NOTE ON THE VALUES OF n WHICH MAKE $\frac{d}{dx}\{P_n^{-m}(x)\}$ VANISH AT $x = a$

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IN a paper "On the Scattering of Sound Waves by a Cone,"* Carslaw uses the fact that the roots of the equation in n, $\frac{d}{dx}P_n^{-m}(x) = 0$ when $x = a$, are real and separate. In a footnote he calls attention to the need for a proof of the theorem.

Such a proof is easily derived from the theory of the Homogeneous Integral Equation.

Consider the differential equation

$$\frac{d}{dt}\left[(1-t^2)\frac{dy}{dt}\right] - m^2(1-t^2)^{-1}y = 0. \tag{1}$$

Solutions are $\{(1-t)/(1+t)\}^{m/2}$, $\{(1+t)/(1-t)\}^{m/2}$

From these are derived

$$Z_1(t) = \{(1+a)(1-t)/(1-a)(1+t)\}^{m/2} + \{(1-a)(1+t)/(1+a)(1-t)\}^{m/2},$$

$$Z_2(t) = \{(1+a)(1-t)/(1-a)(1+t)\}^{m/2}.$$

Consider now $K(x, t)$ defined by the equations

$$K(x, t) = Z_1(t)\,Z_2(x) \quad (t < x),$$

$$K(x, t) = Z_2(t)\,Z_1(x) \quad (t > x).$$

If

$$a < x < 1,$$

$K(x, t)$ is a continuous solution of (1) satisfying the conditions

$$K(x, t) = 0 \quad \text{when} \quad t = 1,$$

$$\frac{\partial}{\partial t}K(x, t) = 0 \quad \text{when} \quad t = a.$$

* *Math. Annalen*, Vol. 75 (1914), p. 143.

Further, $\frac{\partial}{\partial t} K(x, t)$ has a discontinuity at $t = x$, where

$$(1-t^2) \frac{\partial}{\partial t} K(x, t) \Big|_{t=x+}^{t=x-} = 2m.$$

Again, $P_n^{-m}(t)$ is a solution, vanishing at $t = 1$, of

$$\frac{d}{dt}\left[(1-t^2) \frac{d}{dt} P_n^{-m}(t)\right] - m^2(1-t^2)^{-1} P_n^{-m}(t) = -n(n+1) P_n^{-m}(t). \quad (2)$$

Also, if n has one of the required values,

$$\frac{d}{dt} P_n^{-m}(t) = 0 \quad \text{at} \quad t = a.$$

Adopting Hilbert's* well known method we obtain from (1) and (2) the integral equation

$$2mP_n^{-m}(x) = n(n+1) \int_a^1 K(x, t) P_n^{-m}(t)\, dt.$$

If $\lambda_1, \lambda_2, \ldots$ are the characteristic constants of this equation, the required values of n are the roots of the set of quadratic equations

$$n^2+n = 2m\lambda_s \quad (s = 1, 2, 3, \ldots).$$

Now $K(x, t)$ is a symmetric function of x and t, consequently λ_s is real. It follows that possible values of n are real or conjugate complex quantities.

The argument used by Macdonald† in his discussion of the roots of

$$P_n^{-m}(a) = 0,$$

shows that conjugate complex roots or multiple roots are impossible.

Hence the roots are real and separate.

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* See Hilbert, "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen," Chap. VII, or *Gött. Nachr.*, 1904, pp. 213 *et seq.*

† *Proc. London Math. Soc.*, Ser. 1, Vol. XXXI (1899), pp. 265–266.