On the Density of Linear Sets of Points. By W. H. Young.

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In one of T. Brodén's valuable memoirs on the real functions of a real variable there is a small error, to which it is perhaps worth while to call attention, as the point involved is one possessing some interest in itself. The mistake in question will be found on p. 23 of the memoir entitled "Beiträge zur Theorie der stetigen Funktionen einer reellen Variabeln," *Crelle*, CXVIII. It consists in the tacit assumption that, if each point of a linear set of points is a limiting point on both sides, then the set will be *dense everywhere* (*überall dicht*, *überall condensirt*). A set of points of which every point is a limiting point has been called "dense in itself"; and it is known that the terms "dense in itself" and "everywhere dense" are not simply different terms for characterizing the same type of sets of points.* It will be noted, however, that in the case considered by T. Brodén each point of the set is not merely a limiting point of the set, but a limiting point on both sides. Each point therefore possesses the property that its distance from, *so to speak*, the "next" point on either side is a quantity as small as we please. The conclusion that such a set of points must be distributed over the whole segment of the continuum in which we are operating would seem inevitable to a person unfamiliar with the theory of sets, and even Brodén, who has shown himself a master of subtle points of analysis, including this very class of questions, has fallen inadvertently into this error.

Take, however, any set which is *perfect*† and *nowhere dense.* Omit those of its points which are end points of the intervals of its complementary set of intervals, and we at once get a set of points which is *dense in itself* in Brodén's manner and yet nowhere dense.

This set has the potency‡ of the continuum, whereas the sets of points with which Brodén is concerned are countable. We need, however, merely select from our set a countable set which is every-

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* *Cf.* Cantor, *Math. Ann.*, Vol. XXI., p. 575.
† Dense in itself and closed. Cantor, *loc. cit.*
‡ *Mächtigkeit.*

where dense with respect to it, and we shall have a set of points which is at once *countable, dense in itself on both sides of every one of its points, and yet nowhere dense.*

Suppose, for definiteness, we take the set of numbers obtained by taking all the proper fractions expressed in the binary scale (terminating and non-terminating), and interpreting them in the ternary scale. This set is perfect and nowhere dense,* and those points of it which are limits on one side only are the terminating fractions, and those obtained from them by placing the circulating dot over the final 1. If, therefore, we take all the terminating fractions of the ternary scale, and after expressing them in the binary scale interpret them in the ternary scale, we shall get a countable set contained in our perfect set, and everywhere dense with respect to it. Hence each number of our countable set will be a limit on both sides for numbers of that set, and yet the set will be nowhere dense.

This example is sufficient to prove the existence of sets having the property under discussion, and the general method indicated above is at least theoretically sufficient for the construction of unlimited examples. As, however, it is not at once obvious that we could in this case so arrange the construction that the set ultimately ob-

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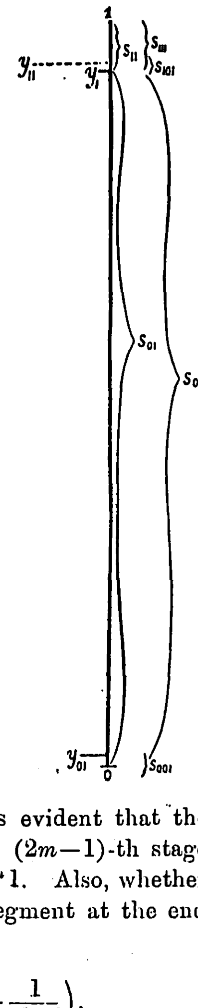
* This is the perfect set of numbers dense nowhere in the segment (0, 1) which is got from H. J. S. Smith's ternary set by derivation (*Proc. Lond. Math. Soc.*, Vol. VI., 1875, p. 148). I take this opportunity of calling attention to two points in the account in Schoenflies' "Bericht ueber die Mengenlehre," p. 101 *et seq.* (*Jahresbericht der deutschen Mathematiker-vereinigung*, Vol. VIII.), which are liable to give a false impression. The first is that H. J. S. Smith's set is not itself perfect, as Schoenflies' introductory remarks would lead one to suppose, the geometrical mode in which it is constructed introducing of necessity isolated points, even if you choose to explicitly include in the set its limiting points, so as to close it, as Volterra subsequently did in a similar example; this Smith does not apparently do; so that his set, as he constructed it is, like his other examples, countable. This oversight renders the account in Schoenflies unnecessarily obscure, commencing as it does with the words "In the interests of history [*sic*] I give here that example of a perfect set of numbers dense nowhere which was constructed by H. J. S. Smith." The interest of the student becomes aroused from more than merely historical motives when he realizes that H. J. S. Smith's ternary *derived* set is to all intents and purposes the same as Cantor's ternary set of numbers introduced by Schoenflies on the following page with the words "the first example of a perfect set dense nowhere which was *consciously* constructed was given by Cantor"; the former set consists in fact of all the ternary fractions involving the figures 0 and 1 only, and the latter set of those involving only 0 and 2. This arithmetical connexion being entirely ignored, this section of the *Bericht* seems wanting in unity of purpose as well as in perspective. Schoenflies' subsequent remarks about the generalization of Cantor's set, when any other base number is adopted, would have their proper place in connexion with Smith's work, who, eight years before Cantor introduced the ideas and the definition of a perfect set, actually adopted a general base $m \overline{\overline{>}} 3$, and whose numbers are the m-ary fractions which do not involve the figure $(m-1)$.

tained satisfied Brodén's special requirements,* I propose to give an example built up in Brodén's manner.



Take the segment (0, 1) of the y-axis, and divide it at the point y_1 into two parts, the lower s_{01}, and the upper s_{11}, so that the ratio

$$s_{01} : s_{11} = 1+j_1 : 1-j_1,$$

where $$j_1 = 1-\tfrac{1}{8}.$$

Next divide each of the two segments so obtained in precisely the same way, j_2 taking the place of j_1, where

$$j_2 = -\left(1-\frac{1}{8.2^2}\right),$$

and so on, j_n being defined by the equation

$$j_n = (-)^{n-1}\left(1-\frac{1}{8n^2}\right).$$

The new points of division at the end of the second stage we denote by y_{01} and y_{11}, y_{01} lying in s_{01} and y_{11} in s_{11}; the new points at the end of the third stage by y_{001}, y_{011}, y_{101}, y_{111}, and so on. Moreover, the intervals themselves at the end of the second stage will be denoted by s_{001}, s_{011}, s_{101}, s_{111}, and y_{001} will lie in s_{001}, and so for the others. The general law of division and notation is now obvious.† The points of division are called by Brodén *primary points*. Then we assert that the set of primary points is of the type required.

From the method of formation of the s's it is evident that the suffix of the maximum segment at the end of the $(2m-1)$-th stage is $(01)^m$, and at the end of the $2m$-th stage is $(01)^m 1$. Also, whether n be even or odd, the length of the maximum segment at the end of the n-th stage is

$$\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{4^2.2^2}\right)\left(1-\frac{1}{4^2.3^2}\right)\cdots\left(1-\frac{1}{4^2.n^2}\right),$$

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* *Loc. cit.*, p. 22, lines 4, 9.

† The indices of the new points of division introduced at the n-th division are such that, prefixing to each a dot, they are all the binary fractions involving n binary places; the last figure is therefore always a 1; *cf.* Brodén, *loc. cit.*, bottom of p. 22.

which is always greater than $\frac{1}{\sqrt{2}}\cdot\frac{4}{\pi}$, but continually approaches this value as n increases. Since each of these maximum segments lies within the preceding one, they form a *sequence*, and determine a definite interval within all of them, free of primary points, and of length $\frac{2\sqrt{2}}{\pi}$. The ends of this interval are, however, never reached by the primary points; they are, in fact, limiting points of the primary set, but not included in it.

Again, starting with any one of the segments left after any number, say n, of stages, we can show in a precisely similar way, by considering the maximum segment in it obtained at each subsequent stage, that it contains within it a definite interval, free of primary points, $\left(\text{whose length is, however, no longer } \frac{2\sqrt{2}}{\pi} \text{ of its own length}\right)$.

Thus we have shown that between every two primary points there is an interval free of primary points, possessing the property that its end points are also not primary points. Moreover, *every primary point is approached on both sides by primary points.*

Hence it follows that any given segment of the segment (0, 1) is either entirely free of primary points or contains an interval entirely free of primary points; so that *the set of primary points is dense nowhere.*

It is evident that the free intervals are the complementary intervals of a perfect set of points having the primary points as a countable set among those points of the perfect set which are limiting points on both sides.

We have purposely taken a definite numerical example, but we might equally well write

$$j_n = (-)^{n-1}\left(1-\frac{2}{p^2n^2}\right),$$

where p is any integer, obtaining in this way a countable set of examples of the type desired, namely, of sets of points nowhere dense and yet consisting entirely of points which are limiting points on both sides and are capable of construction in Brodén's special manner. It will be remarked that our example belongs to the class indicated *a priori* at the commencement of this paper; each set consists of a suitable selection from among the points which are limits on both sides of a certain perfect set nowhere dense. It is easy to see that every example of such a set is theoretically obtainable in this way. For, first, it cannot be closed, as it would then be perfect and nowhere dense, and would therefore involve limiting points on one side only. Next, adding those limiting points of the set not already included, we necessarily get a perfect set nowhere dense, which proves the assertion.

We cannot then obtain the condition that a set of points should be everywhere dense by expressing the fact that the distance of every point from its neighbouring points on either side should be indefinitely small, *unless the set of points obtained is known a priori to be closed.*

In the case of open sets of points, such as those with which Brodén is concerned, constructed by means of binary interpolation in a manner similar to that used in my example, we must express the condition that the length of the maximum segment at the end of the n-th stage in the process of division tends towards the limit zero as n is indefinitely increased.*

Comparing the notation here used with that employed by Brodén (p. 22), we find that

$$j_{n+1} = \frac{v_n - u_n}{v_n + u_n}.$$

The maximum segment at the $(n+1)$-th stage being now evidently

$$\prod_{n=0}^{\infty} \frac{1+|j_{n+1}|}{2},$$

it follows that *the necessary and sufficient condition that the primary set should be dense everywhere is*

$$\prod_{n=0}^{\infty} \frac{1+\left|\dfrac{v_n - u_n}{v_n + u_n}\right|}{2} = 0.$$

This condition must be substituted for those given by Brodén at the bottom of p. 23.†

It will be noticed that Brodén's remarks at the bottom of p. 23 and top of p. 24 down to (46) are still valid, the proper condition

* This is equally easily applied when the binary interpolation is the most general possible.

† $$\prod_{0}^{\infty}\left(1+\frac{v_n}{u_n}\right) = \infty, \quad \prod_{0}^{\infty}\left(1+\frac{u_n}{v_n}\right) = \infty.$$

It must be added that this error does not impair the validity of the examples in the subsequent part of the paper.

being still satisfied; (46) must, however, be replaced by the condition that when

$$(45) \quad \lim_{n=\infty} \frac{u_n}{v_n} = 0 \quad \text{or} \quad \lim_{n=\infty} \frac{v_n}{u_n} = 0,$$

the series $$(46a) \quad \sum_0^\infty \left\{ 1 - \left| \frac{v_n - u_n}{v_n + u_n} \right| \right\}$$

must diverge.

In the example given in this paper

$$\prod_0^{2m-1} \frac{u_n}{s_n} \equiv \prod_1^{2m} \frac{1+j_n}{2} = \frac{1}{4^2(2m)^2} \prod_1^{2m-1} \frac{1+j_n}{2},$$

and $$\prod_0^{2m-2} \frac{u_n}{s_n} \equiv \prod_1^{2m-1} \frac{1+j_n}{2}$$

$$= \left(\frac{1}{4^2}\right)^{m-1} \frac{1}{2^2 \cdot 4^2 \ldots (2m-2)^2} \left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{4^2 \cdot 3^2}\right) \ldots \left(1 - \frac{1}{4^2(2m-1)^2}\right).$$

Hence $$0 < \prod_0^r \frac{u_n}{s_n} < \left(\frac{1}{4^2}\right)^{r-1}$$

for all values of r after $r = 7$; so that

$$\prod_0^\infty \frac{u_n}{s_n} = 0;$$

similarly $$\prod_0^\infty \frac{v_n}{s_n} = 0.$$

Thus both Brodén's conditions are satisfied, although, as we saw, the set is nowhere dense. Indeed in our case $\frac{u_n}{v_n}$ is alternately greater and less than any assignable quantity, so that no finite (>0 and $<\infty$) limit of $\frac{u_n}{v_n}$ is possible, while, on the other hand, the condition (46*a*) is not satisfied, the series being convergent, since

$$\prod_0^\infty \frac{1 + \left| \dfrac{v_n - u_n}{v_n + u_n} \right|}{2} = \frac{2\sqrt{2}}{\pi}.$$