On Pure Ternary Reciprocants, and Functions allied to them.

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1. In the present paper reference will from time to time be made to the two here mentioned. They will be quoted for shortness as Paper I. and Paper II., respectively,—

I. *On Ternary and n-ary Reciprocants* (*Proceedings*, Vol. xvii., pp. 172—196).

II. *On the Linear Partial Differential Equations satisfied by Pure Ternary Reciprocants* (*Proceedings*, Vol. xviii., pp. 142—164).

The notation used will be that of Paper II. Thus, for instance, z_{rs} will throughout denote $\frac{1}{r!\,s!}\frac{d^{r+s}z}{dx^r dy^s}$. The main conclusion of Paper II. was, it will be remembered, that pure ternary reciprocants are those homogeneous and doubly isobaric functions of derivatives, such as z_{rs}, which have four annihilators called Ω_1, Ω_2, V_1, V_2. Of these the first two are

$$\Omega_1 = \Sigma\left\{(m+1)\,z_{m+1,n-1}\frac{d}{dz_{mn}}\right\},\ n \nless 1,\ m+n \nless 2\ \ldots\ldots\ldots(1),$$

$$\Omega_2 = \Sigma\left\{(n+1)\,z_{m-1,n+1}\frac{d}{dz_{mn}}\right\},\ m \nless 1,\ m+n \nless 2\ \ldots\ldots\ldots(2),$$

while the two others [*cf.* Paper II., § 9 (v.)] may be most compactly written

$$V_1 = \Sigma\left\{\Sigma\,(rz_{rs}z_{m+1-r,\,n-s})\frac{d}{dz_{mn}}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

the inner summation in which is limited by $r+s \nless 2$, $r \ngtr m+1$, $s \ngtr n$, $r+s \ngtr m+n-1$, and the outer by $m+n \nless 3$, and

$$V_2 = \Sigma\left\{\Sigma\,(sz_{rs}z_{m-r,\,n+1-s})\frac{d}{dz_{mn}}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

limited by $r+s \nless 2$, $r \ngtr m$, $s \ngtr n+1$, $r+s \ngtr m+n-1$, $m+n \nless 3$. Within limits as stated in each case, the summations are all supposed to be taken over all the range of positive integral (including zero) values of m, n, r, s.

It s proposed to base most of what follows on the consideration of functions, homogeneous and isobaric both in first and second suffixes, which, though subject to annihilation by both V_1 and V_2, are not annihilated also both by Ω_1 and Ω_2, and consequently, not being full invariants of the emanants $(z_{20}, z_{11}, z_{02}\mathfrak{J}(u, v)^2$, &c., are not reciprocants.

2. It will be well to have before us the matrix

$$\left|\begin{array}{lllllllllllllllll}
z_{20} \\
z_{11} \\
z_{02} \\
z_{30}, & z_{20} \\
z_{21}, & z_{11}, & z_{20} \\
z_{12}, & z_{02}, & z_{11} \\
z_{03}, & & z_{02} \\
z_{40}, & z_{30}, & & z_{20} \\
z_{31}, & z_{21}, & z_{30}, & z_{11}, & z_{20} \\
z_{22}, & z_{12}, & z_{21}, & z_{02}, & z_{11}, & z_{20} \\
z_{13}, & z_{03}, & z_{12}, & & z_{02}, & z_{11} \\
z_{04}, & & z_{03}, & & & z_{02} \\
z_{50}, & z_{40}, & & z_{30}, & & & z_{20} \\
z_{41}, & z_{31}, & z_{40}, & z_{21}, & z_{30}, & & z_{11}, & z_{20} \\
z_{32}, & z_{22}, & z_{31}, & z_{12}, & z_{21}, & z_{30}, & z_{02}, & z_{11}, & z_{20} \\
z_{23}, & z_{13}, & z_{22}, & z_{03}, & z_{12}, & z_{21}, & & z_{02}, & z_{11}, & z_{20} \\
z_{14}, & z_{04}, & z_{13}, & & z_{03}, & z_{12}, & & & z_{02}, & z_{11} \\
z_{05}, & & z_{04}, & & & z_{03}, & & & & z_{02} \\
z_{60}, & z_{50}, & & z_{40}, & & & z_{30}, & & & & z_{20} \\
z_{51}, & z_{41}, & z_{50}, & z_{31}, & z_{40}, & & z_{21}, & z_{30}, & & & z_{11}, & z_{20} \\
z_{42}, & z_{32}, & z_{41}, & z_{22}, & z_{31}, & z_{40}, & z_{12}, & z_{21}, & z_{30}, & & z_{02}, & z_{11}, & z_{20} \\
z_{33}, & z_{23}, & z_{32}, & z_{13}, & z_{22}, & z_{31}, & z_{03}, & z_{12}, & z_{21}, & z_{30}, & & z_{02}, & z_{11}, & z_{20} \\
z_{24}, & z_{14}, & z_{23}, & z_{04}, & z_{13}, & z_{22}, & & z_{03}, & z_{12}, & z_{21}, & & & z_{02}, & z_{11}, & z_{20} \\
z_{15}, & z_{05}, & z_{14}, & & z_{04}, & z_{13}, & & & z_{03}, & z_{12}, & & & & z_{02}, & z_{11} \\
z_{06}, & & z_{05}, & & & z_{04}, & & & & z_{03}, & & & & & z_{02} \\
\dots & & \dots & & \dots & & \dots & & \dots & & \dots & & \dots & & \dots & & \dots \\
\dots & & \dots & & \dots & & \dots & & \dots & & \dots & & \dots & & \dots & & \dots
\end{array}\right.$$

From this we can immediately extract any number of determinants which are functions obeying the laws stated at the end of the last article. In fact, we have the

Theorem.—*A determinant of any order n, obtained by selecting from the matrix a row which extends to the n^{th} column and no further, and any $n-1$ preceding rows, is a homogeneous and doubly isobaric function of the derivatives, and is annihilated by both V_1 and V_2.* Call this Prop. I.

For instance, any one of the first three rows is such a determinant of one term, any three of the rows 4 to 7 give such a determinant of the third order, the five rows 8 to 12 with any previous row give one of the fifth order, &c., &c.

That such determinants are all homogeneous needs no proof; that they are separately isobaric in first and second suffixes follows from the fact that the differences of the two partial weights of the constituents in two chosen columns and any the same row are both independent of the particular row; and that they are annihilated by V_1 and V_2 is made clear as follows.

Adopt for the moment the notation c_{rs} to denote the constituent in the r^{th} row and s^{th} column of the matrix. It is easy to see that, the summations extending to all values of the number r,

$$V_1 = \Sigma \left[\left\{ 2z_{20}c_{r2} + z_{11}c_{r3} + 3z_{30}c_{r4} + 2z_{21}c_{r5} + z_{12}c_{r6} + 4z_{40}c_{r7} + 3z_{31}c_{r8} + 2z_{22}c_{r9} + z_{13}c_{r10} + \dots \right\} \frac{d}{dc_{r1}} \right],$$

and $$V_2 = \Sigma \left[\left\{ z_{11}c_{r2} + 2z_{02}c_{r3} + z_{21}c_{r4} + 2z_{12}c_{r5} + 3z_{03}c_{r6} + z_{31}c_{r7} + 2z_{22}c_{r8} + 3z_{13}c_{r9} + 4z_{04}c_{r10} + \dots \right\} \frac{d}{dc_{r1}} \right];$$

whence it follows that V_1, operating on the first column, produces from it a sum of multiples of succeeding columns; and similarly for V_2. Moreover, if any other column than the first be chosen, subsequent columns can be selected in which its constituents are followed by other constituents exactly in the same arrangement as are the same constituents where they appear in the first column. Thus, the operation of V_1 on any column produces a column which is a sum of multiples of following columns; and similarly for V_2. V_1 and V_2 then both annihilate all determinants which can be obtained by associating complete rows of the matrix.

3. It is of great importance to remark that, whatever be the function operated on, the following four surprisingly simple equiva-

lences of operators hold :—

$$\Omega_1 V_1 - V_1 \Omega_1 = 0 \quad\text{.............................}(5),$$

$$\Omega_2 V_2 - V_2 \Omega_2 = 0 \quad\text{.............................}(6),$$

$$\Omega_2 V_1 - V_1 \Omega_2 = V_2 \quad\text{.........................}(7),$$

$$\Omega_1 V_2 - V_2 \Omega_1 = V_1 \quad\text{.........................}(8).$$

To prove the first of these, use the expressions of § 1 for Ω_1 and V_1 Selecting the terms which give $\frac{d}{dz_{mn}}$ in $\Omega_1 V_1 - V_1 \Omega_1$, we find that if $n > 0$ the coefficient of $\frac{d}{dz_{mn}}$ is

$$\Sigma\left\{r(r+1)\, z_{r+1,\,s-1},\ z_{m+1-r,\,n-s}\right\} + \Sigma\left\{r\,(m+2-r)\, z_{rs} z_{m+2-r,\,n-1-s}\right\}$$
$$-(m+1)\,\Sigma\left\{r z_{rs} z_{m+2-r,\ n-1-s}\right\},$$

the first range of summation being limited by

$$r \not> m+1, \quad s \not< 1 \not> n,$$

the second by $\quad r \not> m+1, \quad s \not> n-1,$

the third by $\quad r \not> m+2, \quad s \not> n-1,$

and all three by $\quad r+s \not< 2 \not> m+n-1.$

Now the ranges of the second and third summations, though apparently different, are really the same, since the value $m+2$ of r, which belongs to the third though not to the second, adds to the second only a zero term in virtue of the coefficient $m+2-r$. Thus the coefficient is equal to

$$\Sigma\left\{r(r+1)\, z_{r+1,\,s-1} z_{m+1-r,\,n-s}\right\} \text{ over the range } r \not> m+1,\ s \not< 1 \not> n$$
$$-\Sigma\left\{r(r-1)\, z_{rs}\, z_{m+2-r,\ n-1-s}\right\} \text{ over the range } r \not> m+2,\ s \not> n-1,$$

the ranges being further limited by $r+s \not< 2 \not> m+n-1$. But, if in the latter summation, and the conditions determining its limits, we put $r+1$ for r and $s-1$ for s, we produce exactly the former summation and the conditions by which it is limited. Thus the difference of the summations, *i.e.*, the coefficient of $\frac{d}{dz_{mn}}$ in $\Omega_1 V_1 - V_1 \Omega_1$, vanishes.

The case of $n = 0$ has here been omitted. No such symbol as $\frac{d}{dz_{m0}}$, however, occurs in Ω_1, while in V_1 the coefficients of such symbols contain only such derivatives as $z_{m'0}$, and are consequently annihilated by Ω_1. It follows that the coefficient of $\frac{d}{dz_{m0}}$ in $\Omega_1 V_1 - V_1 \Omega_1$ vanishes.

The proof is therefore complete, that

$$\Omega_1 V_1 - V_1 \Omega_1 = 0 \qquad\qquad (5).$$

In precisely the same way, or merely by interchange of first and second suffixes throughout,

$$\Omega_2 V_2 - V_2 \Omega_2 = 0 \qquad\qquad (6).$$

In reducing $\Omega_2 V_1 - V_1 \Omega_2$ we must consider separately the coefficients of symbols like $\frac{d}{dz_{0n}}$ and those of the more general symbols $\frac{d}{dz_{mn}}$, where m is not zero. We have, firstly,

$$\text{Co. } \frac{d}{dz_{0n}} \text{ in } \Omega_2 V_1 - V_1 \Omega_2$$

$$= \Omega_2 \text{ Co. } \frac{a}{dz_{0n}} \text{ in } V_1$$

$$= \Omega_2 \Sigma (z_{1s} z_{0,n-s})$$

$$= \Sigma \{(s+1) z_{0,s+1} z_{0,n-s}\},\ s \not< 1 \not> n-2;$$

and, secondly, for values of m exceeding zero,

$$\text{Co. } \frac{d}{dz_{mn}} \text{ in } \Omega_2 V_1 - V_1 \Omega_2$$

$$= \Sigma \{r(s+1) z_{r-1,s+1} z_{m+1-r,n-s}\} + \Sigma \{r(n-s+1) z_{rs} z_{m-r,n-s+1}\}$$
$$- (n+1) \Sigma \{r z_{rs} z_{m-r,n-s+1}\},$$

all three summations being limited by

$$r+s \not< 2 \not> m+n-1,$$

the first also by $r \not< 1 \not> m+1,\ s \not> n,$

the second by $r \not> m,\ s \not> n,$

and the third by $r \not> m,\ s \not> n+1.$

The value $n+1$ of s, which belongs to the range of the third but not to that of the second summation, gives rise only to a zero term if added to that range, in virtue of the coefficient $n-s+1$. Thus the difference of the second and third parts of Co. $\dfrac{d}{dz_{mn}}$ is

$$-\Sigma\left\{rs\, z_{rs}\, z_{m-r,\, n-s+1}\right\},$$

over the range limited by

$$r+s \nless 2 \ngtr m+n-1, \quad r \ngtr m, \quad s \ngtr n+1.$$

Now in the first summation put $r+1$ for r, and $s-1$ for s, thus making the limits of that summation identical with these limits. We obtain the result

$$\text{Co. } \frac{d}{dz_{mn}} \text{ in } \Omega_2 V_1 - V_1 \Omega_2 = \Sigma\left[\left\{(r+1)\, s - rs\right\} z_{rs}\, z_{m-r,\, n-s+1}\right]$$
$$= \Sigma\left\{s z_{rs}\, z_{m-r,\, n-s+1}\right\},$$

limited by $r+s \nless 2 \ngtr m+n-1$, $r \ngtr m$, $s \ngtr n+1$, a form with which the previously found coefficient of $\dfrac{d}{dz_{0n}}$ is strictly in accord.

Thus $$\Omega_2 V_1 - V_1 \Omega_2 = \Sigma\left\{\Sigma\left(s z_{rs}\, z_{m-r,\, n-s+1}\right)\right\},$$

over the inner range limited as above, and the outer limited by $m+n \nless 3$,

$$= V_2 \quad\quad\quad\quad (7).$$

Hence also, lastly, by interchange of first and second suffixes throughout,

$$\Omega_1 V_2 - V_2 \Omega_1 = V_1 \quad\quad\quad\quad (8).$$

4. The conclusions which can be drawn from the four symbolical identities (5) to (8) are numerous and important. Attention is in the first place called to one which affects primarily the theory of invariants and seminvariants of a system of quantics, but which will be seen later (§ 12) to have also an important bearing on the theory of reciprocants.

PROP. II.—*From any seminvariant I of the system of quantics*

$$\left(z_{20},\, z_{11},\, z_{02} \between u,\, v\right)^2, \quad \left(z_{30},\, z_{21},\, z_{12},\, z_{03} \between u,\, v\right)^3, \text{ \&c.,}$$

another seminvariant of the system may be generated by operating with V_1 upon it.

For, if $\Omega_1 I = 0$, $V_1 \Omega_1 I = 0$, and therefore, by (5), $\Omega_1 V_1 I = 0$, *i.e.*, $V_1 I$ is annihilated by Ω_1, and is a seminvariant.

The proposition is of course a purely algebraical one with regard to the quantics, whatever be their coefficients, being quite independent of any notion as to those coefficients being derivatives of a function z with regard to x and y.

If the seminvariant I be of degree i, the seminvariant $V_1 I$ thus generated is of degree $i+1$. The first partial weight of $V_1 I$, *i.e.*, the sum of first suffixes in each of its terms, exceeds that of I by unity, and the second partial weight, sum of second suffixes, is the same in I and $V_1 I$. This second partial weight is the weight in the ordinary language of binary quantics. Thus, adopting that ordinary language, the weight of $V_1 I$ is the same as that of I, while its degree exceeds the degree of I by unity.

One example of this use of the operator V_1 will suffice for the present. Choose for I the seminvariant $ac-b^2$ of the n-ic,

$$\left(z_{n0},\ z_{n-1,1}\ \ldots\ z_{0n}\between u,\ v\right)^n,$$

i.e., take $$I = 2nz_{n0}z_{n-2,2}-(n-1)\,z^2_{n-1,1}.$$

We deduce from this the seminvariant

$$\tfrac{1}{2}V_1 I = nz_{n-2,2}\left(\text{Co.}\,\frac{d}{dz_{n0}}\text{ in }V_1\right)+nz_{n0}\left(\text{Co.}\,\frac{d}{dz_{n-2,2}}\text{ in }V_1\right)$$
$$-(n-1)\,z_{n-1,1}\left(\text{Co.}\,\frac{d}{dz_{n-1,1}}\text{ in }V_1\right)$$
$$= nz_{n-2,2}\sum_{r\not<2}^{r\not>n-1}\left(rz_{r0}z_{n+1-r,0}\right)$$
$$+nz_{n0}\left\{\sum_{r\not<2}^{r\not>n-1}\left(rz_{r0}z_{n-1-r,2}\right)+\sum_{r\not<1}^{r\not>n-2}\left(rz_{r1}z_{n-1-r,1}\right)\right.$$
$$\left.+\sum_{r\not<0}^{r\not>n-3}\left(rz_{r2}z_{n-1-r,0}\right)\right\}$$
$$-(n-1)\,z_{n-1,1}\left\{\sum_{r\not<2}^{r\not>n-}\left(rz_{r0}z_{n-r,1}\right)+\sum_{r\not<1}^{r\not>n-2}\left(rz_{r1}z_{n-r,0}\right)\right\}\quad\ldots(9).$$

In particular, taking $n = 3$, from the seminvariant

$$2\,(3z_{30}z_{12}-z^2_{21}),$$

we obtain in this manner

$$6z_{12}z^2_{20}+3z_{30}\,(2z_{20}z_{02}+z^2_{11})-6z_{21}z_{20}z_{11},$$

a seminvariant from which, upon subtraction of

$$3z_{30}\,(z^2_{11}-4z_{20}z_{02}),$$

which is another of the same degree and weights, and division by 6, we obtain another of three terms only, viz.,

$$z_{12}z_{20}^2 - z_{21}z_{20}z_{11} + 3z_{30}z_{20}z_{02},$$

or, again, by adding $z_{30}(z_{11}^2 - 4z_{20}z_{02})$,

the seminvariant $z_{12}z_{20}^2 - z_{21}z_{20}z_{11} + z_{30}(z_{11}^2 - z_{20}z_{02})$ ....(10),

i.e.

$$\begin{vmatrix} z_{30}, & z_{20} & \\ z_{21}, & z_{11}, & z_{20} \\ z_{12}, & z_{02}, & z_{11} \end{vmatrix}$$

of which V_1 and V_2 are annihilators. This we shall meet with again presently.

From any seminvariant formed by the method of this article, repeated operation with Ω_2 will, of course, enable us to write down all the coefficients of a corresponding covariant.

It is almost unnecessary to add that, in virtue of $\Omega_2 V_2 - V_2\Omega_2 = 0$, *$V_2$ is in like manner a generator of seminvariants of the same quantics read from right to left, from other such seminvariants.* This may be quoted as Prop. III.

5. Of other results of the equivalences (5) to (8), the following will be useful for present purposes.

Prop. IV.—*If a function R of the derivatives is annihilated by V_1, so also is $\Omega_1 R$.*

For, by (5), $V_1\Omega_1 R = \Omega_1 V_1 R = 0$. In like manner, by (6)

Prop. V.—*If a function R is annihilated by V_2, so also is $\Omega_2 R$.*

Prop. VI.—*If a function R is annihilated by both V_1 and V_2, so also are both $\Omega_1 R$ and $\Omega_2 R$.*

That $\Omega_1 R$ is annihilated by V_1, and $\Omega_2 R$ by V_2, is told us by the two last propositions. That $\Omega_1 R$ is annihilated by V_2 is true since, by (8),

$$V_2\Omega_1 R = \Omega_1 V_2 R - V_1 R = 0;$$

and, similarly, that $\Omega_2 R$ is annihilated by V_1 is seen to be necessary.

Prop. VII.—*If V_2 and Ω_1 both annihilate a function, so too does V_1.* This is an immediate consequence of (8). Similarly by (7).

Prop. VIII.—*If V_1 and Ω_2 both annihilate a function, so too does V_2.*

6. We are now in a position to construct an important class of covariants of the emanants $(z_{20}, z_{11}, z_{02}\between u, v)^2$, &c.

Take P, a pure function of the derivatives, which is annihilated both by V_1 and V_2, but not by both Ω_1 and Ω_2. If homogeneous and doubly isobaric—and to such functions it will be well to confine attention—it is exceedingly likely to be a coefficient of a covariant of the emanants; but, even if it be not such a coefficient, a covariant, all whose coefficients have the same property as itself, may be obtained from it as follows.

By repeated operation with Ω_1 form the series of pure functions $\Omega_1 P$, $\Omega_1^2 P$, $\Omega_1^3 P$, ... These, like P, will by Prop. VI. be annihilated by V_1 and V_2. Now, each of these functions is of second partial weight one lower than the immediately preceding. One of them, $\Omega_1^n P$, must therefore be presently arrived at, which and all its successors vanish. The last non-vanishing one, $\Omega_1^{n-1} P$, is then annihilated by Ω_1, that is to say, is a seminvariant of the emanants. Call it P_0.

Operate on P_0 repeatedly by Ω_2 till a vanishing result $\Omega_2^{m+1} P_0$ is obtained. Then, writing, in accordance with this fact,

$$\Omega_2 P_0 = mP_1,\ \Omega_2 P_1 = (m-1) P_2,\ \dots\ \Omega_2 P_{m-1} = P_m,\ \Omega_2 P_m = 0,$$

we obtain a covariant of the emanants,

$$(P_0,\ P_1,\ P_2,\ \dots\ P_m)(u,\ v)^m \quad \text{.....................(11)},$$

all whose coefficients are annihilated by V_1 and by V_2.

If the degree and partial weights of P_0 are $i,\ w_1,\ w_2$,

those of P_1 are $i,\ w_1-1,\ w_2+1$,

those of P_2 are $i,\ w_1-2,\ w_2+2$,

&c.,

and finally, those of P_m are $i,\ w_1-m,\ w_2+m$.

Thus, we have $w_1 - m = w_2$ and $w_2 + m = w_1$,

each of which is identical with

$$m = w_1 - w_2.$$

(In particular, if $w_1 = w_2$, $m = 0$; and the covariant reduces to a single term—an invariant of the emanants, and consequently a reciprocant.)

An instance of such covariants is the cubic

$$A_0 u^3 + 3A_1 u^2 v + 3A_2 uv^2 + A_3 v^3 \quad \text{.................(12)},$$

where A_0 is the seminvariant (10)—where A_0, A_1, A_2, A_3 are, in fact, the determinants obtained by omitting the fourth, third, second, first rows, respectively, from the matrix

$$\left\| \begin{matrix} z_{30}, & z_{20} & \\ z_{21}, & z_{11}, & z_{20} \\ z_{12}, & z_{02}, & z_{11} \\ z_{03}, & & z_{02} \end{matrix} \right\| \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13).$$

It may be here remarked that the conditions $A_0 = 0$, $A_1 = 0$, $A_2 = 0$, $A_3 = 0$, two only of which can be independent, are the differential equations of the third order obtained by eliminating the constants from the general equation of a quadric surface.*

7. It is convenient to have a name for covariants of the class introduced in the last article. Let us speak of them as *Reciprocantive Covariants*, and of their leading coefficients, such as P_0, as *Reciprocantive Seminvariants*, of the emanants. The names are justified by the immediately following proposition, as well as by other facts to be adduced later.

PROP. IX.—*Any invariant of a Reciprocantive Covariant of the emanants is a Pure Ternary Reciprocant.*

For, being a function only of $P_0, P_1, \ldots P_m$, all of which are annihilated by V_1 and by V_2, it is itself annihilated by each of those operators; and, being a covariant of a covariant of the emanants, it is a covariant of the emanants themselves.

An example immediately to be considered leads us to supplement this theorem by another which might at first sight appear unnecessary to state, though clearly true; viz.,

PROP. X.—*If any function of seminvariants of the same or different Reciprocantive Covariants be annihilated by Ω_2, it is a Pure Ternary Reciprocant.*

8. In exemplification of this method of constructing pure ternary reciprocants, let us consider two simple cases.

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* In fact the four results of differentiating three times partially the equation

$$a + 2bx + 2cy + dx^2 + 2exy + fy^2 + 2gz + 2hxz + 2kyz + lz^2 = 0$$

may be written

$$\begin{aligned} &(g + hx + ky + lz)\, z_{30} + (h + lz_{10})\, z_{20} &&= 0, \\ &(g + hx + ky + lz)\, z_{21} + (h + lz_{10})\, z_{11} + (k + lz_{01})\, z_{20} &&= 0, \\ &(g + hx + ky + lz)\, z_{12} + (h + lz_{10})\, z_{02} + (k + lz_{01})\, z_{11} &&= 0, \\ &(g + hx + ky + lz)\, z_{03} \qquad\qquad\quad + (k + lz_{01})\, z_{02} &&= 0. \end{aligned}$$

(α) The quadratic emanant

$$z_{20}u^2+z_{11}uv+z_{02}v^2$$

is itself a reciprocantive covariant. Its one invariant,

$$z_{20}z_{02}-\tfrac{1}{4}z^2_{1} \equiv H, \text{ say} \qquad\qquad (14),$$

is the one reciprocant involving second derivatives only (*cf.* Paper I., § 12, or Paper II., § 11).

(β) Take the cubic reciprocantive covariant

$$A_0u^3+3A_1u^2v+3A_2uv^2+A_3v^3 \qquad\qquad (12),$$

where A_0, A_1, A_2, A_3 have the values given in (13) above.

Its coefficients are connected by the linear relations

$$\begin{aligned} z_{30}A_3-z_{21}A_2+z_{12}A_1-z_{03}A_0 &= 0, \\ z_{20}A_3-z_{11}A_2+z_{02}A_1 \qquad &= 0, \\ z_{20}A_2-z_{11}A_1+z_{02}A_0 &= 0, \end{aligned}$$

of which the second and third tell us that

$$\frac{A_0A_2-A_1^2}{z_{20}} = \frac{A_0A_3-A_1A_2}{z_{11}} = \frac{A_1A_3-A_2^2}{z_{02}} = R, \text{ say} \qquad (15).$$

The first of these identical forms of R shows that it is annihilated by Ω_1, and the third that it is annihilated by Ω_2. Thus, by Prop. X., R is a reciprocant. It is of order 5 and of partial weights 6, 6, and is, in fact, the resultant of the quadratic and cubic emanants (*cf.* Paper II., § 11).

The one invariant of (12), its discriminant

$$\begin{aligned} \Delta &= (A_0A_2-A_1^2)(A_1A_3-A_2^2)-\tfrac{1}{4}(A_0A_3-A_1A_2)^2 \\ &= R^2(z_{20}z_{02}-\tfrac{1}{4}z_{11}), \text{ by (15)}, \\ &= R^2H \qquad\qquad (16), \end{aligned}$$

gives no new reciprocant.

9. Facts with regard to the transformation of functions such as we are considering by cyclical changes of dependent and independent variables will now be investigated. In the first place, it is easy to see that—

Prop. XI.—*If Q be any homogeneous isobaric pure function of the derivatives of z, whose degree is i and first partial weight w_1, and which is annihilated by V_1* (not necessarily also by V_2), *the transformed ex-*

pression for $Qz_{10}^{-i-w_1}$ in terms of the derivatives of x is homogeneous and of no dimensions in the first derivatives x_{10}, x_{01}.

For, by Paper II., (11) and (13), we have, under the conditions stated,

$$\frac{d}{dx_{01}}\left(\frac{Q}{z_{10}^{i+w_1}}\right) = -\frac{z_{01}}{z_{10}^{i+w_1}}\Omega_1 Q \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17),$$

and

$$\frac{d}{dx_{10}}\left(\frac{Q}{z_{10}^{i+w_1}}\right) = -\frac{1}{z_{10}^{i+w_1}}\Omega_1 Q \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18),$$

whence

$$\left(z_{01}\frac{d}{dx_{10}} - \frac{d}{dx_{01}}\right)\left(\frac{Q}{z_{10}^{i+w_1}}\right) = 0,$$

i.e., since (Paper I., § 5),

$$\frac{z_{10}}{-1} = \frac{z_{01}}{x_{10}} = \frac{-1}{x_{01}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19),$$

$$\left(x_{10}\frac{d}{dx_{10}} + x_{01}\frac{d}{dx_{01}}\right)\left(\frac{Q}{z_{10}^{i+w_1}}\right) = 0.$$

In like manner, by Paper II., (18) and (21), it is proved that—

Prop. XII.—*If Q' be a homogeneous isobaric pure function, of degree i and second partial weight w_2, of the derivatives of z, which is annihilated by V_2* (not necessarily by V_1), *the y-transform of $Q'z_{01}^{-i-w_2}$ is homogeneous and of no dimensions in the first derivatives y_{10}, y_{01}.*

Now, take for Q a function annihilated by Ω_1 and having other properties as in Prop. XI. Equations (17) and (18) have in this case vanishing right-hand members, and tell us that the x-transform of $Qz_{10}^{-i-w_1}$ is pure.

Again, take for Q' a function having properties as in Prop. XII., and besides annihilated by Ω_2. We see, in like manner, that the y-transform of $Q'z_{10}^{-i-w_2}$ is pure.

These conclusions are, in consequence of the absence of requirement that Q be annihilated by V_2 or Q' by V_1, somewhat more general than their important cases:—

Prop. XIII.—*The x-transform of $P_0 z_{10}^{-i-w_1}$, where P_0 is a reciprocantive seminvariant of degree i and first partial weight w_1, is a pure function.*

Prop. XIV.—*The y-transform of $P_m z_{01}^{-i-w_1}$, where P_m is the result of interchanging first and second suffixes in a reciprocantive seminvariant P_0, and i, w_1 are the degree and second partial weight of P_m, is a pure function.* (N.B.—The second partial weight w_1 of P_m is, of course, the first partial weight of P_0.)

These two propositions are really identical, as will become clear later when we determine the actual expressions of the pure transforms.

10. Let us next employ (17) and (18) to aid in discussing the transformation of coefficients other than the first and last in a reciprocantive covariant

$$(P_0, P_1, P_2, \dots P_m)(u, v)^m.$$

If w_1 be the first partial weight of P_0, $w_1 - r$ is that of P_r. Hence, by (17),

$$\frac{d}{dx_{01}}\left(\frac{P_r}{z_{10}^{i+w_1-r}}\right) = -\frac{z_{01}}{z_{10}^{i+w_1-r}}\Omega_1 P_r$$

$$= x_{10} x_{01}^{i+w_1-r-1} r P_{r-1} \quad \dots\dots\dots\dots (20),$$

by (19) and the law of eduction of one coefficient of a covariant from the preceding. Again, by (18),

$$\frac{d}{dx_{10}}\left(\frac{P_r}{z_{10}^{i+w_1-r}}\right) = -\frac{1}{z_{10}^{i+w_1-r}}\Omega_1 P_r$$

$$= -x_{01}^{i+w_1-r} r P_r \quad \dots\dots\dots\dots (21).$$

Now
$$\frac{1}{z_{10}^{i+w_1}}(P_0, P_1, P_2, \dots P_m)(u, v)^m,$$

may be written

$$\left(\frac{P_0}{z_{10}^{i+w_1}}, \frac{P_1}{z_{10}^{i+w_1-1}}, \frac{P_2}{z_{10}^{i+w_1-2}}, \dots \frac{P_m}{z_{10}^{i+w_2}}\right)\left(u, \frac{v}{z_{10}}\right)^m,$$

w_1 and w_2 meaning the first and second partial weights of P_0, and having their difference equal to m; and, by (19), this may be also written

$$(P_0 x_{01}^{i+w_1}, P_1 x_{01}^{i+w_1-1}, P_2 x_{01}^{i+w_1-2}, \dots P_m x_{01}^{i+w_2})(u, vx_{01})^m \quad \dots\dots (22).$$

It suggests itself, in connection with Props. XIII. and XIV., and results (20) and (21) above, to seek values of u and v that this may be annihilated by $\frac{d}{dx_{01}}$ and $\frac{d}{dx_{10}}$, *i.e.*, that its x-transform may be a pure function.

Now, by (20), which is best made use of in the form

$$\frac{d}{dx_{01}}(P_r x_{01}^{i+w_1-r}) = r\frac{x_{10}}{x_{01}^2}(P_{r-1} x_{01}^{i+w_1-r+1}),$$

the result of differentiating (22) partially with regard to x_{01} is

$$\left\{\frac{d}{dx_{01}}(u^m)+m\frac{x_{10}}{x_{01}^2}u^{m-1}(vx_{01})\right\}P_0x_{01}^{i+w_1}$$

$$+\left\{m\frac{d}{dx_{01}}(u^{m-1}vx_{01})+m(m-1)\frac{x_{10}}{x_{01}^2}u^{m-2}(vx_{01})^2\right\}P_1x_{01}^{i+w_1-1}$$

$$+\left\{\frac{m(m-1)}{1.2}\frac{d}{dx_{01}}\left[u^{m-2}(vx_{01})^2\right]\right.$$

$$+\ldots\ldots; \qquad \left.+\frac{m(m-1)(m-2)}{1.2}\frac{x_{10}}{x_{01}^2}u^{m-3}(vx_{01})^3\right\}P_2x_{01}^{i+w_1-2}$$

and, using (21) in the form

$$\frac{d}{dx_{10}}(P_rx_{01}^{i+w_1-r})=-\frac{r}{x_{01}}(P_{r-1}x_{01}^{i+w_1-r+1}),$$

the result of partially differentiating (22) with regard to x_{10} is

$$\left\{\frac{d}{dx_{10}}(u^m)-m\frac{1}{x_{01}}u^{m-1}(vx_{01})\right\}P_0x_{01}^{i+w_1}$$

$$+\left\{m\frac{d}{dx_{10}}(u^{m-1}vx_{01})-m(m-1)\frac{1}{x_{01}}u^{m-2}(vx_{01})^2\right\}P_1x_{01}^{i+w_1-1}$$

$$+\left\{\frac{m(m-1)}{1.2}\frac{d}{dx_{10}}\left[u^{m-2}(vx_{01})^2\right]\right.$$

$$+\ldots\ldots; \qquad \left.-\frac{m(m-1)(m-2)}{1.2}\frac{1}{x_{01}}u^{m-3}(vx_{01})^3\right\}P_2x_{01}^{i+w_1-2}$$

and in both these results of differentiation, it is readily seen that the coefficients of $P_0x_{01}^{i+w_1}$, $P_1x_{01}^{i+w_1-1}$, $P_2x_{01}^{i+w_1-2}$, ... are all made to vanish upon putting

$$u=\frac{x_{10}}{x_{01}}, \quad v=\frac{1}{x_{01}},$$

i.e., $$u=-z_{01}, \quad v=z_{10}.$$

Hence the conclusion—

Prop. XV.—*If* $(P_0, P_1, \ldots P_m)(u, v)^m$ *be a reciprocantive covariant, the x-transform of*

$$\frac{1}{z_{10}^{i+w_1}}(P_0, P_1, \ldots P_m)(-z_{01}, z_{10})^m$$

is a pure function, i being the degree and w_1 *the first partial weight of* P_0.

Also we have, in like manner,—

PROP. XVI.—*Under exactly the same circumstances the y-transform of*

$$\frac{1}{z_{01}^{i+w_1}}(P_0, P_1, \dots P_m)(-z_{01}, z_{10})^m$$

is a pure function.

11. Required now the pure function of the derivatives of x to which, by Prop. XV.,

$$\frac{1}{z_{10}^{i+w_1}}(P_0, P_1, \dots P_m)(-z_{01}, z_{10})^m$$

is equal. This may, as above, be written

$$\left(\frac{P_0}{z_{10}^{i+w_1}}, \frac{P_1}{z_{10}^{i+w_1-1}}, \dots \frac{P_m}{z_{10}^{i+w_2}}\right)\left(\frac{x_{10}}{x_{01}}, 1\right)^m \dots\dots\dots\dots\dots\dots (23).$$

Now, in Paper II., § 10, it was seen that

$$\frac{z_{rs}}{z_{10}^{1+r}} = -x_{sr} + \text{terms with } \frac{1}{x_{01}} \text{ as a factor.}$$

Consequently, if P be a homogeneous isobaric function of the suffixed z's, whose degree and first partial weight are i and w_1, and if $P'(x)$ denote the same function of the suffixed x's with suffixes reversed in order (x_{sr} for z_{rs}, &c.),

$$\frac{P}{z_{10}^{i+w_1}} = (-1)^i P'(x) + \text{terms with } \frac{1}{x_{01}} \text{ as a factor.}$$

Now, P'_r is P_{m-r}. Thus (23) becomes

$$(-1)^i \left[P_m(x) + \dots, P_{m-1}(x) + \dots, \dots P_0(x) + \dots\right]\left(\frac{x_{10}}{x_{01}}, 1\right)^m,$$

each $+\dots$ in which indicates that terms with $\frac{1}{x_{01}}$ as factor are omitted where it occurs.

But it has been proved (Prop. XV.) that this form is independent of x_{10} and x_{01}. Thus we may give these first derivatives any values we please. Make x_{01} then infinitely great compared with other magnitudes occurring in the expression. The form taken is

$$(-1)^i P_0(x).$$

This, consequently, is the transform of

$$\frac{1}{z_{10}^{i+w_1}}(P_0, P_1, \dots P_m)(-z_{01}, z_{10})^m$$

required.

In exactly the same way, the y-transform of

$$\frac{1}{z_{01}^{i+w_1}}(P_0, P_1, \dots P_m)(-z_{01}, z_{10})^m,$$

already proved to be pure, is

$$(-1)^{i+m} P_m(y).$$

The two results together are most compactly stated

$$(P_0, P_1, \dots P_m)(-z_{01}, z_{10})^m = (-1)^i \frac{P_0(x)}{x_{01}^{i+w_1}} = (-1)^{i+m} \frac{P_m(y)}{y_{10}^{i+w_1}} \dots (24),$$

by use of (19), and the analogous qualities

$$\frac{y_{10}}{-1} = \frac{y_{01}}{z_{10}} = \frac{-1}{z_{01}}.$$

In (24) is also contained information as to the pure functions to which Props. XIII. and XIV. have told us that $P_0 z_{10}^{-i-w_1}$ and $P_m z_{01}^{-i-w_1}$ are equal. The first, by putting z, x, y for x, y, z, is seen to be $(-1)^{i+w_2} P_m(x)$; and the second, by putting y, z, x for x, y, z, to be $(-1)^{i+w_2} P_0(y)$.

In words, the results (24) may be stated as follows:—

Prop. XVII.—*A first cyclical transformation of dependent and independent variables in a reciprocantive covariant with* $-z_{10}$, z_{01} *inserted for its variables, produces from it, but for a sign factor and a power of a first derivative, the reciprocantive seminvariant which is its leading coefficient; and a second cyclical transformation produces, but for factors as before, the same reciprocantive seminvariant with first and second suffixes interchanged throughout.* Or, of course, the facts may be stated beginning from the seminvariant P_0, or from the reversed seminvariant P_m.

One of many conclusions from (24) with regard to mixed ternary reciprocants seems worth mentioning. Taking the product of the three members of (24), all written with z as the dependent variable, we see that

$$\frac{P_0 P_m (P_0, P_1, \dots P_m)(-z_{01}, z_{10})^m}{(z_{10}, z_{01})^{i+w_1}}$$

is an absolute mixed ternary reciprocant, or its numerator a mixed ternary reciprocant of index $i+w_1$. For instance,

$$z_{20}z_{02}(z_{20}z_{01}^2-z_{11}z_{01}z_{10}+z_{02}z_{10}^2)$$

is a ternary reciprocant of index 3; and, referring to (12) and (13),

$$A_0A_3(A_0z_{01}^3-3A_1z_{01}^2z_{10}+3A_2z_{01}z_{10}^2-A_3z_{10}^3),$$

i.e.

$$\begin{vmatrix} z_{30}, & z_{20} & \\ z_{21}, & z_{11}, & z_{20} \\ z_{12}, & z_{03}, & z_{11} \end{vmatrix} \times \begin{vmatrix} z_{21}, & z_{11}, & z_{20} \\ z_{12}, & z_{02}, & z_{11} \\ z_{03}, & & z_{02} \end{vmatrix} \times \begin{vmatrix} z_{10}^3, & z_{30}, & z_{20} & \\ 3z_{10}^2z_{01}, & z_{21}, & z_{11}, & z_{20} \\ 3z_{10}z_{01}^2, & z_{12}, & z_{02}, & z_{11} \\ z_{01}^3, & z_{03}, & & z_{02} \end{vmatrix},$$

is one of index 8.

12. Let us return from this discussion of the extent to which what the present paper has defined as reciprocantive seminvariants and covariants possess the fundamental property of ternary reciprocants, to the consideration of methods afforded by the propositions of § 5 for the determination of pure ternary reciprocants.

The method of §§ 6—8 is a powerful one; but it is based upon the knowledge of homogeneous isobaric functions annihilated by V_1 and V_2, and though in § 2 we have before us an infinite number of such functions, we have no indication that the system is a complete one. The method in question is not then yet rendered thoroughly systematic for the determination of *all* pure ternary reciprocants.

Another process to be now briefly explained follows closely one of known power for the systematic calculation of invariants of ternary quantics, and has the advantage of theoretical completeness. I exemplified it, without full confidence in or any statement of its generality, in Paper II., § 12, by obtaining the pure ternary reciprocant of type 3, 4, 4,

$$6(z_{02}^2z_{40}+z_{20}^2z_{04})-3z_{11}(z_{02}z_{31}+z_{20}z_{13})+z_{22}(z_{11}^2+2z_{02}z_{20})$$
$$-3\left\{2z_{02}(3z_{30}z_{12}-z_{21}^2)-z_{11}(9z_{30}z_{03}-z_{12}z_{21})+2z_{20}(3z_{03}z_{21}-z_{12}^2)\right\}.$$

(I would remark that V in the article referred to is mis-written for V_1.)

Required the pure ternary reciprocants of degree i and (equal) partial weights w_1, w_1.

Let R be such a reciprocant. It is (Paper I.) an invariant of the emanants

$$(z_{20}, z_{11}, z_{02}\between u, v)^2, \quad (z_{30}, z_{21}, z_{12}, z_{03}\between u, v)^3, \text{ \&c.}$$

If, then, $I_1, I_2, I_3, \ldots$ be a complete system of linearly independent invariants of the given type of the system, it is necessary that

$$R \equiv a_1 I_1 + a_2 I_2 + a_3 I_3 + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

for some values or other of the numerical multipliers $a_1, a_2, a_3, \ldots$

Now (Prop. II.), V_1 operating on any seminvariant I produces another seminvariant, of degree and first partial weight exceeding those of I by unity, and of second partial weight equal to that of I. If, then, $J_1, J_2, J_3, \ldots$ be a complete system of linearly independent seminvariants of the emanants whose degree and partial weights are $i+1$, w_1+1, w_1, we must have

$$V_1 R \equiv (\lambda_1 a_1 + \mu_1 a_2 + \nu_1 a_3 + \ldots) J_1 + (\lambda_2 a_1 + \mu_2 a_2 + \nu_2 a_3 + \ldots) J_2$$
$$+ (\lambda_3 a_1 + \mu_3 a_2 + \nu_3 a_3 + \ldots) J_3 + \ldots$$

for some known, vanishing or non-vanishing, values of the multipliers λ, μ, ν. For R to be annihilated by V_1, we have then the conditions

$$\left.\begin{array}{l} \lambda_1 a_1 + \mu_1 a_2 + \nu_1 a_3 + \ldots = 0 \\ \lambda_2 a_1 + \mu_2 a_2 + \nu_2 a_3 + \ldots = 0 \\ \lambda_3 a_1 + \mu_3 a_2 + \nu_3 a_3 + \ldots = 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (26).$$

If these be satisfied, not only will V_1 but also V_2 be an annihilator of R. For (Prop. VIII.), since $\Omega_2 R = 0$ and $V_1 R = 0$, $V_2 R = 0$. The linear conditions (26) are consequently all those which $a_1, a_2, a_3, \ldots$ have to satisfy in order that R may be a reciprocant. If the number of these conditions be less by one or any greater deficiency than the number of a's, *i.e.*, than the number of invariants I in (25), a pure ternary reciprocant R, or a number having that deficiency for a superior limit of linearly independent pure ternary reciprocants, is determined. Hence

PROP. XVIII.—*If the number of linearly independent invariants of degree i and partial weights w_1, w_1 of the quadratic cubic, &c. emanants exceed the number of seminvariants of degree $i+1$ and partial weights w_1+1, w_1, the excess is equal to, or at any rate a superior limit to, the number of linearly independent pure ternary reciprocants of type i, w_1, w_1.*