



## LX. On stationary waves in flowing water.—Part III

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LX. *On Stationary Waves in Flowing Water.*—Part III.

By Sir WILLIAM THOMSON, F.R.S.

[Continued from p. 452.]

In No. 138 (November), p. 446, equation (6), for  $(gD + q^2)$  read  $(gD + q^2)$ .

AS promised in Part I., we may now consider the application of the principles developed in it and in Part II. to the question of towing in a canal, and we shall find almost surprisingly a theoretical anticipation,  $49\frac{1}{2}$  years after date, of Scott Russell's brilliant "Experimental Researches into the Laws of certain Hydrodynamical Phenomena that accompany the Motion of Floating Bodies, and have not previously been reduced into Conformity with the known Laws of the Resistance of Fluids"\*, which had led to the Scottish system of "fly-boat," carrying passengers on the Glasgow and Ardrossan Canal, and between Edinburgh and Glasgow on the Forth and Clyde Canal, at speeds of from 8 to 12 or 13 miles an hour† by a horse, or a pair of horses, galloping along the bank. The practical method originated from the accident of a spirited horse, whose duty it was to drag a boat along at a slow speed (I suppose a walking speed), taking fright and running off, drawing the boat after him, and so discovering that when the speed exceeded  $\sqrt{gD}$  the resistance was less than at lower speeds. Mr. Scott Russell's description of the incident, and of how Mr. Houston took advantage for his Company of his horse's discovery, is so interesting that I quote it *in extenso*:—"Canal navigation furnishes at once the most interesting illustrations of the interference of the wave, and most important opportunities for the application of its principles to an improved system of practice. It is to the diminished anterior section of displacement, produced by raising a vessel with a sudden impulse to the summit of the progressive wave, that a very great improvement recently introduced into canal transport owes its existence. As far as I am able to learn, the isolated fact was discovered accidentally on the Glasgow and Ardrossan Canal of small dimensions. A spirited horse in the boat of William Houston

\* By John Scott Russell, Esq., M.A., F.R.S.E. Read before the Royal Society of Edinburgh on April 3, 1837, and published in the 'Transactions' in 1840.

† One mile an hour is English and American reckoning of velocity, which, when not at sea, signifies 1.60933 kilometres per hour, or .44704 metre per second.

Esq., one of the proprietors of the works, took fright and ran off, dragging the boat with it, and it was then observed, to Mr. Houston's astonishment, that the foaming stern surge which used to devastate the banks had ceased, and the vessel was carried on through water comparatively smooth, with a resistance very greatly diminished. Mr. Houston had the tact to perceive the mercantile value of this fact to the Canal Company with which he was connected, and devoted himself to introducing on that canal vessels moving with this high velocity. The result of this improvement was so valuable in a mercantile point of view, as to bring, from the conveyance of passengers at a high velocity, a large increase of revenue to the Canal Proprietors. The passengers and luggage are conveyed in light boats, about sixty feet long and six feet wide, made of thin sheet iron and drawn by a pair of horses. The boat starts at a slow velocity behind the wave, and at a given signal it is by a sudden jerk of the horses drawn up on the top of the wave, where it moves with diminished resistance, at the rate of 7, 8, or 9 miles an hour”\*.

The “diminished anterior section of displacement produced by raising a vessel with a sudden impulse to the summit of the progressive wave” is no doubt a correct observation of an essential feature of the phenomenon; but it is the annulment of “the foaming stern surge which [at the lower speeds] used to devastate the banks” that gives the direct explanation of the diminished resistance. It is in fact easy to see that when the motion is steady, no waves can be left astern of a boat towed through a canal at a speed greater than  $\sqrt{gD}$ , the velocity of an infinitely long wave in the canal; and therefore (the water being supposed inviscid) the resistance to towage must be *nil* when the velocity exceeds  $\sqrt{gD}$ . This holds true also obviously for towage in an infinite expanse of open water of depth  $D$  over a plane bottom.

The formula (25) of Part II. for the whole horizontal component force upon an inequality or succession of inequalities on the bottom allows us to calculate the resistance on a boat of any dimensions and any shape provided we know the height of the regular waves which follow it steadily at its own speed in the canal, at a sufficiently great distance behind it to be sensibly uniform across the breadth of the canal, according to the principle explained in the middle of p. 354 of Part I. The principles upon which the values of  $\delta$  [the  $h$  of formula (25), Part II.] may be calculated are partly given in the remainder

\* Trans. Roy. Soc. Edinb. vol. xiv. (1840) p. 79.

of the present article, and will be more fully developed in Part IV.

To find the steady motion of water flowing in a rectangular channel over a bottom with geometrically specified inequalities, it is convenient, after the manner of Fourier, to first solve the problem for the case in which the profile of the bottom is a curve of sines deviating infinitesimally from a horizontal plane.

For convenience, take OX along the mean level of the bottom, positive in the direction of U the mean velocity of the stream; and OY vertical, positive upwards. Let

$$h = H \cos mx \quad . \quad . \quad . \quad . \quad . \quad (1)$$

be the equation of the bottom; and

$$y - D = \mathfrak{h} = \mathfrak{h} \cos mx \quad . \quad . \quad . \quad . \quad . \quad (2)$$

be the equation of the free surface,  $\mathfrak{h}$  being height above its mean level. Let  $\phi$  be the velocity potential;  $u, v$  the velocity components; and  $p$  the pressure at any point  $(x, y)$  of the water at time  $t$ : so that we have

$$u = \frac{d\phi}{dx} \quad \text{and} \quad v = \frac{d\phi}{dy} \quad . \quad . \quad . \quad . \quad . \quad (3),$$

and

$$p = C - gy - \frac{1}{2}(u^2 + v^2) \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Now the deviation from uniform horizontal velocity is infinitesimal, and therefore  $v$  and  $u - U$ , are infinitely small. Hence (4) gives

$$p = C - gy - \frac{1}{2}U^2 + U(u - U) \quad . \quad . \quad . \quad . \quad . \quad (5).$$

$\phi$  must be a solution of the equation of continuity  $\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0$ , and the proper one for our present case clearly is

$$\phi = Ux + \sin mx (K\epsilon^{my} + K'\epsilon^{-my}) \quad . \quad . \quad . \quad (6),$$

where, because the motion is steady,  $K$  and  $K'$  are constants. This, in virtue of (3), gives

$$u - U = m \cos mx (K\epsilon^{my} + K'\epsilon^{-my}), \quad . \quad . \quad (7);$$

$$v = m \sin mx (K\epsilon^{my} - K'\epsilon^{-my}) \quad . \quad . \quad . \quad (8).$$

Hence, as the values of  $y$  at the bottom and at the surface are infinitely nearly 0 and  $D$  respectively, we find respectively for the vertical component velocity at the bottom and at the surface,

$$m \sin mx (K - K'), \quad \text{and} \quad m \sin mx (K\epsilon^{mD} - K'\epsilon^{-mD}).$$

Hence, to make the bottom-stream-lines and surface-stream-

lines agree respectively with the assumed forms (1) and (2), we clearly have

$$m(K - K') = mHU \quad . \quad . \quad . \quad (9),$$

and

$$m(K\epsilon^{mD} - K'\epsilon^{-mD}) = m\mathfrak{H}U \quad . \quad . \quad . \quad (10);$$

whence

$$\left. \begin{aligned} K &= U \frac{\mathfrak{H} - H\epsilon^{-mD}}{\epsilon^{mD} - \epsilon^{-mD}}, \\ K' &= U \frac{\mathfrak{H} - H\epsilon^{mD}}{\epsilon^{mD} - \epsilon^{-mD}} \end{aligned} \right\} \quad . \quad . \quad . \quad (11).$$

Now at the free surface the pressure is constant, and hence, by (5), we have

$$-gy + U(u - U) = \text{constant} \quad . \quad . \quad . \quad (12):$$

from which, by (2), (7), and (11), we find

$$0 = -g\mathfrak{H} + mU^2 \frac{\mathfrak{H}(\epsilon^{mD} + \epsilon^{-mD}) - 2H}{\epsilon^{mD} - \epsilon^{-mD}},$$

whence

$$\mathfrak{H} = \frac{2H}{\epsilon^{mD} + \epsilon^{-mD} - \frac{g}{mU^2}(\epsilon^{mD} - \epsilon^{-mD})} \quad . \quad . \quad (13),$$

which is the solution of our problem, for the case of the bottom a simple harmonic curve.

Suppose now the equation of the bottom to be

$$h = (\kappa \cos mx + \kappa^2 \cos 2mx + \kappa^3 \cos 3mx + \&c.) m\Delta/\pi \quad . \quad (14);$$

the equation of the surface, found by superposition of solutions given by (13), allowable because the motion deviates infinitely little from horizontal uniform motion throughout the water, is

$$y - D = \mathfrak{y} = \sum_{i=1}^{i=\infty} \frac{2\kappa^i \cos imx \cdot m\Delta/\pi}{\epsilon^{imD} + \epsilon^{-imD} - \frac{g}{imU^2}(\epsilon^{imD} - \epsilon^{-imD})} \quad (15).$$

To interpret the equation (14) by which the bottom is defined, remark that, by the well-known summation of its second member, it is equivalent to

$$h = \frac{\frac{1}{2}m\Delta/\pi \cdot (1 - \kappa^2)}{1 - 2\kappa \cos mx + \kappa^2} - \frac{1}{2}m\Delta/\pi = \frac{m\Delta/\pi \cdot \kappa (\cos mx - \kappa)}{1 - 2\kappa \cos mx + \kappa^2} \quad (16).$$

The series (14) is convergent for all values of  $\kappa$  less than unity. According to the method of Fourier, Cauchy, and Poisson, the extreme case of  $\kappa$  infinitely little less than unity will be made the foundation of our practical solutions. By

(14) we see that 
$$\int_{-\pi/m}^{\pi/m} dx \, h = 0 \quad . \quad . \quad . \quad . \quad . \quad (17);$$

and hence by the first of equations (16) we see that

$$\int_{-\pi/m}^{\pi/m} dx \, \frac{\frac{1}{2} m A / \pi \cdot (1 - \kappa^2)}{1 - 2\kappa \cos mx + \kappa^2} = A \quad . \quad . \quad . \quad (18).$$

Now when  $\kappa$  is infinitely little short of unity the factor of  $dx$  in the first member of (18) is zero for all values of  $x$  differing finitely from zero or  $2i\pi/m$ , ( $i$  being an integer); and it is infinitely great when  $x=0$  or  $2i\pi/m$ . Hence we infer from (17) and (18) that a vertical longitudinal section of the bottom presents a regular row of symmetrical elevations and depressions above and below its mean level; the elevations being confined to very small spaces on the two sides of each of the points  $x=0$  and  $x=2i\pi/m$ , and the profile-area of each elevation being  $A$ . The depths of the depressions below the average level in the intermediate spaces between the elevations, are of course extremely small because of the exceeding shortness of the spaces over which are the elevations. For our complete analytical solution, not only must  $A$  be infinitely small, but the steepness of the slope up to the summit of  $A$  must everywhere be an infinitely small fraction of a radian; and of course therefore the infinitesimal lowering of the bottom between the ridges, which the adoption of a mean bottom-level for our datum line has necessarily introduced, may be left out of account in our dynamical problem.

If the slope of the ridge is not an infinitely small fraction of a radian our solution will still hold, provided its height is very small in comparison with the depth of the water over it. But the effective potency of the ridge would then not be its profile-area  $A$ , but something much greater; of which the amount would be found by taking a stream-line over it, far enough above it to have nowhere more than an infinitesimal slope, and finding the profile-area of such a stream-line above its own average level considered as the virtual bottom. With these explanations we shall speak of a ridge for brevity instead of an "irregularity" or "obstacle," and call its profile-area  $A$ , simply the "magnitude of the ridge;" this being, as we see by (15), the measure of its potency in disturbing the surface. When instead of a ridge we have a hollow,  $A$  is negative; and when convenient we may, of course, call a hollow a negative ridge.

It is clear that (15) converges, and does not depend for its convergence on  $\kappa$  being less than unity; so that in it we may take  $\kappa$  absolutely equal to unity, and we shall do so accordingly.

To find now the effect of a single ridge, remark that if  $l$  be the length from ridge to ridge,

$$m = 2\pi/l \quad . \quad . \quad . \quad . \quad . \quad (19).$$

After the manner of Fourier now suppose  $l$  infinitely large; which makes  $m$  infinitely small; and put

$$im = q \text{ and } m = dq \quad . \quad . \quad . \quad . \quad (20);$$

then with  $\kappa = 1$ , (15) becomes

$$\psi = \int_0^\infty dq \frac{2A/\pi \cdot \cos qx}{\epsilon^{qD} + \epsilon^{-qD} - \frac{1}{qb}(\epsilon^{2D} - \epsilon^{-2D})} \quad . \quad . \quad (21);$$

where

$$b = U^2/g \quad . \quad . \quad . \quad . \quad . \quad (22).$$

Equation (21) will be shortened, and for some interpretations simplified, by making  $qD = \sigma$ , when it becomes

$$\psi = \int_0^\infty d\sigma \frac{2A/D\pi \cdot \cos(\sigma x/D)}{\epsilon^\sigma + \epsilon^{-\sigma} - \frac{D}{b\sigma}(\epsilon^\sigma - \epsilon^{-\sigma})} \quad . \quad . \quad (23).$$

The definite integral (21) or (23) seemed rather intractable, and the quadratures required to evaluate it, for many and widespread enough values of  $x$  to show the shape of the surface for any one particular value of  $D/b$ , would be very laborious. But I had found a method of evaluating it from the periodic solution for an endless succession of equidistant equal ridges (15), wholly analogous to analytical deductions from corresponding solutions for cases of thermal conduction and of signalling through submarine cables, to be found in vol. ii. pp. 49 and 56 of my *Collected Mathematical and Physical Papers*; and, towards applying this method to a particular case of the disturbance due to a single ridge, I had fully worked out the periodic solution for the case represented by the diagram of curves (fig. 3, p. 529), when I found a direct and complete analytical solution for the single-ridge problem in a form exceedingly convenient for arithmetical computation, except for the case of  $x$  equal to zero, or from zero to a quarter or a half of the depth. The previous method happily gives the solution for small values of  $x$ , and indeed for values up to two or three times the depth, by very rapidly converging series, and thus between the two methods we have a remarkably satisfactory solution of the whole problem.

Before explaining the curves and their relation to the problem of the single ridge, I shall give the new direct solution

of this problem. It is founded on a well-known analytical method of Cauchy's, of which examples are given in the Eighteenth note (p. 284) to his Memoir on the Theory of Waves\*.

First, bring the denominator of (23) to the form of the product of an infinite number of quadratic factors, as follows:—  
Let

$$W = \frac{1}{2\left(1 - \frac{D}{b}\right)} \left\{ \epsilon^\sigma + \epsilon^{-\sigma} - \frac{D}{b\sigma} (\epsilon^\sigma - \epsilon^{-\sigma}) \right\} \quad (24).$$

Expanding in powers of  $\sigma$ , we have

$$W = 1 + \frac{1}{1 - \frac{D}{b}} \left\{ \frac{1}{1.2} \left(1 - \frac{1}{3} \frac{D}{b}\right) \sigma^2 + \frac{1}{1.2.3.4} \left(1 - \frac{1}{5} \frac{D}{b}\right) \sigma^4 + \&c. \right\} \quad (25).$$

Hence, when  $b$  is greater than  $D$ ,  $W$  is positive for all real values of  $\sigma$ . But when  $b$  has any positive value less than  $D$ ,  $W$  (which is always positive for small values of  $\sigma^2$ ) is negative for large values of  $\sigma^2$ ; and therefore at least one positive value of  $\sigma^2$  makes  $W$  zero. We shall see presently that only one positive value of  $\sigma^2$  does so. We shall see that all the zeros of  $W$  when  $b$  is less than  $D$ , and all but one when  $b$  is greater than  $D$ , correspond to real negative values of  $\sigma^2$ . This indeed is obvious if for  $\sigma^2$  we put  $-\theta^2$ , which gives

$$W = \frac{2}{1 - \frac{D}{b}} \left( \cos \theta - \frac{D}{b} \frac{\sin \theta}{\theta} \right) \quad (26);$$

and which shows that the zeros of  $W$  are given by the roots of the well-known transcendental equation

$$\frac{\tan \theta}{\theta} = \frac{b}{D} \quad (27).$$

When  $b$  is greater than  $D$  this equation has all its roots real, and in the first, third, fifth, &c. quadrants. When  $b$  is less than  $D$  the root in the first quadrant is lost, and in its stead we clearly have a pure imaginary; while the roots in the third, fifth, &c. quadrants remain real. Let  $\theta_1, \theta_2, \theta_3$ , &c. be the roots of the first, third, fifth, &c. quadrants. As the first term of equation (25) is unity, we have

\* *Mémoires de l'Académie Royale de l'Institut de France, savans étrangers*, tome i. (1827).



$$\text{or } \left. \begin{aligned} W &= \left(1 - \frac{\theta^2}{\theta_1^2}\right) \left(1 - \frac{\theta^2}{\theta_2^2}\right) \left(1 - \frac{\theta^2}{\theta_3^2}\right) \&c. \\ W &= \left(1 + \frac{\sigma^2}{\theta_1^2}\right) \left(1 + \frac{\sigma^2}{\theta_2^2}\right) \left(1 + \frac{\sigma^2}{\theta_3^2}\right) \&c. \end{aligned} \right\} \quad (28);$$

where  $\theta_2^2$ ,  $\theta_3^2$ , &c. are real positive numerics, while  $\theta_1^2$  is real positive or real negative according as  $b$  is greater than  $D$  or less than  $D$ .

Resolving now the reciprocal of  $W$  into partial fractions, we find

$$\frac{1}{W} = \frac{N_1}{1 + \frac{\sigma^2}{\theta_1^2}} + \frac{N_2}{1 + \frac{\sigma^2}{\theta_2^2}} + \frac{N_3}{1 + \frac{\sigma^2}{\theta_3^2}} + \&c. \quad (29);$$

where

$$\begin{aligned} N_i &= \frac{-1}{\theta_i^2 \left[ \frac{dW}{d(\theta^2)} \right]_i} = \frac{-2}{\theta_i \left( \frac{dW}{d\theta} \right)_i} = \frac{(1-D/b) \cos \theta_i}{D/b - \cos^2 \theta_i} \\ &= \frac{(1-D/b) \sin \theta_i}{\theta_i (1-b/D \cdot \cos^2 \theta_i)} \quad (30). \end{aligned}$$

For  $i=1$  and  $D > b$ ,  $\theta_i$  is, as we have seen, imaginary (its square real negative), and for this case the formula (30) may be conveniently written

$$N_1 = -\frac{\frac{1}{2}(D/b-1)(\epsilon^{\sigma_1} + \epsilon^{-\sigma_1})}{D/b - \frac{1}{2} - \frac{1}{4}(\epsilon^{2\sigma_1} + \epsilon^{-2\sigma_1})} \quad (31);$$

and the equation for finding  $\sigma_1$  is

$$\epsilon^{\sigma_1} + \epsilon^{-\sigma_1} - \frac{D}{b\sigma_1}(\epsilon^{\sigma_1} - \epsilon^{-\sigma_1}) = 0 \quad (32),$$

an equation which has one, and only one, real root when  $D > b$ , and no real root when  $D < b$ .

When  $b/D$  is given, it is easy to find, as the case may be,  $\sigma_1$  of (32) or  $\theta_1$  the first-quadrant root of (27), by arithmetical trial and error; and the successive roots  $\theta_2$ ,  $\theta_3$ , &c. more and more easily, by the solution of (27). It is to be remarked that, whatever be the value of  $b/D$ , these roots approach more and more nearly to the superior limits of the quadrants in which they lie: thus if we put

$$\theta_i = (i - \tfrac{1}{2})\pi - \alpha_i \quad (33),$$

we have

$$\begin{aligned} N_i \theta_i &= (-1)^{i+1} \frac{(1-D/b) \sin \alpha_i}{D/b - \sin^2 \alpha_i} [(i - \tfrac{1}{2})\pi - \alpha_i] \\ &= (-1)^{i+1} \frac{(1-D/b) \cos \alpha_i}{1 - b/D \sin^2 \alpha_i} \quad (34); \end{aligned}$$

and

$$\sin \alpha_i [(i - \frac{1}{2})\pi - \alpha_i] = D/b \cdot \cos \alpha_i \quad . \quad . \quad . \quad (35);$$

or, as is convenient for approximation, when  $i$  is very large,

$$\alpha_i [(i - \frac{1}{2})\pi - \alpha_i] = D/b \cdot \frac{\alpha_i}{\tan \alpha_i} \quad . \quad . \quad . \quad (36),$$

which shows that as  $i$  increased to infinity, the value of  $\alpha_i$  approaches asymptotically to  $D/b (i - \frac{1}{2})\pi$ . Hence when  $i$  is very large, the second member of (36) becomes approximately  $D/b \cdot (1 - \frac{1}{2} \alpha_i^2)$ ; and the equation becomes

$$(1 - \frac{1}{2} D/b) \alpha_i^2 - (i - \frac{1}{2}) \pi \alpha_i = -D/b \quad . \quad . \quad (37);$$

a quadratic, of which the smaller root when  $D$  is less than  $3b$ , and the positive root when  $D$  is greater than  $3b$ , is the required value of  $\alpha_i$ .

Going back now to (23) and modifying it by (24) and (29), we have

$$\eta = \frac{A/D\pi}{1-D/b} \cdot \sum N_i \int_0^\infty d\sigma \frac{\cos \frac{x\sigma}{D}}{1 + \frac{\sigma^2}{\theta_i^2}} \quad . \quad . \quad . \quad (38);$$

or, according to the well-known evaluation (attributed by Cauchy to Laplace) of the definite integral indicated,

$$\eta = \frac{\frac{1}{2} A/D}{1-D/b} \cdot \sum \theta_i N_i e^{-\frac{\theta_i x}{D}} \quad . \quad . \quad . \quad (39);$$

or with  $\theta_i$ ,  $N_i$  eliminated by (33) and (34),

$$\eta = \frac{1}{2} A/D \cdot \sum \frac{(-1)^{i+1} \cos \alpha_i}{(1-b/D \cdot \sin^2 \alpha_i)} e^{-\frac{[(i-\frac{1}{2})\pi - \alpha_i]x}{D}} \quad . \quad . \quad (40),$$

where  $\alpha_1, \alpha_2, \dots \alpha_i$  denote all the positive roots of (35).

This series converges with exceeding rapidity when  $x$  is any thing greater than  $D$ , and with very convenient rapidity for calculation when  $x$  is even as small as a tenth of  $D$ . When  $x=0$ , the convergence has the same order as that of  $1-e+e^2-\&c.$ , when  $e=1$ ; and we find the sum by taking as remainder half the term after the last term included. The true value of the sum is intermediate between the values which we obtain by this rule for a certain number of terms, and then for one term more. When it is desired to obtain the result with considerable accuracy, a large number of terms would be required; and it will no doubt be preferable to use my first method as indicated above.

It remains to deal with the first term for the case  $D > b$ , which makes it imaginary in the form (39), but real in the form (38) with  $-\sigma_1^2$  substituted for  $\theta_1^2$ . For this case we

have, by the well-known definite integral, first, I believe, evaluated by Cauchy,

$$\eta_1 = \frac{\frac{1}{2}A/D}{1-D/b} \cdot \sigma_1 N_1 \sin \frac{\sigma_1 x}{D} \quad . \quad . \quad . \quad (41);$$

where  $\sigma_1$  and  $N_1$  are given by (32) and (31).

It is to be remarked that, inasmuch as (38) has the same value for equal positive and negative values of  $x$ , the evaluations expressed in (39) and (41) are essentially discontinuous at  $x=0$ ; and when  $x$  is negative,  $-x$  must be substituted for  $x$  in the second member of the formulas. I hope in Part IV. to give numerical illustrations; but with or without numerical illustrations, the analytical formula (39), with (41) for its first term and the sign of  $x$  changed throughout when  $x$  is negative, is particularly interesting as a discontinuous expression for a curve passing continuously from one to the other of the two curves

$$\left. \begin{aligned} y &= \frac{\frac{1}{2}A/D}{1-D/b} \cdot \sigma_1 N_1 \sin \frac{\sigma_1 x}{D} \text{ for large positive values of } x \\ \text{and} \\ y &= -\frac{\frac{1}{2}A/D}{1-D/b} \cdot \sigma_1 N_1 \sin \frac{\sigma_1 x}{D} \text{ for large negative values of } x \end{aligned} \right\} (42).$$

For the case of  $b > D$  every term of (39) is real, and (remembering that the sign of  $x$  is changed when  $x$  is negative) we see that it makes  $\eta$  equal for equal positive and negative values of  $x$ , and diminish asymptotically to zero as  $x$  becomes greater and greater in either direction. It expresses unambiguously the solution (clearly unique when  $b > D$ ) of the problem of steady motion of water in a uniform rectangular canal interrupted only by a single ridge of magnitude  $A$  across the bottom. This is the case of velocity of flow greater than that acquired by a body in falling through a height equal to half the depth.

It is otherwise in respect to uniqueness of the solution when the velocity of flow is less than that acquired by a body in falling through a height equal to half the depth ( $b < D$ ). For this case the formulas (39) and (41) express a particular solution of the problem of steady motion through a rectangular canal, when regularity of the canal is only interrupted by the single ridge of magnitude  $A$ . But we clearly have an infinite number of solutions of this problem; because in still water in a canal of depth  $D$  we can have free waves of any velocity from zero to  $\sqrt{gD}$ , which is the velocity of an infinitely long wave in water of depth  $D$ . In our flowing water then superimpose upon the solution (39) (41), any

wave-motion of arbitrary magnitude, and arbitrarily chosen position for one of the zeros, with wave-length such that the velocity of wave-propagation is  $U$ , and the direction of motion such as to cause the progression of the wave to be up-stream. The wave-motion thus instituted constitutes a set of free stationary waves, and the superposition of this upon the case of motion represented by our symmetrical solution constitutes the general solution of the problem of single-ridge steady motion. To find the arbitrary addition which we must thus make to our symmetrical solution to find the general solution, put (13) into the following form :

$$\frac{2H}{\mathfrak{H}} = \epsilon^{mD} + \epsilon^{-mD} - \frac{g}{mU^2} (\epsilon^{mD} - \epsilon^{-mD}) \quad \dots \quad (43).$$

This shows that if  $H=0$ ,  $\mathfrak{H}$  may have any value (that is to say, we may have stationary waves of any magnitude over a plane bottom) if

$$\epsilon^{mD} + \epsilon^{-mD} - \frac{g}{mU^2} (\epsilon^{mD} - \epsilon^{-mD}) = 0 \quad \dots \quad (44).$$

This is in fact the well-known equation to find the velocity  $U$  relatively to the water, of periodic waves of wave-length  $2\pi/m$  in a canal of depth  $D$ . For us at present equation (44) is to be looked upon as a transcendental equation for determining the wave-length corresponding to  $U$  a given velocity of progress ; and it has, as we have seen, only one real root when  $U < \sqrt{gD}$  ; but no real root when  $U > \sqrt{gD}$ . Putting now in (43)  $U^2 = gb$ , and comparing with (32), we see that  $mD = \sigma_1$  ; and going back to equation (2) above we see that

$$\mathfrak{H} \cos \frac{\sigma_1(x-a)}{D} \quad \dots \quad (45) ;$$

where  $\mathfrak{H}$  and  $a$  are arbitrary constants, is the addition which we must make to (39) to give the general solution for the case  $b < D$ . Putting together this and (39) and (40), we accordingly have for the general solution of the single-ridge steady-motion problem, for the case of  $U < \sqrt{gD}$ ,

$$\left. \begin{aligned} \mathfrak{h} &= C \cos \frac{\sigma_1 x}{D} + (C' + \frac{\frac{1}{2}A/D}{1-D/b} \cdot \sigma_1 N_1) \sin \frac{\sigma_1 x}{D} + \frac{\frac{1}{2}A/D}{1-D/b} \cdot \sum_2^{\infty} \theta_i N_i \epsilon^{-\frac{\theta_i x}{D}} \\ &\quad \text{when } x \text{ is positive, and} \\ \mathfrak{h} &= C \cos \frac{\sigma_1 x}{D} + (C' - \frac{\frac{1}{2}A/D}{1-D/b} \cdot \sigma_1 N_1) \sin \frac{\sigma_1 x}{D} + \frac{\frac{1}{2}A/D}{1-D/b} \cdot \sum_2^{\infty} \theta_i N_i \epsilon^{-\frac{\theta_i x}{D}} \\ &\quad \text{when } x \text{ is negative} \end{aligned} \right\} \quad (46) ;$$

where  $C$  and  $C'$  denote arbitrary constants.

The motion represented by this solution, with any values of  $C$  and  $C'$ , is steady and stable throughout any finite length of the canal on each side of the ridge, provided the water is introduced at one end of the portion considered and taken away at the other conformably. If the canal extends to infinity in both directions, and if the water throughout be given in the state of motion corresponding to the solution (46); the motion throughout any finite distance on each side of the ridge will continue for an infinite time conformable to (46). The water, if given at rest, might be started into this state of motion in the following manner:—First displace its surface to the shape represented by equation (46), and apply a rigid corrugated lid to keep it exactly in this shape, so that it is now enclosed as it were in a rectangular tube with one side corrugated, two sides plane, and the fourth side (the bottom) plane, except at the place of the ridge. Next by means of a piston set the water gradually in motion in this tube. To begin with, the pressure on the lid will, in virtue of gravity, be non-uniform; less at the high parts and greater at the low parts. If too great a velocity be given to the water by the piston the pressure will, in virtue of fluid motion, be greater at the high parts and less at the low parts. If the average velocity be made exactly  $U$ , the pressure will be uniform over the lid, which may then be dissolved; thus the liquid is left moving steadily under the surface represented by equation (46) as free surface. But it is only in virtue of this motion being given to the fluid throughout an infinite length of the canal on each side of the ridge, that the motion can remain steady on each side of the ridge conformable to (46), except for the particular case of this general solution, corresponding to

$$C=0 \quad \text{and} \quad C' = \frac{\frac{1}{2}A/D}{1+D/b} \cdot \sigma_1 N_1 \quad . \quad . \quad . \quad (47),$$

which reduces (46) to

$$\left. \begin{aligned} \eta &= \frac{A/D}{1+D/b} \left( \sigma_1 N_1 \sin \frac{\sigma_1 x}{D} + \frac{1}{2} \sum_2^{\infty} \theta_i N_i \epsilon^{\frac{\theta_i x}{D}} \right) \quad \text{when } x \text{ is positive} \\ \text{and,} \quad \eta &= \frac{\frac{1}{2}A/D}{1+D/b} \sum_2^{\infty} \theta_i N_i \epsilon^{\frac{\theta_i x}{D}} \quad \text{when } x \text{ is negative} \end{aligned} \right\} (48);$$

this being the practical solution for the case of water flowing from the side of  $x$  negative over the single ridge and towards the side of  $x$  positive. It is the mathematical realization, for the case of a single ridge, of the circumstances described in Part I. No. 1 (*ante*, pp. 356–357), and is the mathematical

solution promised in the last sentence of Part II. The demonstration that this is the practically unique solution for inviscid water flowing in a canal with a single ridge, and the explanation of how any other state of motion, such, for example, as that represented by (46) with any value of  $C$  and  $C'$ , but given to the water throughout only a finite distance on each side of the ridge, settles into the permanent steady motion represented by (48), must be reserved for Part IV., which I hope will appear in the January number.

Meantime the accompanying diagram represents by two curves two cases of the solution (46) for the particular value  $2.456$  of  $D/b$ ; that is to say, for velocity  $= .6381$  of the critical velocity  $\sqrt{gD}$ . The faint curve represents the solution (39) (41), or, which is the same, (46) with  $C=0$  and  $C'=0$ . The heavy curve represents the practical solution (48). These curves were drawn from calculations of a periodic solution, according to the first of the two methods indicated above, before I had found the analytical solution (39) by which the desired result could have been arrived at with much less labour. The faint curve was drawn first by direct calculation from the periodic solution: the letters  $\frac{1}{2}l$ ,  $\frac{1}{4}l$ ,  $-\frac{1}{2}l$ ,  $-\frac{1}{4}l$ , &c., show on the two sides of one ridge quarters of the distance from ridge to ridge in the periodic solution, one of the ridges being in the middle of the diagram. The heavy curve is found by adding to the ordinates of the faint curve the ordinates of a curve of sines, found by trial to as nearly as possible annul on the one side, and to double on the other side, the ordinates of the original curve. How nearly perfect was the annulment on the one side and the doubling on the other is illustrated by the small-scale diagram annexed (fig. 3), which has been drawn by the engraver from a six times larger copy. How nearly perfect the annulment and the

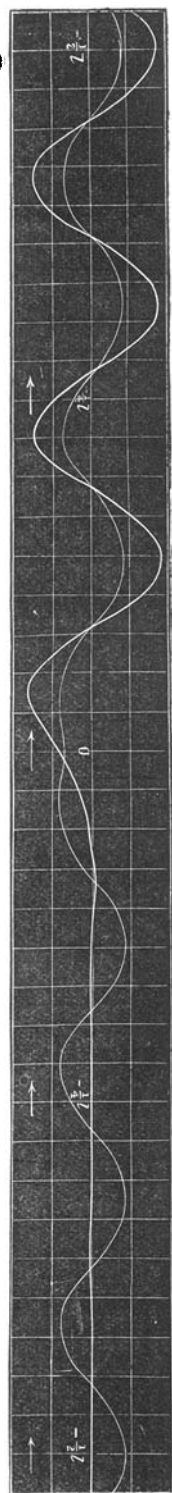


Fig. 3.

doubling ought to be at any particular distance from a single ridge is now easily calculated from the second line of equation (48), and will be actually calculated for the case of these curves, and probably also for some other cases for numerical illustrations, which I hope to give in Part IV.

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LXI. *Decomposition of Glass by Carbon Dioxide held in Solution in Capillary Films of Water.* By Prof. R. BUNSEN\*.

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IN an earlier publication † I have given my investigations of the phenomena which present themselves when carbon dioxide is allowed to act on capillary glass threads covered with an extremely thin film of moisture. According to these investigations, it appears that 49·453 grammes of such capillary threads are able in 109 days to take up so much carbon dioxide, that on heating not less than 236·9 cubic centim. of this gas is set free. The gas so retained in the water-film, showed towards pressure and temperature precisely the relations which are presented in the ordinary phenomena of gas absorption by liquids. In these experiments, as in all that have been previously carried out, it has been assumed, both from the result of direct observation and on theoretical grounds, that the action of carbon dioxide on glass may be entirely disregarded. And, indeed, experiments were carried out in my laboratory seventeen years ago by Dr. Emmerling, which showed that glass vessels in which an 11 per cent. solution of hydrochloric acid was boiled for hours together did not lose 0·0005 grm. in weight. If, in addition to this, we bear in mind that under ordinary atmospheric pressure, at 15° C., water dissolves only 0·2 per cent. by weight of carbon dioxide, an acid which is set free from all its compounds even by the weakest acids, and, further, that repeated observations show that dry carbon dioxide has practically no action upon dry glass, then it must appear almost absurd to attempt to explain the gradual fixation of carbon dioxide on glass dried by calcium chloride by a chemical decomposition of the glass.

But the matter presents itself under quite a different aspect when we have regard to the phenomena of absorption as occurring in capillary films. Water which at 15° C. and

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† Wied. *Ann.* xxiv. p. 321 (1885).