Quadrature-Formulae in Relation to Moments
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that where each operand in one quantity corresponds to the same operand in the other. This correspondence is called one, and its direct and inverse are the same. Next to the function one, it is natural to take that where each of two corresponding operands is the same function of the other. This type is called a transposition, and here again the direct and inverse are the same. Then we come to the case of three operands, where one corresponds to another, this other to a third, and the third corresponds to the first. This type of correspondence may be called a treposition, and here the direct and inverse are not the same. Transpositions and trepositions, like other types of correspondence, can be illustrated by models or by a double column of symbols. Many other correspondences of this kind, consisting of perhaps several trepositions, or a transposition and a treposition, and so on, are considered in books on the Theory of Groups.

In the letter already referred to it was suggested that the words multiplication and addition, and terms such as factor ought to have the same meaning for all correspondences, and that it should agree with their meaning for the case of numbers.

That this is not at present the case is sufficiently shown by the large number of ways in which the product of two vectors is understood. If the word was used in the same way as for numbers, the term product of two vectors would be given to what is generally called their sum.

The meaning of product of two correspondences was defined in the letter. A multiplication or addition table of functions is an example of one-to-two correspondence of functions, and is therefore outside the range of this paper. But we may consider the question of choice of an intermediate quantity when a number of quantities have a one-to-one correspondence, and the meaning of "power" and "root" for some of the correspondences already referred to.

For example, in the case of a correspondence of words in two languages, this may be factorized into the product of the correspondence of one language to Esperanto and of Esperanto to the other. In the case of correspondence by value, the intermediate quantity in most countries consists of amounts of gold. In the case of correspondence of simultaneous states of changes, the intermediate quantity adopted is called the time.

A function must be of the kind having the same operands in both quantities in order that it may be multiplied by itself. Thus we can consider the square or cube of a function, such as "father," or of a vector, but not of the correspondence tabulated in an English-French dictionary. The square of "father" is grandfather, its cube is great-grandfather. If the crossclassification has been given by performing an operation, the square of the correspondence is given by a repetition of the operation. The square of a transposition is one, and the cube of a treposition is one. It may be noticed that if "root" is taken to mean the inverse of "power," then it is somewhat misleading to employ it also to mean a solution of an equation. By a cube root of one ought strictly to be meant a treposition, and not such an expression as $-\frac{1+\sqrt{-3}}{2}$.

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QUADRATURE-FORMULAE IN RELATION TO MOMENTS.

WRITERS on mensuration do not seem to call attention to the convenience of quadrature-formulae for determining positions of centres of gravity, magnitudes of moments of inertia, etc.

Let u be the ordinate, for abscissa x, of a plane figure. Then the first moment of the figure with regard to $x=0$ is $\int uxdx$. If u is a rational integral algebraical function of x, then so also is ux; and the methods of quadrature apply to $\int uxdx$ as well as to $\int udx$. The principle can be extended to higher moments, and also to cases in which u is the area of the section of a solid by a plane whose abscissa is x.

In most cases what we really require is not the pth moment $\int ux^p dx$, but the mean value of x^p, i.e. $\int ux^p dx \Big/ \int udx$. It is then convenient to use the same quadrature-formula for $\int udx$ as for $\int ux^p dx$. Suppose that u is of degree m in x, and that $\lambda_0 v_0 + \lambda_1 v_1 + \ldots + \lambda_n v_n$ is an expression which gives the value of $\int vdx$ when v is of degree not greater than $m+p$ in x. Then $\int ux^p dx \Big/ \int udx = \Sigma\lambda ux^p \big/ \Sigma\lambda u$, and we have only to take account of the mutual ratios of the λ's or the u's, not of their absolute values.

Suppose, for example, that we require the centre of gravity of a hemisphere of radius a. Here ux (x being measured along the axis) is of the third degree in x, and therefore Simpson's (the "prismoidal") formula applies. Taking the plane surface of the hemisphere to be $x=0$, so that the u's in the formula are the areas of the sections by $x=0$, $x=\frac{1}{2}a$, $x=a$, we have

λ (proportional to)	1	4	1
u (proportional to)	$2.2=4$	$3.1=3$	$4.0=0$
x	0	$\frac{1}{2}a$	a

and therefore, for the centre of gravity,

$$\bar{x} = \frac{1.4.0+4.3.\frac{1}{2}a+1.0.a}{1.4+4.3+1.0} = \frac{6a}{16} = \frac{3}{8}a.$$

Again, suppose we require the moment of inertia of a triangular lamina with regard to one side. In this case, also, ux is of the third degree in x. Measuring x from the particular side, and taking the distance of the opposite angle to be h, we have

λ	1	4	1
u	2	1	0
x^2	0	$\frac{1}{4}h^2$	h^2

so that

$$\text{mean value of } x^2 = \frac{4.\frac{1}{4}h^2}{2+4} = \frac{1}{6}h^2.$$

As a third example, let us find the moment of inertia of an ellipsoid with regard to a principal plane. Here ux is of the fourth degree, and Weddle's formula is convenient. The coefficients in the complete formula are

$$1, 5, 1, 6, 1, 5, 1;$$

but, since everything is symmetrical about the principal plane, we need only consider half the figure. We have then, if the axis is $2a$,

λ	3	1	5	1
u	3.3	4.2	5.1	6.0
x^2	0	$\frac{1}{9}a^2$	$\frac{4}{9}a^2$	a^2

and

$$\text{mean value of } x^2 = \frac{8+100}{27+8+25} \cdot \frac{1}{9}a^2 = \frac{1}{5}a^2.$$

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