+\frac{a(r,\theta'')}{r}\right\}\right.$$

$$\left.-\sqrt{\frac{2}{(r+1)\,\pi\sin\theta''}}\left\{\cos(r+\tfrac{3}{2})\,\theta''-\frac{\pi}{4}+\frac{a(r,\theta'')}{r}\right\}\right]$$

or $$\Sigma P_r(x)\sqrt{\frac{2}{r\pi\sin\theta''}}\left[\sin\theta''\sin\left\{(r+\tfrac{1}{2})\,\theta''-\frac{\pi}{4}\right\}+\frac{a(r,\theta'')}{r}\right].$$

When the value of $P_r(x)$ is substituted, we obtain an aggregate of terms, of which the only one which requires special examination is

$$\frac{2}{\pi}\,\frac{1}{\sqrt{\sin\theta\sin\theta''}}\sin\theta''\sum_1^n\frac{\cos\left\{(r+\tfrac{1}{2})\,\theta-\frac{\pi}{4}\right\}\sin\left\{(r+\tfrac{1}{2})\,\theta''-\frac{\pi}{4}\right\}}{r}$$

or $$\frac{1}{\pi}\sqrt{\frac{\sin\theta''}{\sin\theta}}\sum_1^n\frac{\sin(r+\tfrac{1}{2})(\theta''-\theta)-\cos(r+\tfrac{1}{2})(\theta''+\theta)}{r}.$$

Now $\sum_1^n\frac{\sin(r+\frac{1}{2})(\theta''-\theta)}{r}$, although it does not converge uniformly in the neighbourhood of $\theta''-\theta=0$, can easily be shewn to have a value which is numerically less than a fixed number, for all values of n, θ, θ''. The series $\sum_1^n\frac{1}{r}\cos(r+\frac{1}{2})(\theta''+\theta)$ converges uniformly in an interval of $\theta''+\theta$, which is interior to the interval $(0, 2\pi)$, and is therefore numerically less than a fixed number.

It has now been shewn that $\int_{x+\mu_1}^{x+\mu}F(x', x, n)\,dx'$ is numerically less than a fixed number independent of n, x, μ, and μ_1, provided $0\leqslant\mu_1\leqslant\mu<\epsilon$, if x is in the interior of the interval $(-1+\epsilon, 1-\epsilon)$.

That $\int_{x-\mu}^{x-\mu_1}F(x', x, n)\,dx'$ has the same property, can be proved in the same manner. It has now been shewn that the theorems of §§ 4, 5 are applicable to an interval enclosed in the interior of the interval $(-1, 1)$.

19. The investigations in §§ 16–18, are sufficient to establish the following theorem:—

Let $f(x)$ be a function which, whether limited or unlimited, has a Lebesgue integral in the interval $(-1, 1)$, and is such that in sufficiently *small neighbourhoods of the points* -1, 1, *the function is of limited total fluctuation* (*à variation bornée*).

The series

$$\Sigma \frac{2n+1}{2} P_n(x) \int_{-1}^{1} f(x') P_n(x')\, dx'$$

converges at any point x *interior to the interval* $(-1, 1)$ *to the value* $\frac{1}{2}\{f(x+0)+f(x-0)\}$, *if a neighbourhood of the point* x *exists in which the function is of limited total fluctuation.*

In any interval in which $f(x)$ *is continuous, and which is contained in the interior of another interval in which the function has limited total fluctuation, the convergence of the series to the value* $f(x)$ *is uniform.*

The condition that the function should be of limited total fluctuation in neighbourhoods of the points -1, 1, although sufficient, is not necessary. I propose, in a later communication, to replace this condition by a much less stringent one.