ON CERTAIN FUNCTIONS DEFINED BY TAYLOR'S SERIES OF FINITE RADIUS OF CONVERGENCE

By E. W. BARNES.

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1. The function $g_\beta(x;\theta)$ is defined when $|x|<1$ by the Taylor's series

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+\theta)^\beta}.$$

When β is a positive integer the function can be derived from the case when $\beta = 1$ by differentiation with regard to θ. The function

$$g(x;\theta) = \sum_{n=0}^{\infty} \frac{x^n}{n+\theta}$$

has been separately studied by the author.*

We shall therefore assume in the present investigation that β is not equal to zero or a positive integer. The subsequent theory is a development of the investigation given in the author's memoir "On the Asymptotic Expansion of Integral Functions defined by Taylor's Series."† Some of the following results were originally communicated in that paper. On account of its length they were merely stated ‡ in brief without proofs; the complete investigation, with some extensions, is now given. I refer to the introduction to that paper for an account of the general history and literature of the subject.

We shall assume that θ is not zero or a negative integer; in such cases the function $g_\beta(x;\theta)$ evidently does not exist.

We shall also assume that, in the definition of $g_\beta(x;\theta)$,

$$(n+\theta)^\beta = \exp\{\beta \log(n+\theta)\},$$

wherein
$$0 < |I\{\log(n+\theta)\}| < \pi.$$

This definition completely specifies the function when $|x|<1$ and θ is not real and negative. In the latter case we may conveniently take

$$I\{\log(n+\theta)\} = \pi$$

when $(n+\theta)$ is negative. We thus arbitrarily specify at most only a finite number of terms of the series.

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* *Quarterly Journal of Mathematics*, Vol. XXXVII., pp. 289–313.

† *Philosophical Transactions of the Royal Society* (A), Vol. 206, pp. 249–297

‡ *Loc. cit.*, Parts IV. and VI.

We use throughout $I\{f(x)\}$ to denote the imaginary part of $f(x)$, $R\{f(x)\}$ denoting its real part. Thus the condition

$$0 < |I\{\log(n+\theta)\}| < \pi$$

is equivalent to
$$-\pi < \frac{1}{\iota} I\{\log(n+\theta)\} < \pi.$$

2. I propose to establish the following propositions:—

(1) The function $g_\beta(x;\theta)$ has a single singularity in the finite part of the plane. The singularity occurs at $x=1$, and is not an essential singularity.

(2) The function $g_\beta(x;\theta)-g_\beta(x;1)/x^{\theta-1}$ has no singularities in the finite part of the plane, and, if $|\log x| < 2\pi$, it admits the expansion

$$\frac{1}{x^\theta}\sum_{n=0}^{\infty}\frac{(\log x)^n}{n!}\{\zeta(\beta-n,\theta)-\zeta(\beta-n,1)\}.$$

(3) Near $x=1$, $g_\beta(x;\theta)$ is many-valued.

(4) The function $g_\beta(x;\theta)-\Gamma(1-\beta)(-\log x)^{\beta-1}x^{-\theta}$ is one-valued near $x=1$, and in the vicinity of this point admits the convergent expansion

$$\sum_{n=0}^{\infty}\frac{(x-1)^n}{x^{n+1}}\bar{\zeta}_{n+1}(\beta,\theta),$$

where $\bar{\zeta}_{n+1}(\beta,\theta)$ denotes the $(n+1)$-ple Riemann ζ function of equal parameters unity.

(5) If θ be not real, the function

$$g_\beta(x;\theta)+g_\beta(x^{-1};-\theta)\,e^{\mp\pi\iota\beta},$$

the negative or positive sign being taken as $I(\theta) >$ or < 0, is one-valued near $x=1$, and has no singularity at this point.

(6) If θ be not real and a positive or negative integer (zero included), $g_\beta(x;\theta)$ admits, when $|x|$ is very large, the asymptotic expansion

$$\left\{\frac{1}{(-\theta)^\beta}-g_\beta\left(\frac{1}{x};-\theta\right)\right\}e^{\mp\pi\iota\beta}+\frac{[\log(-x)]^{\beta-1}}{(-x)^\theta}\sum_{n=0}^{\infty}\frac{(-)^n\left(\dfrac{\pi}{\sin\pi\theta}\right)^{(n)}}{n!\,\Gamma(\beta-n)[\log(-x)]^n}.$$

The modification of the previous theorem, when θ is zero or a negative integer, will be indicated.

Spence's formulæ connecting the functions $\sum_{m=1}^{\infty}\frac{x^m}{m^n}$ and $\sum_{m=1}^{\infty}\frac{1}{x^m m^n}$ when n is an integer will be deduced.

It will be shown that the proposition (4) leads to the result previously obtained when $\beta=1$.

In Part II. of the paper similar results are established for the more general function $f_\beta(x;\theta)$, defined, when $|x|<1$, by the series

$$\sum_{n=0}^{\infty} \frac{x^n \chi(n+\theta)}{(n+\theta)^\beta},$$

when, outside a circle outside which the points $n+\theta$ ($n=0, 1, \ldots, \infty$) all lie, $\chi(x)$ admits the convergent expansion $\sum_{r=0}^{\infty} b_r/x^r$.

PART I.—*The Function* $g_\beta(x;\theta)$.

3. *To shew that* $g_\beta(x;\theta)$ *has no singularities except possibly on the real axis between* $x=1$ *and* $x=+\infty$, *the limits included.*

We have

$$\sum_{n=0}^{N-1} \frac{x^n}{(n+\theta)^\beta} = \int_0^\infty e^{-z} \sum_{n=0}^{N-1} \frac{(xz)^n}{(n+\theta)^\beta\, n!}\, dz$$

$$= \int_0^\infty e^{-z} G_\beta(xz;\theta)\, dz - \int_0^\infty e^{-z} \sum_{n=N}^{\infty} \frac{(xz)^n}{(n+\theta)^\beta\, n!}\, dz,$$

where
$$G_\beta(xz;\theta) = \sum_{n=0}^{\infty} \frac{(xz)^n}{(n+\theta)^\beta\, n!},$$

and the integration is along the real axis.

Now, when N is large

$$\left| \sum_{n=N}^{\infty} \frac{(xz)^n}{(n+\theta)^\beta\, n!} \right| < \eta_N \frac{|xz|^N}{(N-k)!}\, e^{|xz|},$$

where η_N tends to zero as N tends to infinity, if $k > R(-\beta)$. Hence

$$\left| \int_0^\infty e^{-z} \sum_{n=N}^{\infty} \frac{(xz)^n}{(n+\theta)^\beta\, n!}\, dz \right| < \eta_N |x|^N \int_0^\infty e^{-(1-|x|)z} \frac{z^N}{(N-k)!}\, dz,$$

and, if $\dfrac{|x|}{1-|x|} < 1$, this expression tends to zero as N tends to infinity.

Therefore, if $|x|$ be sufficiently small,

$$g_\beta(x;\theta) = \int_0^\infty e^{-z} G_\beta(xz;\theta)\, dz.$$

Now, when $|xz|$ is large, both Mr. Hardy and I have shown that

$$G_\beta(xz;\theta) = \left\{ \frac{e^{xz}}{(xz)^\beta} P(xz) + (-xz)^{-\theta} [\log(-xz)]^{\beta-1} Q(xz) \right\},$$

where $|P(xz)|$ and $|Q(xz)|$ tend to definite finite limits as $|xz|$ tends to infinity.

Therefore the integral $\int_0^\infty e^{-z} G_\beta(xz;\theta)\,dz$

is finite for all values of x such that $R(x) < 1$. It is evidently an analytic function of x for all such values, and therefore it represents the continuation of $g_\beta(x;\theta)$ for all such values of x.

Again, if $\int_0^\infty (B)$ denote an integral along an axis in the positive half of the z-plane, $\int_0^\infty (A)$ denoting the original integral along the positive half of the real axis,

$$\int_0^\infty (A) = \int_0^\infty (B)$$

when $R(x) < 1$; for they differ by an integral along a contour at infinity which vanishes. Therefore $\int_0^\infty (B)$ represents the continuation of $g_\beta(x;\theta)$ for all values of x for which it is finite and continuous.

By taking suitable directions for the B-integral, we see that $g_\beta(x;\theta)$ can be continued for all values of x such that $|\arg(1-x)| < \pi$, and that it has no singularities in this region. We therefore have the given theorem. The line $(1, \infty)$ serves as a cross-cut to render the function $g_\beta(x;\theta)$ one-valued.

4. We will now shew that *the function*

$$g_\beta(x;\theta) - \frac{g_\beta(x;1)}{x^{\theta-1}}$$

has no singularities in the finite part of the plane except $x = 0$, *and that, near* $x = 1$, *it admits the expansion*

$$\frac{1}{x^\theta} \sum_{n=0}^{\infty} \frac{(\log x)^n}{n!} \{\zeta(\beta-n,\theta) - \zeta(\beta-n,1)\}.$$

Let $1/L$ be an axis from the origin within 90° of the axis to the point a, and let L be the image of $1/L$ in the real axis. Then, if the integral be taken round a Gamma function contour embracing the axis L,

$$\frac{\iota\Gamma(1-\beta)}{2\pi} \int_L (-y)^{\beta-1} e^{-ay}\,dy = \frac{1}{a^\beta},$$

where $(-y)^{\beta-1}$ has a cross-cut along the axis L, and $\log(-y)$ is real when y is real and negative, and where a^β has a cross-cut along $-1/L$ (*i.e.*, the negative direction of the axis $1/L$), and is real when a is real and positive. We assume that a is not real and negative.

Consider the integral

$$I = \frac{\iota\Gamma(1-\beta)}{2\pi} \int_L (-y)^{\beta-1} \frac{e^{-y\theta}}{1-x e^{-y}}\,dy,$$

where the contour excludes the poles $\log x \pm 2n\pi\iota$ of the subject of integration. If θ be not real and negative, we can determine L so that it is within 90° of the axes to the points $\theta+n$, $n = 0, 1, 2, \ldots, \infty$. We have

$$I = \frac{\iota\Gamma(1-\beta)}{2\pi}\int_L (-y)^{\beta-1} e^{-y\theta} \sum_{n=0}^{N-1} x^n e^{-ny}\, dy + \frac{\iota\Gamma(1-\beta)}{2\pi}\int_L (-y)^{\beta-1} \frac{x^N e^{-y(\theta+N)}}{1-xe^{-y}}\, dy;$$

and therefore $$I = \sum_{n=0}^{N-1} \frac{x^n}{(n+\theta)^\beta} + I_N, \text{ let us say.}$$

If $|x| < 1$, the series tends to a definite finite limit as N tends to infinity.

Also, if $|x| < 1$, we shall have $R(\log x) < 0$.

If $\log x$ lies outside the contour and $|1-x|$ be small, we may near $y = 0$ deform the contour so that the minimum value of $|1-xe^{-y}|$ is finite and occurs when $y = \log x + \eta$, where $\eta > 0$, and so that for other values of y on the contour we have $R(y - \log x) > \eta$.

Then we shall have $$I_N = K e^{-\eta N} + L x^N,$$

where $|K|$ tends to a finite limit as N tends to infinity and $|L|$ tends to zero.

Therefore, if $\log x$ be outside the contour and $|1-x|$ be small, $|I_N|$ tends to zero as N tends to infinity, provided $|x| < 1$.

Hence, when these conditions hold,

$$I = g_\beta(x;\theta).$$

Hence

$$g_\beta(x;\theta) - \frac{g_\beta(x;1)}{x^{\theta-1}} = \frac{\iota\Gamma(1-\beta)}{2\pi}\int_L (-y)^{\beta-1} \frac{e^{-\theta y} - e^{-y}x^{1-\theta}}{1-xe^{-y}}\, dy$$

$$= -\frac{1}{x^{\theta-1}} \frac{\iota\Gamma(1-\beta)}{2\pi}\int_L (-y)^{\beta-1} e^{-y} \frac{1-(xe^{-y})^{\theta-1}}{1-xe^{-y}}\, dy. \quad \text{(A)}$$

For it is evident that any axis L as previously chosen is in the positive half of the y plane, and is therefore a possible axis when $\theta = 1$.

But in the latter integral the points $y = \log x \pm 2n\pi\iota$ are no longer singularities of the subject of integration: therefore we may drop the condition that such points shall lie outside the contour of integration. We shall assume that $x^{\theta-1}$ is completely specified, as will be the case if we assign a cross-cut along the negative half of the real axis.

The integral (A) is evidently finite and continuous when x takes any range of values limited by this cross-cut. It represents, therefore, the continuation of $g_\beta(x;\theta)-g_\beta(x;1)/x^{\theta-1}$ for all values of x so limited. Therefore this function has no singularities in the finite part of the plane except the singularity at the origin due to $x^{\theta-1}$.

5. Put $y=\log x+t$, and suppose that $|\log x|$ is small. The integral (A) may be written

$$-\frac{1}{x^\theta}\frac{\iota\Gamma(1-\beta)}{2\pi}\int(-\log x-t)^{\beta-1}\frac{e^{-t}-e^{-\theta t}}{1-e^{-t}}\,dt.$$

Expand the original contour so that it includes P, a parallel to the axis L from the point $\log x$, and so that it also includes a circle of radius $|\log x|$ whose centre is $y=\log x$. Change the specification of $(-y)^{\beta-1}$ so that it is unaltered on the contour, but has a cross-cut along the parallel inside the contour, so that

$$(-\log x-t)^{\beta-1}=(-t)^{\beta-1}\sum_{n=0}^{\infty}\frac{\beta-1)\ldots(\beta-n)}{n!}\left(\frac{\log x}{t}\right)^n$$

when $|t|>|\log x|$, and is the continuation of the function represented by the series when $|t|<|\log x|$. Now close up the contour till it embraces P, as the original contour embraced L. The integral in (A) will be unaltered in value by these operations.

Hence

$$g_\beta(x;\theta)-g_\beta(x;1)/x^{\theta-1}=-\frac{1}{x^\theta}\frac{\iota\Gamma(1-\beta)}{2\pi}\int_L(-\log x-y)^{\beta-1}\frac{e^{-y}-e^{-\theta y}}{1-e^{-y}}\,dy. \tag{B}$$

If now the bulb of the contour be a circle of radius $>|\log x|$ and centre $y=0$, and if the remainder of the contour be the double description of that part of the axis L outside this circle, we have on the contour

$$(-\log x-y)^{\beta-1}=(-y)^{\beta-1}\sum_{n=0}^{N}\frac{(\beta-1)\ldots(\beta-n)}{n!}\left(\frac{\log x}{y}\right)^n+R_N,$$

where $|R_N|$ tends to zero as N tends to infinity. Thus the integral (B) is equal to

$$-\frac{1}{x^\theta}\sum_{n=0}^{N}\frac{(\beta-1)\ldots(\beta-n)}{n!}(-\log x)^n\frac{\iota\Gamma(1-\beta)}{2\pi}\int_L(-t)^{\beta-n-1}\frac{e^{-t}-e^{-\theta t}}{1-e^{-t}}\,dt$$
$$-\frac{1}{x^\theta}\frac{\iota\Gamma(1-\beta)}{2\pi}\int_L R_N\frac{e^{-y}-e^{-\theta y}}{1-e^{-y}}\,dy$$
$$=\frac{1}{x^\theta}\sum_{n=0}^{N}\frac{(\beta-1)\ldots(\beta-n)}{n!}(-\log x)^n\frac{\Gamma(1-\beta)}{\Gamma(1-\beta+n)}\{\zeta(\beta-n,\theta)-\zeta(\beta-n,1)\}$$
$$-J_N\quad\text{(say)}$$
$$=\frac{1}{x^\theta}\sum_{n=0}^{N}\frac{(\log x)^n}{n!}[\zeta(\beta-n,\theta)-\zeta(\beta-n,1)]-J_N,$$

where $\zeta(s, \theta)$ denotes the simple Riemann ζ function of parameter unity.

6. Now, if r be so chosen that $R(\theta+r)$ is positive,

$$\frac{\zeta(\beta-n, \theta)}{n!} = \frac{\sum_{m=0}^{r-1}(\theta+m)^{n-\beta}}{n!} + \frac{\iota\Gamma(1+n-\beta)}{2\pi\Gamma(1+n)}\int_L (-x)^{\beta-n-1}\frac{e^{-(\theta+r)x}}{1-e^{-x}}\,dx.$$

The contour of the integral embraces the axis L and excludes the points $2n\pi\iota$ $(n \neq 0)$. Hence on the contour we may take the minimum value of $|x|$ to be k, where $k < 2\pi$.

Hence, when n is large and θ not real and a negative integer,

$$\left|\frac{\zeta(\beta-n, \theta)}{n!}\right| = K\frac{n^{-R(\beta}}{k^n}$$

where K is finite when n is very large.

The series

$$\sum_{n=0}^{N}\frac{(\log x)^n}{n!}\zeta(\beta-n, \theta)$$

therefore tends to a finite limit as n tends to infinity, provided

$$|\log x| < k < 2\pi.$$

Finally, therefore, if $\log x$ is defined by a cross-cut along the negative half of the real axis, if θ be not real and negative, and if $|\log x| < 2\pi$,

$$g_\beta(x;\theta) - g_\beta(x;1)/x^{\theta-1} = \frac{1}{x^\theta}\sum_{n=0}^{\infty}\frac{(\log x)^n}{n!}\{\zeta(\beta-n,\theta)-\zeta(\beta-n,1)\}.$$

By means of the relation

$$g_\beta(x;\theta-1) = \frac{1}{(\theta-1)^\beta} + xg_\beta(x;\theta)$$

we may enunciate the previous theorem with the narrower restriction that θ shall not be zero or a negative integer. Compare the investigation in § 9.

7. We proceed now to shew that $g_\beta(x;\theta)$ *has a single singularity in the finite part of the plane, that this singularity occurs at* $x = 1$, *and that at this point the function branches infinitely often.*

We have seen in § 4 that, provided β be not a positive integer, $|x| < 1$, and the contour excludes the points $\log x \pm 2n\pi\iota$ $(n = 0, 1, \ldots, \infty)$,

$$g_\beta(x;\theta) = \frac{\iota\Gamma(1-\beta)}{2\pi}\int_L (-y)^{\beta-1}\frac{e^{-y\theta}}{1-xe^{-y}}\,dy.$$

Suppose now that the contour includes $\log x$, but excludes the points

$$\log x \pm 2n\pi\iota \quad (n \neq 0),$$

so that, if $x = re^{\iota\phi}$,

$$|(\phi \pm 2\pi)/\log r - \arg L| > |\phi/\log r - \arg L|;$$

then the integral is equal to

$$g_\beta(x;\theta) - \Gamma(1-\beta)(-\log x)^{\beta-1}x^{-\theta}.$$

Now suppose that $|x| > 1$, and that $(-\log x)^{\beta-1}$ is made one-valued by a cross-cut along the axis of integration, $\log(-\log x)$ being real when $\log x$ is real and negative. The integral remains finite and continuous. Hence the equality

$$g_\beta(x;\theta) - \Gamma(1-\beta)(-\log x)^{\beta-1}x^{-\theta} = \frac{\iota\Gamma(1-\beta)}{2\pi}\int_L (-y)^{\beta-1}\frac{e^{-y\theta}}{1-xe^{-y}}\,dy \quad \text{(A)}$$

continues to hold good, even for values of x which are real and greater than unity, provided we regard $g_\beta(x;\theta)$ as representing the continuation of the function defined by the original Taylor's series where $|x| < 1$. Hence the function $g_\beta(x;\theta)$ has no singularities on the positive part of the real axis between $x = 1$ and $x = \infty$. It has, therefore (§ 3), a single singularity in the finite part of the plane, viz., at $x = 1$. Near this point the function is many-valued.

8. We proceed now to show that, *near* $x = 1$, *the function*

$$g_\beta(x;\theta) - \Gamma(1-\beta)(-\log x)^{\beta-1}x^{-\theta}$$

is one-valued and admits the expansion

$$\sum_{n=0}^{\infty}\frac{(x-1)^n}{x^{n+1}}\bar{\zeta}_{n+1}(\beta,\theta),$$

valid when $\left|\dfrac{x-1}{x}\right| < 1$. *We thus see that* $x = 1$ *is a singularity of specifiable branching of* $g_\beta(x;\theta)$.

From the equality (A) of the previous paragraph we obtain

$$g_\beta(x;\theta) - \Gamma(1-\beta)(-\log x)^{\beta-1}x^{-\theta}$$
$$= \sum_{n=0}^{N-1}\frac{\iota\Gamma(1-\beta)}{2\pi}\int_L (-y)^{\beta-1}\frac{e^{-y\theta}}{(1-e^{-y})^{n+1}}\frac{(x-1)^n}{x^{n+1}}\,dy$$
$$+ \frac{\iota\Gamma(1-\beta)}{2\pi}\int_L (-y)^{\beta-1}\frac{e^{-y\theta}}{1-xe^{-y}}\left\{\frac{x-1}{x(1-e^{-y})}\right\}^N dy.$$

The first series may be written

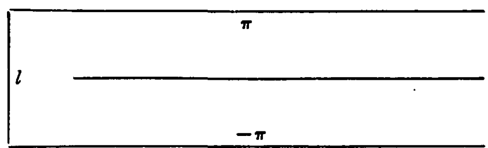
$$\sum_{n=0}^{N-1}\frac{(x-1)^n}{x^{n+1}}\bar{\zeta}_{n+1}(\beta,\theta),$$

where $\bar{\zeta}_{n+1}(\beta, \theta)$ is the $(n+1)$-ple Riemann ζ function of equal parameters unity defined by the integral

$$\frac{\iota \Gamma(1-\beta)}{2\pi} \int_L (-y)^{\beta-1} \frac{e^{-y\theta}}{(1-e^{-y})^{n+1}}\, dy.$$

We have assumed that θ is not real and negative.

Suppose now that $R(\theta) > 0$; then we may replace the contour L by a contour C, embracing the positive half of the real axis, which now serves as a cross-cut to make the function $(-\log x)^{\beta-1}$ one-valued.



Deform this contour till it consists of two lines above and below the real axis and distant π from it, and a line l parallel to the imaginary axis, cutting the real axis in a point whose distance from the origin is $> \log 2$ on the negative side of the origin.

Since $I\{\log x\}$ lies between $\pm \pi\iota$, the point $\log x$ can always be taken to lie within the contour.

The minimum value of $|1-e^{-y}|$ on the contour will be unity. For, if $y = (\cos\phi + \iota \sin\phi) r$, we have on the two infinite lines $r \sin\phi = \pm\pi$, and therefore $\cos(r \sin\phi) = -1$. Hence

$$|1-e^{-y}| = \sqrt{[1-2e^{-r\cos\phi}\cos(r\sin\phi)+e^{-2r\cos\phi}]} = 1+e^{-r\cos\phi} > 1;$$

and on the line l $\qquad e^{-r\cos\phi} > 2,$

and therefore $\qquad |1-e^{-y}| > \sqrt{[(2-1)^2]} > 1.$

Hence, when n is large

$$|\bar{\zeta}_{n+1}(\beta, \theta)| \leqslant \frac{\Gamma(1-\beta)}{2\pi\mu^{n+1}} \int |(-y)^{\beta-1}|\,|e^{-y\theta}|\,|dy| < \mu^{-(n+1)} K,$$

where $\mu > 1$ and K is finite and independent of n if β be not a positive integer and $R(\theta) > 0$.

Hence the series $\qquad \sum_{n=0}^{N-1} \frac{(x-1)^n}{x^{n+1}} \bar{\zeta}_{n+1}(\beta, \theta)$

tends to a definite finite limit as N tends to infinity if $|(x-1)/x| < 1$.

This can be otherwise seen since the integral

$$\frac{\iota\Gamma(1-\beta)}{2\pi} \int_C (-y)^{\beta-1} \frac{e^{-y\theta}}{1-xe^{-y}} \left\{\frac{x-1}{x(1-e^{-y})}\right\}^N dy$$

will tend to zero as N tends to infinity if $|(x-1)/x| < 1$.

Therefore, if $R(\theta) > 0$ and $|(x-1)/x| > 1$, and if the principal value of $(-\log x)^{\beta-1}$ [which is such that $\log(-\log x)$ is real when $\log x$ is real and negative and has a cross-cut along the positive half of the real axis,

$|I(\log x)|$ being less than π, since $|(x-1)/x| < 1]$ be taken,

$$g_\beta(x;\theta)-\Gamma(1-\beta)(-\log x)^{\beta-1}x^{-\theta} = \sum_{n=0}^{\infty}\frac{(x-1)^n}{x^{n+1}}\bar{\zeta}_{n+1}(\beta,\theta).$$

Thus the nature of the singularity of $g_\beta(x;\theta)$ at $x=1$ is given by

$$\Gamma(1-\beta)(-\log x)^{\beta-1}x^{-\theta}.$$

This singularity is not essential*, and is not even an infinity unless $R(\beta)<1$, or β is a positive integer, or we wind infinitely often round the point. When $\beta=0$, $g_\beta(x;\theta)=(1-x)^{-1}$, and the nature of its singularity near $x=1$ is given by $-x^{-\theta}/\log x$. This result, though somewhat paradoxical at first sight, is evidently true.

9. We will now *remove the limitation* $R(\theta)>0$ *introduced into the proof of the preceding proposition, and show that the theorem is true if* θ *be not zero or a negative integer.*

We evidently have

$$g_\beta(x;\ \theta-1) = \sum_{n=0}^{\infty}\frac{x^n}{(n-1+\theta)^\beta} = (\theta-1)^{-\beta}+xg_\beta(x;\ \theta).$$

Hence, by the preceding theorem, if $R(\theta)>0$,

$$g_\beta(x;\ \theta-1)-\Gamma(1-\beta)(-\log x)^{\beta-1}x^{-(\theta-1)}$$
$$= \frac{1}{(\theta-1)^\beta}+\sum_{n=0}^{\infty}\frac{(x-1)^n}{x^n}\bar{\zeta}_{n+1}(\beta,\theta).$$

For brevity put

$$u_{n+1}=\bar{\zeta}_{n+1}(\beta,\theta-1),\qquad v_{n+1}=\bar{\zeta}_{n+1}(\beta,\theta),$$

and
$$z=\frac{x-1}{x}.$$

I have elsewhere shewn† that

$$v_{n+1}=u_{n+1}-u_n.$$

[Let
$$S_N=\sum_{n=0}^{N}v_{n+1}z^n,\qquad S'_N=\sum_{n=0}^{N}u_{n+1}z^n.$$

* It must, of course, be counted as an essential singularity if we say that $\log x$ has an essential singularity at $x=0$. Essential singularity is defined in such a negative manner that it will probably be ultimately convenient to class such points as the one in question under another title.

† "The Theory of the Multiple Gamma Function," *Transactions of the Cambridge Philosophical Society*, Vol. XIX., pp. 374–425, § 26.

Then $S_N = \sum_{n=0}^{N} (u_{n+1} - u_n) z^n = -u_0 + \sum_{n=0}^{N} u_{n+1}(z^n - z^{n+1}) + u_{N+1} z^{N+1}$

$$= -u_0 + (1-z) S'_N + u_{N+1} z^{N+1}.$$

But $u_{N+1} = v_{N+1} + u_N = \ldots = v_{N+1} + v_N + \ldots + v_1 + u_0.$

Therefore $$|u_{N+1}| \leqslant \sum_{r=1}^{N+1} |v_r| + |u_0|.$$

We have seen that, when r is large and $R(\theta) > 0$,

$$|v_r| < K/\mu^{r+1},$$

where $\mu \geqslant 1$.* Hence, when N is large,

$$|u_{N+1}| < K'N,$$

where K' is finite, if β be not an integer and $R(\theta) > 0$.

Hence, if $|z| < 1$, $|u_{N+1} z^{N+1}|$ tends to zero as N tends to infinity. But S_N tends to a definite finite limit as N tends to infinity. Therefore the same is true of S'_N. Hence

$$\frac{1}{(\theta-1)^\beta} + \sum_{n=0}^{\infty} \left(\frac{x-1}{x}\right)^n \bar{\zeta}_{n+1}(\theta, \beta)$$
$$= \frac{1}{(\theta-1)^\beta} - u_0 + \frac{1}{x} \sum_{n=0}^{\infty} \left(\frac{x-1}{x}\right)^n \bar{\zeta}_{n+1}(\beta, \theta-1).]$$

And the latter series is convergent.

Now $u_0 = (\theta-1)^{-\beta}$. Therefore, if $R(\theta) > -1$, and θ be not zero, the theorem of the preceding paragraph is valid. Proceeding by successive stages, we shew that it is valid for all values of θ, provided θ be not zero or a negative integer.

10. If we compare the results of the preceding paragraphs with the expansion obtained in § 4, we see that when x is in the immediate vicinity of the point 1 we have the equality of the two expansions

$$\frac{1}{x^\theta} \sum_{n=0}^{\infty} \frac{(\log x)^n}{n!} \{\zeta(\beta-n, \theta) - \zeta(\beta-n, 1)\},$$

and $$\sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \{\bar{\zeta}_{n+1}(\beta, \theta) - x^{1-\theta} \bar{\zeta}_{n+1}(\beta, 1)\},$$

for each is convergent when $|x-1|$ is small and equal to

$$g_\beta(x;\ \theta) - x^{1-\theta} g_\beta(x;\ 1).$$

* The previous argument can be used to show that, when $R(\theta) > 0$ and β is not an integer, $|v_r|$ tends to zero as r tends to infinity.

The Case of $\beta = 1$.

11. I have previously* shewn that, when $\beta = 1$,

$$x^\theta g(x; \theta) + \log(1-x)$$
$$= \psi(1) - \psi(\theta) + x^\theta \sum_{n=1}^{\infty} \frac{(1-x)^n}{x^{n+1}} \frac{(\theta-1)\dots(\theta-n)}{n!} \left(\frac{1}{1} + \dots + \frac{1}{n}\right),$$

provided $|(1-x)/x| < 1$.

This result, at first sight, seems very different from that previously obtained for general values of β. It is now proposed to shew that as β tends to unity the result of § 8 leads to that just quoted.

It is necessary to introduce certain properties of $(n+1)$-ple Riemann ζ functions of equal parameters: these are taken from an unpublished chapter of a forthcoming book by the author on *Gamma Functions and Allied Transcendents*. The reader will, however, find little difficulty in deducing them from the author's memoirs dealing with the general multiple Riemann ζ function.†

If we put $\beta = 1-\epsilon$ in the result of the preceding paragraph, we obtain

$$g_{1-\epsilon}(x; \theta) - \Gamma(\epsilon)(-\log x)^{-\epsilon} x^{-\theta} = \sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \bar{\zeta}_{n+1}(1-\epsilon, \theta). \qquad \text{(A)}$$

Now $$\operatorname*{Lt}_{\epsilon=0} \left\{ \bar{\zeta}_{n+1}(1-\epsilon, \theta) + (-)^n \frac{{}_{n+1}\bar{S}_1^{(2)}(\theta)}{\epsilon} \right\} = -\bar{\psi}'_{n+1}(\theta),$$

when ${}_{n+1}\bar{S}_1(\theta)$ denotes the first $(n+1)$-ple Bernoullian function of θ of equal parameters unity, and $\bar{\psi}'_{n+1}(\theta)$ denotes

$$\frac{d}{d\theta} \log \{\bar{\Gamma}_{n+1}(\theta)\},$$

$\bar{\Gamma}_{n+1}(\theta)$ denoting the $(n+1)$-ple gamma function of equal parameters unity.

I have elsewhere shewn that‡

$${}_{n+1}\bar{S}_1^{(2)}(\theta) = \frac{(\theta-1)\dots(\theta-n)}{n!}.$$

Hence, if we expand the result (A) in ascending powers of ϵ, as is

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* *Quarterly Journal of Mathematics*, Vol. XXXVII., p. 308.

† *Loc. cit.*, § 9.

‡ *Transactions of the Cambridge Philosophical Society*, Vol. XIX., p. 431.

evidently legitimate if ϵ be very small, we get, on equating coefficients of $1/\epsilon$,

$$-x^{-\theta} = -\sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \, {}_{n+1}\bar{S}_1^{(2)}(\theta),$$

and, on equating the terms independent of ϵ,

$$g(x, \theta) + x^{-\theta} \log(-\log x) - \psi(1) x^{-\theta} = -\sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \bar{\psi}'_{n+1}(\theta).$$

The result of equating higher powers of ϵ is to give us the nature of the behaviour of functions

$$\sum_{n=0} \frac{x^n [\log(n+\theta)]}{n+\theta}$$

near $x = 1$.

12. The first result is equivalent to

$$x^{-\theta} = \sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \frac{(\theta-1)\dots(\theta-n)}{n!}$$

$$= \frac{1}{x}\left(1 + \frac{1-x}{x}\right)^{\theta-1}, \tag{1}$$

and is evidently true.

The second result may be written

$$g(x; \theta) + x^{-\theta} \log(1-x) - \psi(1) x^{-\theta}$$
$$= x^{-\theta} \log\left(\frac{1-x}{-\log x}\right) - \sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \bar{\psi}'_{n+1}(\theta).$$

We have then to prove that the right-hand side of this equality is equal to

$$-x^{-\theta}\psi(\theta) + \sum_{n=1}^{\infty} \frac{(1-x)^n}{x^{n+1}} \frac{(\theta-1)\dots(\theta-n)}{n!} \left(\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n}\right).$$

Put now $(x-1)/x = z$; then we have to shew, when $|z| < 1$, that

$$(1-z)^{\theta-1}\left[\psi(\theta) - \log\left\{\frac{(1-z)\log(1-z)}{-z}\right\}\right]$$
$$= \psi(\theta) + \sum_{n=1}^{\infty} z^n \left\{\bar{\psi}'_{n+1}(\theta) + \frac{(1-\theta)\dots(n-\theta)}{n!}\left(\frac{1}{1} + \dots + \frac{1}{n}\right)\right\}.$$

Now $\quad (-)^n \bar{\psi}'_{n+1}(\theta) = \sum_{k=1}^{n} \frac{(-)^k}{k!} \, {}_{n+1}\bar{S}_0^{(k+1)}(\theta) \frac{S'_k(\theta)}{k} + {}_{n+1}\bar{S}'_0(\theta)\, \psi'(\theta),$

where $\qquad S_k(\theta) = {}_1S_k(\theta) \quad$ and $\quad \psi(\theta) = \psi_1(\theta)$.

Therefore

$$\psi'_{n+1}(\theta) = \psi_1(\theta) \frac{(1-\theta)\dots(n-\theta)}{n!} + (-)^n \sum_{k=1}^{n} \frac{(-)^k}{k!} \, {}_{n+1}\bar{S}_0^{(k+1)}(\theta) \frac{S'_k(\theta)}{k}.$$

Using the identity (1), we see that the terms involving $\psi_1(\theta)$ vanish from the equality, and we have to establish that, if $|z| < 1$,

$$-(1-z)^{\theta-1}\log\left\{\frac{(1-z)\log(1-z)}{-z}\right\}$$
$$=\sum_{n=1}^{\infty}(-z)^n\left\{\sum_{k=1}^{n}\frac{(-)^k}{k!}\,{}_{n+1}\bar{S}_0^{(k+1)}(\theta)\,\frac{S_k'(\theta)}{k}+{}_{n+1}\bar{S}_0'(\theta)\left(\frac{1}{1}+\ldots+\frac{1}{n}\right)\right\}.$$

13. We will first shew that

$${}_{n+1}\bar{S}_0'(\theta)\left(\frac{1}{1}+\ldots+\frac{1}{n}\right)+\sum_{k=1}^{n}\frac{(-)^k}{k!}\,{}_{n+1}\bar{S}_0^{(k+1)}(\theta)\,\frac{S_k'(\theta)-S_k'(0)}{k}$$
$$+{}_{n+1}\bar{S}_0^{(2)}(\theta)=0.$$

Denote the function on the left-hand side of this equality by $F_{n+1}(\theta)$.

Then, since $${}_{n+1}\bar{S}_0'(\theta+1)={}_{n+1}\bar{S}_0'(\theta)+{}_n\bar{S}_0'(\theta)$$

and $$S_k'(\theta)-S_k'(0)=kS_{k-1}(\theta),$$

we have

$$F_{n+1}(\theta)+F_n(\theta)$$
$$=F_{n+1}(\theta+1)+\frac{1}{n}\,{}_n\bar{S}_0'(\theta)-\sum_{k=1}^{n}\frac{(-)^{k-1}}{k!}\,{}_{n+1}\bar{S}_0^{(k+1)}(\theta+1)\,\theta^{k-1}$$
$$=F_{n+1}(\theta+1)+\frac{1}{n}\,{}_n\bar{S}_0'(\theta)+\frac{1}{\theta}\left\{1-\exp\left(-\theta\frac{d}{d\theta}\right)\right\}{}_{n+1}\bar{S}_0'(\theta+1)$$

[for ${}_{n+1}\bar{S}_0'(\theta+1)$ is a polynomial in θ]

$$=F_{n+1}(\theta+1)+\frac{1}{n}\,{}_n\bar{S}_0'(\theta)-\frac{1}{\theta}\,{}_{n+1}\bar{S}_0'(\theta+1)$$
$$=F_{n+1}(\theta+1),$$

for $${}_{n+1}\bar{S}_0'(\theta)=\frac{(\theta-1)\ldots(\theta-n)}{n!}.$$

Now, $F_n(\theta)$ is a polynomial in θ. Therefore, if

$$F_n(\theta)=0,$$

we have $$F_{n+1}(\theta)=\text{constant}.$$

But, when $\theta=0$,

$$F_n(\theta)={}_n\bar{S}_0'(0)\left\{\frac{1}{1}+\ldots+\frac{1}{n-1}\right\}+{}_n\bar{S}_0^{(2)}(0)=0.$$

Therefore, if $$F_n(\theta) = 0,$$

we have $$F_{n+1}(\theta) = 0.$$

Now
$$\begin{aligned} F_1(\theta) &= {}_2\bar{S}_0'(\theta) + {}_2\bar{S}_0^{(2)}(\theta) - {}_2\bar{S}_0^{(2)}(\theta)\,S_0(\theta) \\ &= \theta - 1 + 1 - \theta \\ &= 0. \end{aligned}$$

Therefore, by induction, $$F_{n+1}(\theta) = 0.$$

14. We now have to shew, if $|z| < 1$, that

$$-(1-z)^{\theta-1}\log\left\{\frac{(1-z)\log(1-z)}{-z}\right\}$$
$$= \sum_{n=1}^{\infty}(-z)^n\left\{\sum_{k=2}^{n}(-)^k\,{}_{n+1}\bar{S}_0^{(k+1)}(\theta)\,\frac{S_k(0)}{k\,.\,k!} + S_1'(0)\,{}_{n+1}\bar{S}_0^{(2)}(\theta)\right\}.$$

The function on the left-hand side can evidently be expanded in a series of ascending powers of z, if $|z|$ be sufficiently small, and the coefficient of $(-z)^n$ is given by

$$\frac{1}{2\pi\iota}\int\frac{(1-z)^{\theta-1}}{(-z)^{n+1}}\log\left\{\frac{(1-z)\log(1-z)}{-z}\right\}dz,$$

taken round a small circle including $z = 0$, on which the subject of integration is one-valued.

Put $1-z = e^y$; then, when z makes a small circuit round the origin, y will do the same, and the integral becomes

$$-\frac{1}{2\pi\iota}\int\frac{e^{\theta y}}{(e^y-1)^{n+1}}\log\left\{\frac{e^y y}{e^y-1}\right\}dy.$$

Now, when y is small, $y+\log\dfrac{y}{e^y-1}$ admits the expansion $\sum\limits_{n=1}^{\infty} c_n y^n$.

Differentiating, we have

$$\sum_{n=1}^{\infty} nc_n y^{n-1} = 1 + \frac{1}{y} - \frac{e^y}{e^y-1} = 1 + \frac{1}{y} + \sum_{n=0}^{\infty}(-y)^{n-1}\frac{S_n'(0)}{n!}.$$

Hence $$1 + S_1'(0) = c_1;$$

and therefore $$c_1 = -S_1'(0),$$

and $$c_n = (-)^{n-1}\frac{S_n'(0)}{n\,.\,n!},\ \text{when } n > 1.$$

The integral is therefore equal to

$$-\frac{1}{2\pi\iota}\int\frac{e^{\theta y}}{(e^y-1)^{n+1}}\left\{-S_1'(0)\,y + \sum_{k=2}^{\infty}(-)^{k-1}\frac{S_k'(0)}{k\,.\,k!}\,y^k\right\}dy.$$

Now* $$\frac{e^{\theta y}}{(e^y-1)^{n+1}} = \sum_{s=1}^{n+1} \frac{{}_{n+1}\bar{S}_0^{(k)}(\theta)}{y^k}$$
+terms which are finite when y vanishes.

Therefore the integral is equal to

$$S_1'(0)\; {}_{n+1}\bar{S}_0^{(2)}(\theta) + \sum_{k=2}^{n} (-)^k\; {}_{n+1}\bar{S}_0^{(k+1)}(\theta)\, \frac{S_k'(0)}{k\,.\,k\,!}.$$

We thus have the required equality.

15. We proceed now to shew that, *if θ be not real, the function*

$$g_\beta(x\,;\,\theta) + g_\beta\left(\frac{1}{x};\,-\theta\right) e^{\mp\pi\iota\beta},$$

the positive or negative sign being taken according as $I(\theta)$ is $<$ or >0, is one-valued† *near $x=1$, and has no singularity at this point.*

In the investigation of § 7, we have seen that

$$g_\beta(x\,;\,\theta) - \Gamma(1-\beta)(-\log x)_L^{\beta-1} x^{-\theta} = \frac{\iota\Gamma(1-\beta)}{2\pi}\int_L (-y)^{\beta-1}\frac{e^{-y\theta}}{1-xe^{-y}}\,dy,$$

where $1/L$ is an axis within 90° of the points $\theta, \theta+1, \ldots, \theta+n, \ldots,$ and where $\arg(-\log x)_L$ lies between $-(\pi-\psi)\iota$ and $(\pi+\psi)\iota$, ψ being the angle between L and the positive half of the real axis, and ranging in value from $-\pi$ to π.

In exactly the same way we may prove that

$$g_\beta\left(\frac{1}{x};\,-\theta\right) - \Gamma(1-\beta)\left(-\log\frac{1}{x}\right)_\lambda^{\beta-1} x^{-\theta}$$
$$= \frac{\iota\Gamma(1-\beta)}{2\pi}\int_\lambda (-y)^{\beta-1}\frac{e^{y\theta}}{1-x^{-1}e^{-y}}\,dy,$$

where $1/\lambda$ is an axis within 90° of the points $-\theta, -\theta+1, \ldots, -\theta+n, \ldots,$ and where $\arg(-\log 1/x)_\lambda$ lies between $-(\pi-\phi)\iota$ and $(\pi+\phi)\iota$, ϕ being the angle between λ and the positive half of the real axis, and ranging in value from $-\pi$ to π, $-\log 1/x$ being real when x is real and positive.

In the former case L is a cross-cut for $\log x$ to make $g_\beta(x;\,\theta)$ one-valued; in the latter case $-\lambda$ is a cross-cut for $\log x$ to make $g_\beta(x^{-1};\,-\theta)$ one-valued.

These cross-cuts are in general not the same, but within the region common to the two expansions it is readily seen by constructing a

* *Cambridge Philosophical Transactions*, Vol. XIX., p. 378.

† Its actual value depends, of course, on which branches of the original functions we choose.

figure that, if $I(\theta) > 0$,

$$\arg(-\log 1/x)_\lambda = \arg(-\log x)_L + \pi\iota,$$

and, if $I(\theta) < 0$, $\arg(-\log 1/x)_\lambda = \arg(-\log x)_L - \pi\iota.$

When the final integrals are expressed by convergent series in powers of $(1-x)/x$, we may rotate the cross-cuts till they coincide along an imaginary axis; and then, since

$$(-\log x)_L^{\beta-1} + (-\log 1/x)_\lambda^{\beta-1} e^{\mp\beta\pi\iota} = 0$$

$$[-, I(\theta) > 0; \ +, I(\theta) < 0],$$

we see that $$g_\beta(x;\theta) + g_\beta(x^{-1}; -\theta)\, e^{\mp\pi\iota\beta}$$

may be represented by a series convergent when $|1-x|$ is small. Therefore this function is uniform near $x = 1$, and has no singularity at this point.

16. The preceding proposition indicates a close connection between the functions $g_\beta(x;\theta)$ and $g_\beta(x^{-1}; -\theta)$, the former of which can be expressed by a Taylor's series when $|x| < 1$, and the latter by a Taylor's series when $|x| > 1$.

We may readily shew that, when $|x| > 1$,

$$g_\beta\left(\frac{1}{x}; -\theta\right) e^{\mp\pi\iota\beta} - \frac{1}{\theta^\beta} = \sum_{n=1}^{\infty} \frac{1}{x^n(\theta-n)^\beta},$$

the $-$ or $+$ sign being taken according as $I(\theta) >$ or < 0, provided $(\theta-n)$ has values which correspond to a cross-cut along the negative half of the real axis.

For, by definition, when $|x| > 1$,

$$g_\beta\left(\frac{1}{x}; -\theta\right) = \sum_{n=0}^{\infty} \frac{1}{x^n(n-\theta)^\beta},$$

where $$|\arg(n-\theta)| < \pi$$

and $$(\theta-n)^\beta = (n-\theta)^\beta e^{\pm\pi\iota\beta},$$

according as $I(\theta) >$ or < 0.

17. We may now shew that, $g_\beta(x;\theta)$ *admits, when* $|x|$ *is very large, the asymptotic expansion*

$$-\sum_{n=1}^{\infty} \frac{1}{x^n(\theta-n)^\beta} + \frac{[\log(-x)]^{\beta-1}}{(-x)^\theta} \sum_{n=0}^{\infty} \frac{(-)^n \left(\dfrac{\pi}{\sin\pi\theta}\right)^{(n)}}{n!\,\Gamma(\beta-n)\,[\log(-n)]^n},$$

provided θ be not real, and $\log(-x)$ *be defined with respect to a cross-cut along the positive half of the real axis, being real when x is real and negative. If the argument of* $\log(-x)$ *is ϕ ($|\phi| < \pi$), so that*

$$\log(-x) = |\log(-x)|\, e^{\iota\phi},$$

the argument of $[\log(-x)]^{\beta-1}$ *is* $\phi[R(\beta)-1]+\log\{|\log(-x)|\}\, I(\beta)$.

If $|x| < 1$,

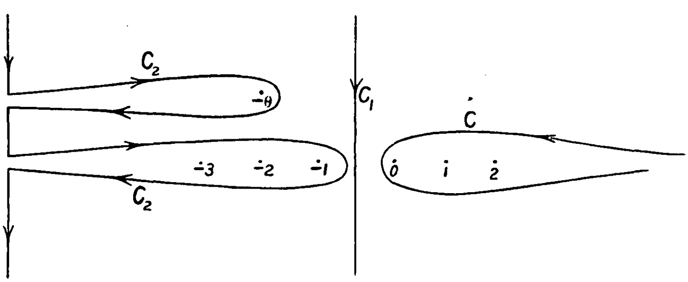
$$g_\beta(x\,;\,\theta) = \sum_{n=0}^{\infty} \frac{x^n}{(n+\theta)^\beta} = \frac{1}{2\pi\iota}\int_C \frac{\pi(-x)^s\, ds}{\sin \pi s\,(s+\theta)^\beta} \qquad \text{(A)}$$

where the contour C encloses the origin but not the points -1 or $-\theta$, and embraces the positive half of the real axis.

Let C_1 be a straight contour parallel to the imaginary axis, cutting the real axis between 0 and -1, with a loop, if necessary, to ensure that $-\theta$ lies to the left of this contour. Then, if $|\arg(-x)| < \pi$,

$$\int_C = \int_{C_1};$$

for the integral vanishes along the infinite contour which is the difference of the contours C and C_1. Hence the integral (A) taken along the contour C_1 represents the continuation of the function $g_\beta(x\,;\,\theta)$ for all values of x such that $|\arg(-x)| < \pi$, $|x|$ having any value greater than, equal to, or less than unity.



Suppose now that $|x| > 1$, and that C_2 is the contour of the figure. Then, if $|\arg(-x)| < \pi$, the integral along the contour C_1 will equal that along the contour C_2, and the integral along the straight parts of this contour at infinity will vanish.

We shall therefore have, if $|x| > 1$ and $|\arg(-x)| < \pi$,

$$g\ (x;\,\theta) = -\sum_{n=1}^{\infty} \frac{1}{x^n(\theta-n)^\beta} + \frac{\iota}{2\pi}\int_C (-x)^{-y-\theta}(-y)^{-\beta}\frac{\pi}{\sin\pi(y+\theta)}\, dy.$$

In obtaining the final integral we have employed the transformation $s = -y-\theta$. The values of $(\theta-n)^\beta$ correspond to a cross-cut ($-\theta$ to

$-\infty$) parallel to the real axis. In the final integral $\pi/\sin\pi(y+\theta)$ has no poles within or on the contour of integration. It is therefore represented by the summable divergent series

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\pi}{\sin\pi\theta}\right)^{(n)} y^n$$

on the contour, and, in fact, over the whole of the y-plane dissected by lines passing from the zeros of $\sin\pi(y+\theta)$ away from the origin to infinity.

Therefore, by a proposition proved in my original memoir,* this integral may be represented by the asymptotic series

$$\frac{\iota(-x)^{-\theta}}{2\pi}\sum_{n=0}^{N}\frac{(-)^n}{n!}\left(\frac{\pi}{\sin\pi\theta}\right)^{(n)}\int_C e^{-y\log(-x)}(-y)^{n-\beta}\,dy+J_N,$$

where $|J_N\{\log(-x)\}^{n-\beta+1}|$ tends to zero as $|\log(-x)|$ tends to infinity; and this series is equal to

$$(-x)^{-\theta}\sum_{n=0}^{N}\frac{(-)^n}{n!}\left(\frac{\pi}{\sin\pi\theta}\right)^{(n)}\frac{1}{\Gamma(\beta-n)[\log(-x)]^{n-\beta+1}}\tag{1}$$

where $$[\log(-x)]^{n-\beta}=\exp\{(n-\beta)\log[(-x)]\},$$

where, when $\log(-x)$ is specified, $\arg\{\log(-x)\}$ lies between $\pm\pi$, and is zero when $\log(-x)$ is real and positive.

We thus have the given expansion.

We notice that, when $R(\theta)<1$, the series (1) gives the asymptotic expansion near $|x|=\infty$ of $g_\beta(x;\theta)$. When $R(\theta)>1$, let ν be the integer next greater than $R(\theta)$. Then the asymptotic expansion of $g_\beta(x;\theta)$ is

$$\sum_{n=1}^{\nu-1}\frac{1}{x^n(\theta-n)^\beta}+\text{the series (1).}$$

18. From the previous theorem we see that, when $|x|$ is very large and θ is not real, we have the asymptotic equality

$$g_\beta(x;\theta)+\left\{g_\beta\left(\frac{1}{x};\,-\theta\right)-\frac{1}{(-\theta)^\beta}\right\}e^{\mp\pi\iota\beta}$$
$$=\frac{[\log(-x)]^{\beta-1}}{(-x)^\theta}\sum_{n=0}^{\infty}\frac{(-)^n\left(\dfrac{\pi}{\sin\pi\theta}\right)^{(n)}}{n!\,\Gamma(\beta-n)[\log(-x)]^n},\tag{1}$$

the $-$ or $+$ sign being taken as $I(\theta)>$ or <0.

* *Philosophical Transactions of the Royal Society* (A), Vol. 206, pp. 249–297, § 5.

Changing x into $1/x$, we get the asymptotic expansion of $g(1/x;\theta)$ at the origin. The relation (1) holds when $|\log(-x)|$ is very large, that is, whether $|x|$ or $1/|x|$ be very large. It is easy to verify that the preceding formula remains unchanged when we change x into $1/x$ and θ into $-\theta$.

The series (1) becomes convergent when we multiply the general term by $1/n!$: it is therefore what I have elsewhere* proposed to call an asymptotic series in $\log(-x)$ of the second order, similar to the well known series for $\log\Gamma(z+a)$ when $|z|$ is large.

19. *When β is a positive integer and θ is not real, the previous investigation will hold.*

But now the asymptotic series (1) is replaced by the finite series

$$\frac{[\log(-x)]^{\beta-1}}{(-x)^{\theta}}\sum_{n=0}^{\beta-1}\frac{(-)^n\left(\dfrac{\pi}{\sin\pi\theta}\right)^{(n)}}{n!\,\Gamma(\beta-n)[\log(-x)]^n};$$

for this series is the residue of $\dfrac{\pi(-x)^s}{\sin\pi s(s+\theta)^{\beta}}$ at $s=-\theta$. Thus, when θ is not real and β is a positive integer, we have the absolute equality

$$g_{\beta}(x;\theta)+\left\{g_{\beta}\left(\frac{1}{x};\ -\theta\right)-\frac{1}{(-\theta)^{\beta}}\right\}e^{\mp\pi i\beta}$$
$$=\frac{[\log(-x)]^{\beta-1}}{(-x)^{\theta}}\sum_{n=0}^{\beta-1}\frac{(-)^n\left(\dfrac{\pi}{\sin\pi\theta}\right)^{(n)}}{n!\,\Gamma(\beta-n)[\log(-x)]^n}.$$

20. We must now *investigate the asymptotic expansion of $g_{\beta}(x;\theta)$ when θ is real.*

In this case difficulties arise from the fact that the specification of $(s+\theta)^{-\beta}$ when s is real and less than $-\theta$ is arbitrary. We have (§ 1) adopted the convention that in this case we will take $\arg(s+\theta)=+\pi$. We must therefore take a cross-cut from $-\theta$ to $-\infty$ inclined at a small angle to the real axis and work with the contours of the modified figure (p. 304).

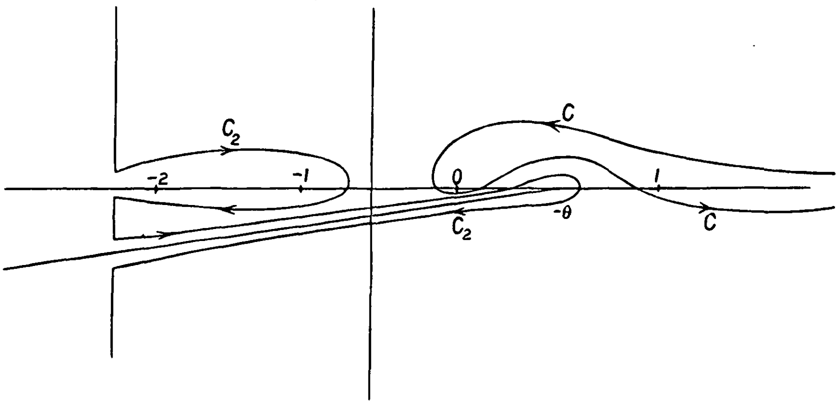
We see that, if $|\arg(-x)|<\pi$ and $|x|>1$, and if θ be not zero or a positive or negative integer,

$$g_{\beta}(x;\theta)=-\sum_{n=1}^{\infty}\frac{1}{x^n(\theta-n)^{\beta}}-\frac{1}{2\pi i}\int_{C'}(-x)^{-y-\theta}(-y)^{-\beta}\frac{\pi}{\sin\pi(y+\theta)}\,dy,$$

the contour C' being derived from the part of the contour C_2 which

* "A Memoir on Integral Functions," *Phil. Trans. Roy. Soc.* (A), Vol. 199, pp. 411–500, § 32.

encloses $-\theta$ by the transformation $s = -(y+\theta)$, so that C' embraces an axis from the origin which lies above the real axis and is inclined at a small angle to it.



We obtain the same result as before for the asymptotic expansion of this integral. And this is to be expected, since the terms of

$$\sum_{n=-\infty}^{\infty} \frac{x^n}{(n+\theta)^\beta},$$

for which $n < -\theta$, are, when $|x|$ is large, of order less than that of any term of the asymptotic expansion.

21. Consider next *the case where θ is a positive integer.*

If we put $\theta = p$, we now obtain

$$g_\beta(x\,;\theta) = -\sum_{n=1}^{\infty}{}' \frac{1}{x^n(p-n)^\beta} - \frac{1}{2\pi\iota}\int_C (-x)^{-y-p}(-y)^{-\beta}\frac{(-)^p\pi}{\sin \pi y}\,dy,$$

the accent denoting that the term corresponding to $n = p$ is to be omitted from the summation.

On the contour C' we have the summable divergent expansion

$$\frac{\pi y}{\sin \pi y} = 1 + \sum_{n=1}^{\infty}\left(\frac{\pi x}{\sin \pi x}\right)^{(n)}_{x=0}\frac{y^n}{n\,!}.$$

Therefore the integral gives rise to the asymptotic series

$$\frac{1}{x^p}\sum_{n=0}^{\infty}\left(\frac{\pi x}{\sin \pi x}\right)^{(n)}_{x=0}\frac{(-)^{n-1}}{n!}\frac{1}{2\pi\iota}\int e^{-y\log(-x)}(-y)^{n-\beta-1}\,dy.$$

We therefore have the asymptotic expansion

$$g_\beta(x\,;p) = -\sum_{n=1}^{\infty}{}' \frac{1}{x^n(p-n)^\beta}$$
$$+\frac{[\log(-x)]^\beta}{x^p}\sum_{n=0}^{\infty}\left(\frac{\pi x}{\sin \pi x}\right)^{(n)}_{x=0}\frac{(-)^{n-1}}{n!\,\Gamma(1+\beta-n)[\log(-x)]^n},$$

22. Suppose, finally, *that $\theta = p$, a positive integer, and that β is a positive integer.*

The result just obtained becomes the actual equality

$$g_\beta(x; p) = -\sum_{n=1}^{\infty}{}' \frac{1}{x^n (p-n)^\beta}$$

$$-\frac{[\log(-x)]^\beta}{x^p} \sum_{n=0}^{\beta-1} \left(\frac{\pi x}{\sin \pi x}\right)^{(n)}_{x=0} \frac{(-)^n}{n!\, \Gamma(1+\beta-n) [\log(-x)]^n}.$$

Suppose now that $\theta = 1$.

Then, if we put

$$P_\beta(x) = \sum_{n=1}^{\infty} n^{-\beta} x^n,$$

so that $P_\beta(x)$ is one of the series which Leau makes fundamental in his researches, we have

$$g_\beta(x; \theta) = x^{-1} P_\beta(x).$$

Also $$\sum_{n=1}^{\infty}{}' \frac{1}{x^n (1-n)^\beta} = e^{-\pi\iota\beta} \sum_{n=2}^{\infty} \frac{1}{x^n (n-1)^\beta} = \frac{e^{-\pi\iota\beta}}{x} P_\beta\left(\frac{1}{x}\right).$$

Now we know that, when $|x|$ is small, we have the expansion

$$\frac{\pi x}{\sin \pi x} = 1 + \sum_{n=1}^{\infty} \frac{2^{2n-1}-1}{2^{2n-2}} S_{2n} x^{2n}$$

where $$S_{2n} = \frac{1}{1^{2n}} + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \dots.$$

Hence $$\left(\frac{\pi x}{\sin \pi x}\right)^{(n)}_{x=0} = 1 \qquad \text{(when } n = 0)$$

$$= 0 \qquad \text{(when } n \text{ is odd)}$$

$$= (2m)!\, \frac{2^{2m-1}-1}{2^{2m-2}} S_{2m} \qquad \text{(when } n = 2m).$$

Hence, if β be even,

$$P_\beta(x) + P_\beta\left(\frac{1}{x}\right) = -\frac{[\log(-x)]^\beta}{\beta!} - \sum_{m=1}^{\frac{1}{2}\beta} \frac{[\log(-x)]^{\beta-2m}}{(\beta-2m)!} \frac{2^{2m-1}-1}{2^{2m-2}} S_{2m};$$

and, if β be odd,

$$P_\beta(x) - P_\beta\left(\frac{1}{x}\right) = -\frac{[\log(-x)]^\beta}{\beta!} - \sum_{m=1}^{\frac{1}{2}(\beta-1)} \frac{[\log(-x)]^{\beta-2m}}{(\beta-2m)!} \frac{2^{2m-1}-1}{2^{2m-2}} S_{2m}.$$

These are two little known formulæ due to Spence, and quoted by De Morgan.*

23. As has been suggested in § 11, it is evident that we may apply the preceding methods to series of the type

$$h_\beta(x) = \sum_{n=0}^{\infty} \frac{x^n \{\log(n+\theta)\}^k}{(n+\theta)^\beta},$$

where k is a positive integer. We have merely to write $\beta+\epsilon$ for β and expand the various functions of β in ascending powers of ϵ. The analysis will evidently be laborious.

When k is not restricted to be an integer, but is any complex quantity of finite modulus, the function may be expressed by the contour integral

$$\frac{1}{2\pi\iota}\int_C \frac{[\log(s+\theta)]^k \pi(-x)^s ds}{(s+\theta)^\beta \sin\pi s}.$$

And this in turn, as in § 17, can, when $|x| > 1$ and $|\arg(-x)| < \pi$, be expressed in the form

$$-\sum_{n=1}^{\infty} \frac{[\log(\theta-n)]^k}{x^n(\theta-n)^\beta} - \frac{1}{2\pi\iota}\int_C \frac{[\log(-y)]^k \pi(-x)^{-y-\theta}}{\sin\pi(y+\theta)(-y)^\beta}\, dy$$

provided θ be not real.

If $R\log(-x) > 0$, the integral is convergent. Thus the function $h_\beta(x)$ has no singularities outside a circle of radius unity except possibly on the line $(1, +\infty)$. An asymptotic expansion in the neigbourhood of this singularity at infinity can be obtained.

We can also shew that $x = 1$ and $x = \infty$ are the only singularities of the function.

PART II.—*The Function* $f_\beta(x;\theta)$.

24. We proceed now to obtain analogous theorems for the very general function $f_\beta(x;\theta)$. This function is defined when $|x| < 1$ by the expansion

$$\sum_{n=0}^{\infty} \frac{x^n \chi(n+\theta)}{(n+\theta)^\beta}$$

* De Morgan, *Differential and Integral Calculus* (1842), p. 659, Formulæ I. and II. In the notation of De Morgan

$$2s_n = \frac{2^{n-1}-1}{2^{n-2}} S_n.$$

where outside a circle of radius $l < \mu$, where μ is the least of the quantities $|n+\theta|$, $n = 0, 1, 2, \ldots, \infty$, $\chi(x)$ admits the absolutely convergent expansion

$$\sum_{r=0}^{\infty} \frac{b_r}{x^r}.$$

As before, we assume that β is not a positive integer, and we make the further restriction that it shall not be zero or a negative integer. Such limiting cases may be dealt with by more elementary methods, or by applying the calculus of limits to the formulæ which arise.

25. We will first shew *that, when* $|x| < 1$, $f_\beta(x;\theta)$ *can be written in the form*

$$\sum_{r=0}^{\infty} b_r g_{\beta+r}(x;\theta).$$

We have $$f_\beta(x;\theta) = \sum_{n=0}^{\infty} x^n \left\{ \sum_{r=0}^{R} \frac{b_r}{(n+\theta)^{\beta+r}} + \sum_{r=1}^{\infty} \frac{b_{R+r}}{(n+\theta)^{\beta+R+r}} \right\}.$$

Now, for values of r greater than an assignable quantity R,

$$|b_r| < l'^r,$$

where $l' = l+\epsilon$, and ϵ may be as small as we please.

Hence $$\left| \sum_{r=1}^{\infty} \frac{b_{R+r}}{(n+\theta)^{\beta+R+r}} \right| \leqslant \sum_{r=1}^{\infty} \frac{l'^{R+r}}{|(n+\theta)^{\beta+R}|\,|n+\theta|^r}$$

$$< \frac{l'^R}{|(n+\theta)^{\beta+R}|} \sum_{r=1}^{\infty} \frac{l'^r}{\mu^r}$$

$$< \frac{l'^{R+1}}{|(n+\theta)^{\beta+R}|\,(\mu-l')}, \qquad \text{if } \mu > l'.$$

Therefore $$f_\beta(x;\theta) = \sum_{r=0}^{R} b_r g_{\beta+r}(x;\theta) + J_R,$$

where $$|J_R| < \sum_{n=0}^{\infty} \frac{|x^n|\, l'^{R+1}}{|(n+\theta)^{\beta+R}|\,(\mu-l')}.$$

Thus $|J_R|$ tends to zero as R tends to infinity if $|x| < 1$.
We thus have the theorem stated.

26. We will now shew that $f_\beta(x;\theta)$ *has no singularities except possibly when x lies on the positive part of the real axis between* $1-\epsilon$ $(\epsilon > 0)$ *and* $+\infty$.

Suppose that $F_\beta(x;\theta)$ denotes the integral function

$$\sum_{n=0}^{\infty} \frac{x^n \chi(n+\theta)}{n!\,(n+\theta)^\beta}.$$

Then I have shewn, in my fundamental memoir,* that, when $R(x) > 0$,

$$F_\beta(x;\theta) = \frac{e^x J(x)}{x^\beta},$$

where $|J(x)|$ tends to a definite finite limit as $|x|$ tends to infinity.

And, when $R(x) < 0$, $F_\beta(x;\theta) = (-x)^{l-\theta} J_1(x)$, where l is the radius of convergence of $\chi(y)$, and where $|J_1(x)|$ is at most finite when $|x|$ tends to infinity.

As in § 3, we may therefore show that

$$f_\beta(x;\theta) = \int_0^\infty e^{-z} F_\beta(xz;\theta)\, dz.$$

The integral may be taken along any axis for which $R(z) > 0$ and $R[(1-x)z] > 0$. We thus obtain continuations of $f_\beta(x;\theta)$ which are finite and continuous for all values of x such that $|\arg(1-x)| < \pi$.

We thus have the given theorem, which is true for all values of β of finite modulus.

27. We will now show that, *provided* β *be not an integer, and if* $R(\theta) > 0$, *and if* θ *lies outside the circle of convergence of* $\chi(x)$, *and provided* $\left|\frac{x-1}{x}\right| < 1$,

$$f_\beta(x;\theta) - (-\log x)_{1/\theta}^{\beta-1} x^{-\theta}\Gamma(1-\beta) \sum_{r=0}^{\infty} b_r \frac{(\log x)^r \Gamma(\beta)}{\Gamma(\beta+r)} = \sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \psi_{n+1}(\beta,\theta)$$

where† $$\psi_{n+1}(\beta,\theta) = \frac{\iota\Gamma(1-\beta)}{2\pi}\int (-y)^{\beta-1}\Phi(y)\frac{e^{-y\theta}}{(1-e^{-y})^{n+1}}\,dy$$

and $$\Phi(y) = \sum_{r=0}^{\infty}\frac{\Gamma(\beta)\, b_r y^r}{\Gamma(\beta+r)}.$$

The contour of the integral embraces the axis to $1/\theta$, *which is the cross-cut which makes* $(-\log x)_{1/\theta}^{\beta-1}$ *one-valued.*

If $|x| < 1$, we have, by § 4,

$$\begin{aligned} f_\beta(x;\theta) &= \sum_{r=0}^{\infty} b_r g_{\beta+r}(x;\theta) \\ &= \sum_{r=0}^{\infty}\frac{\iota\Gamma(1-\beta-r)\, b_r}{2\pi}\int (-y)^{\beta+r-1}\frac{e^{-y\theta}}{1-xe^{-y}}\,dy, \end{aligned}$$

provided $R(\theta) > 0$, and provided the contour of the integral excludes the

* *Loc. cit.*, § 1, Part V., §§ 31 and 36.

† The reader will, of course, not confuse this function with the logarithmic derivates of the multiple gamma function considered in §§ 11 *et seq.*

poles $y = \log x \pm 2n\pi\iota$ ($n = 0, 1, 2, \ldots, \infty$). Hence, if $|I\{\log x\}| < \pi$,

$$f_\beta(x;\theta) - \sum_{r=0}^{\infty} b_r \Gamma(1-\beta-r)(-\log x)^{\beta+r-1} x^{-\theta}$$

$$= \sum_{r=0}^{\infty} \frac{\iota\Gamma(1-\beta-r)\, b_r}{2\pi} \int (-y)^{\beta+r-1} \frac{e^{-y\theta}}{1-xe^{-y}}\, dy, \quad (1)$$

where now the contour of the integral includes $\log x$, but not $\log x \pm 2n\pi\iota$ ($n = 1, 2, \ldots, \infty$).

In this equality $(-\log x)^{\beta+r-1}$ has its principal value with respect to a cross-cut along the axis of integration. And the first series converges if $|-\log x|$ is finite. The second series is therefore convergent if $|-\log x|$ be finite and $|x| < 1$.

28. We will next shew that, if the contour of integration embraces the axis to $1/\theta$, and if $R(\theta) > 0$ and $|\theta| > l$, the integral

$$\frac{\iota\Gamma(1-\beta)}{2\pi} \int \sum_{r=0}^{R} \frac{b_r \Gamma(1-\beta-r)(-y)^{\beta+r-1}}{\Gamma(1-\beta)} \frac{e^{-y\theta}}{1-xe^{-y}}\, dy \quad (2)$$

tends to a definite finite limit as R tends to infinity.

Evidently $$\sum_{r=0}^{\infty} \frac{b_r(-y)^r \Gamma(1-\beta-r)}{\Gamma(1-\beta)} = \Phi(y),$$

an integral function of y.

Also, if k be an integer such that $R(\beta+k) > 0$, we have

$$|\beta+k+r| > r.$$

Hence $$|\Phi(y)| < \sum_{r=0}^{k-1} \frac{\Gamma(\beta)\, b_r y^r}{\Gamma(\beta+r)} + \frac{1}{|\beta(\beta+1)\ldots(\beta+k)|} \sum_{r=0}^{\infty} \frac{b_{r+k} y^{r+k}}{r!}.$$

If, now, k be sufficiently large, $|b_{r+k}| < l'^{r+k}$, where $l' = l+\epsilon$, and ϵ is a positive quantity as small as we please.

Therefore

$$|\Phi(y)| < \text{a polynomial in } |y| + \frac{l'^k |y|^k}{|\beta(\beta+1)\ldots(\beta+k)|} e^{l'|y|}.$$

Therefore the integral (2) tends to a definite finite limit under the conditions assigned. Therefore under these conditions we obtain from (1)

$$f_\beta(x;\theta) - \sum_{r=0}^{\infty} b_r \Gamma(1-\beta-r)(-\log x)_{1/\theta}^{\beta+r-1} x^{-\theta}$$

$$= \frac{\iota\Gamma(1-\beta)}{2\pi} \int (-y)^{\beta-1} \Phi(y) \frac{e^{-y\theta}}{1-xe^{-y}}\, dy. \quad (3)$$

Provided $\log x$ is inside and $\log x \pm 2n\pi\iota$ $(n \neq 0)$ is outside the contour, and provided $(-\log x)_{1/\theta}^{\beta-1}$ has its principal value with respect to the axis of integration, the integral remains finite and continuous when $|x| > 1$, provided $|-\log x|$ is finite. Thus, under these limitations coupled with $R(\theta) > 0$ and $|\theta| > l' > l$, we obtain a continuation of $f_\beta(x;\theta)$ when $|x| > 1$.

It is obvious that, by taking $R(\theta)$ sufficiently large, we may find an infinite number of lines in the positive half of the y-plane which may serve as axes of integration and cross-cuts for $(-\log x)^{\beta-1}$. For instance, if $R(\theta) > l'$, the positive half of the real axis will so serve. We are not limited to the particular cross-cut chosen if $R(\theta)$ be very large.

29. Consider now the integral in the formula (3). It will be equal to

$$\sum_{n=0}^{N-1} \frac{(x-1)^n}{x^{n+1}} \psi_{n+1}(\beta;\theta) + J_N$$

where
$$\psi_{n+1}(\beta;\theta) = \frac{\iota\Gamma(1-\beta)}{2\pi} \int_{1/\theta} (-y)^{\beta-1} \Phi(y) \frac{e^{-y\theta}}{(1-e^{-y})^{n+1}}\, dy$$

and
$$J_N = \frac{\iota\Gamma(1-\beta)}{2\pi} \int_{1/\theta} \frac{(-y)^{\beta-1}\Phi(y)\, e^{-y\theta}}{1-xe^{-y}} \left[\frac{x-1}{x(1-e^{-y})}\right]^N dy.$$

By deforming the contour embracing $1/\theta$, so that it consists near the origin of the contour figured in § 8 and further away embraces a parallel to the axis of $1/\theta$, we can ensure that upon it the minimum value of $|1-e^{-y}|$ is μ, where $\mu = 1-\epsilon$, and ϵ is > 0, but as small as we please.

Then evidently $|J_N|$ tends to zero as N tends to infinity, if

$$|(x-1)/x| < 1-\epsilon.$$

The series
$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \psi_{n+1}(\beta;\theta)$$

is therefore absolutely convergent under the same limitation.

We therefore have, if $|(x-1)/x| < 1$,

$$f_\beta(x;\theta) - (-\log x)^{\beta-1} x^{-\theta} \frac{\pi}{\sin\pi\beta} \sum_{r=0}^{\infty} \frac{b_r(\log x)^r}{\Gamma(\beta+r)} = \sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \psi_{n+1}(\beta;\theta).$$

If $(-\log x)^{\beta-1}$, *qua* function of $\log x$, have a cross-cut along the axis to $1/\theta$, we have proved the formula under the limitation $R(\theta) > 0$ and $|\theta| > l' > l$.

And, if $R(\theta)$ be sufficiently large, we may take the cross-cut for $(-\log x)^{\beta-1}$ to be in an infinite number of positions in the positive half of the y plane.

30. From the previous theorem we deduce at once that $f_\beta(x;\theta)$ *has, when* $R(\theta)$ *is sufficiently large, a single singularity in the finite part of*

the plane. This singularity occurs at $x = 1$, and is specifiable. It is a multiform point, and near it $f_\beta(x;\theta)$ behaves like*

$$\frac{\pi}{\sin \pi\beta}(-\log x)^{\beta-1} x^{-\theta} \sum_{r=0}^{\infty} \frac{b_r (\log x)^r}{\Gamma(\beta+r)}.$$

To establish these results the reader has merely to recall § 26, and to notice that the condition $|(x-1)/x| < 1$ is equivalent to $R(x) > \frac{1}{2}$.

31. We give some further theorems before we remove from the previous theorem the condition that $R(\theta)$ must be positive and sufficiently large.

We will first shew that, *if $R(\theta) > 0$, and if θ have any value of finite modulus such that the points $\theta+m$ ($m = 0, 1, 2, \ldots, \infty$) lie outside the circle of convergence of $\chi(x)$, and if n be finite,*

$$\psi_{n+1}(\beta;\theta) = \sum_{r=0}^{\infty} b_r \bar{\zeta}_{n+1}(\beta+r, \theta).$$

If $|\theta| > l' > l$ and $R(\theta) > 0$,

$$\begin{aligned}\psi_{n+1}(\beta;\theta) &= \frac{\iota\Gamma(1-\beta)}{2\pi}\int_{1/\theta} (-y)^{\beta-1}\Phi(y)\frac{e^{-y\theta}}{(1-e^{-y})^{n+1}}\,dy \\ &= \sum_{r=0}^{R-1} \frac{\iota\Gamma(1-\beta-r)\,b_r}{2\pi}\int (-y)^{\beta+r-1}\frac{e^{-y\theta}}{(1-e^{-y})^{n+1}}\,dy \\ &\qquad + \frac{\iota}{2\sin\pi\beta}\int (-y)^{\beta-1}\sum_{r=R}^{\infty}\frac{b_r y^r}{\Gamma(\beta+r)}\frac{e^{-y\theta}}{(1-e^{-y})^{n+1}}\,dy \\ &= \sum_{r=0}^{R-1} b_r\bar{\zeta}_{n+1}(\beta+r, \theta) + J_R \quad \text{(say)}.\end{aligned}$$

Now
$$(-)^n \bar{\zeta}_{n+1}(s, \theta) = \sum_{k=0}^{n} (-)^k \frac{{}_{n+1}\bar{S}_1^{(k+2)}(\theta)}{k!}\,\zeta(s-k, \theta);$$

and, if β be not an integer and θ be not equal to $0, -1, -2, \ldots$, each of the multiple $\bar{\zeta}$ functions is finite.

Hence the series $\sum_{r=0}^{R-1} b_r \bar{\zeta}_{n+1}(\beta+r, \theta)$ tends to a definite finite limit as R tends to infinity, provided series of the type

$$\sum_{r=0}^{\infty} b_r \zeta(\beta+r-k, \theta) \quad (k = 0, 1, \ldots, n)$$

are convergent. But when $R(\beta+r-k)$ is very large and positive

$$|\zeta(\beta+r-k, \theta)| < \mu^{-r} L$$

where L is finite and independent of r, and μ is defined in § 24.

* See the note to § 8.

The series are therefore convergent if $\mu > l' > l$, where μ is the minimum value of $|\theta+m|$ $(m = 0, 1, 2, \ldots, \infty)$.

This will appear again, and we prove the theorem by considering the integral J_R.

Divide up the contour of integration into two parts :—(1) a circle of radius less than unity round the origin ; (2) the double description of the axis of the contour outside this circle.

On (1) the subject of integration is finite, and can be made as small as we please by sufficiently increasing R.

On (2), if we choose k so that $R(\beta+k) > 0$, we have, if $r > k$,

$$|\Gamma(\beta+r)| > K(r-k)!$$

where K is finite, non-zero, and independent of r.

Hence $$\sum_{r=R}^{\infty} \frac{b_r y^r}{\Gamma(\beta+r)} < \frac{(l'|y|)^R}{K(R-k)!} e^{l'|y|}.$$

Hence the modulus of the integral along this part of the contour tends to zero as R tends to infinity, provided $|\theta| > l'$ and $R(\theta) > 0$.

We thus have the given theorem.

32. We will next shew that, *if* $R(\theta) > l' > l$ *and* $\left|\frac{x-1}{x}\right| < 1$,

$$f_\beta(x;\theta) - x^{-\theta} \sum_{r=0}^{\infty} b_r \Gamma(1-\beta-r)(-\log x)^{\beta+r-1} = \sum_{r=0}^{\infty} b_r \sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \bar{\zeta}_{n+1}(\beta+r,\theta).$$

By the result of § 28, the function on the left-hand side of this equality, which for brevity we will denote by $P(x)$, is, if $R(\theta) > l' > l$, equal to

$$\frac{\iota\Gamma(1-\beta)}{2\pi}\int (-y)^{\beta-1}\Phi(y)\frac{e^{-y\theta}}{1-xe^{-y}}\,dy$$

where the contour of integration embraces the positive half of the real axis and includes $\log x$.

This integral may be written

$$\sum_{n=0}^{N-1} \frac{(x-1)^n}{x^{n+1}} \psi_{n+1}(\beta;\theta) + \frac{\iota\Gamma(1-\beta)}{2\pi}\int (-y)^{\beta-1}\Phi(y)\frac{e^{-y\theta}}{1-xe^{-y}}\left\{\frac{x-1}{x(1-e^{-y})}\right\}^N dy$$
$$= \sum_{r=0}^{\infty} b_r \sum_{n=0}^{N-1} \frac{(x-1)^n}{x^{n+1}} \bar{\zeta}_{n+1}(\beta+r,\theta) + J_N \text{ (say)}. \quad (1)$$

Evidently $|J_N|$ tends to zero as N tends to infinity if $\left|\frac{x-1}{x}\right| < 1$.

Again, as in § 8, with the contour there employed,

$$|\bar{\zeta}_{n+1}(\beta+r,\theta)| < \frac{|\Gamma(1-\beta-r)|}{2\pi\mu^{n+1}}\int |(-y)^{r+\beta-1}|\,|e^{-y\theta}|\,dy < \frac{k(1+\epsilon)^r}{\mu^{n+1}[R(\theta)]^r} \quad (2)$$

where k is finite and independent of r and n, and $\epsilon > 0$.

Hence $$\left|\sum_{n=0}^{\infty}\frac{(x-1)^n}{x^{n+1}}\bar{\zeta}_{n+1}(\beta+r,\theta)\right| < \frac{K(1+\epsilon)^r}{[R(\theta)]^r}$$

where K is finite, if $|(1-x)/x| < 1$.

Hence, if $R(\theta) > l'(1+\epsilon)$, the first series in (1) tends to a definite finite limit as N tends to infinity.

We thus obtain the given theorem.

33. We may now shew that *the theorem of* § 30 *is valid for all values of* θ *provided* $\theta+m$ ($m = 0, 1, 2, \ldots, \infty$) *lies outside the circle of convergence of* $\chi(x)$.

If $R(\theta) > l'$, where $l' > l$, and $|(x-1)/x| < 1$, we have

$$\begin{aligned} f_\beta(x;\theta-1) &= \frac{\chi(\theta-1)}{(\theta-1)^\beta} + xf_\beta(x;\theta) \\ &= x^{1-\theta}\sum_{r=0}^{\infty} b_r\Gamma(1-\beta-r)(-\log x)^{\beta+r-1} + \frac{\chi(\theta-1)}{(\theta-1)^\beta} \\ &\qquad + \sum_{r=0}^{\infty} b_r \sum_{n=0}^{\infty}\frac{(x-1)^n}{x^{n+1}}\bar{\zeta}_{n+1}(\beta+r,\theta) \end{aligned}$$

Hence

$$\begin{aligned} &f_\beta(x;\theta-1) - x^{1-\theta}\sum_{r=0}^{\infty} b_r\Gamma(1-\beta-r)(-\log x)^{\beta+r-1} \\ &= \frac{\chi(\theta-1)}{(\theta-1)^\beta} + \sum_{r=0}^{\infty} b_r\left[\sum_{n=0}^{\infty}\left\{-\bar{\zeta}_n(\beta+r,\theta-1)+\bar{\zeta}_{n+1}(\beta+r,\theta-1)\right\}\frac{(x-1)^n}{x^{n+1}}\right] \\ &= \frac{\chi(\theta-1)}{(\theta-1)^\beta} + \sum_{r=0}^{\infty} b_r\Bigg[-\bar{\zeta}_0(\beta+r,\theta-1) \\ &\qquad + \sum_{n=0}^{\infty}\bar{\zeta}_{n+1}(\beta+r,\theta-1)\left\{\frac{(x-1)^n}{x^n} - \frac{(x-1)^{n+1}}{x^{n+1}}\right\}\Bigg] \\ &= \frac{\chi(\theta-1)}{(\theta-1)^\beta} + \sum_{r=0}^{\infty} b_r\left[\frac{-1}{(\theta-1)^{\beta+r}} + \sum_{n=0}^{\infty}\frac{(x-1)^n}{x^{n+1}}\zeta_{n+1}(\beta+r,\theta-1)\right].^* \end{aligned}$$

Now $\sum_{r=0}^{\infty}\frac{b_r}{(\theta-1)^{\beta+r}}$ is convergent and equal to $\frac{\chi(\theta-1)}{(\theta-1)^\beta}$ if $\theta-1$ lies outside the circle of convergence of $\chi(x)$.

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* The analysis may be made formally rigorous by coupling § 32 (2) with the results of § 9.

Also, if $\sum_{r=0}^{\infty} b_r(u_r+v_r)$ and $\sum_{r=0}^{\infty} b_r u_r$ are absolutely convergent under any assigned limitations, then will $\sum_{r=0}^{\infty} b_r v_r$ be convergent under the same limitations. For

$$\sum_{r=R}^{\infty} |b_r v_r| = \sum_{r=R}^{\infty} |b_r(u_r+v_r-u_r)| \leqslant \sum_{r=R}^{\infty} \{|b_r(u_r+v_r)|+|b_r u_r|\},$$

and the latter series can be made as small as we please by sufficiently increasing R.

Hence, if $(\theta-1)$ lies outside the circle of convergence of $\chi(x)$, and if $R(\theta) > l'$ where $l' > l$, and $|(1-x)/x| < 1$, the series

$$\sum_{r=0}^{\infty} b_r \sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \bar{\zeta}_{n+1}(\beta+r, \theta-1)$$

is absolutely convergent and equal to

$$f_\beta(x; \theta-1) - x^{1-\theta} \sum_{r=0}^{\infty} b_r \Gamma(1-\beta-r)(-\log x)^{\beta+r-1}.$$

Continue this process indefinitely, and we see that the theorem of § 30 is valid for all values of θ provided $\theta+m$ $(m = 0, 1, 2, \ldots, \infty)$ lies outside the circle of convergence of $\chi(x)$.

34. Finally, let us consider the result of applying the process of § 17 to the function $f_\beta(x; \theta)$.

Assume that the points $\theta \pm n$, $n = 0, 1, 2, \ldots, \infty$ all lie outside the circle of convergence of $\chi(x)$.

If the contour C of § 17 do not enclose any part of the circle of radius l and centre $-\theta$, we evidently have

$$f_\beta(x; \theta) = \frac{1}{2\pi\iota} \int_C \frac{\pi(-x)^s \chi(s+\theta)}{\sin \pi s (s+\theta)} ds$$

when $|x| < 1$.

If, now, $|\arg(-x)| < \pi$, the integral will vanish when taken round an infinite contour for which $R(s)$ is greater than a finite negative quantity.

Hence, if the contour C_1 of § 17 have, if necessary, a loop to ensure that the circle round $-\theta$ lies to the left of the contour, we have

$$f_\beta(x; \theta) = \frac{1}{2\pi\iota} \int_{C_1} \frac{\pi(-x)^s \chi(s+\theta)}{\sin \pi s (s+\theta)^\beta} ds.$$

The latter integral is valid for all values of $|x|$ provided $|\arg(-x)| < \pi$; and therefore represents the continuation of $f_\beta(x; \theta)$ over the whole plane, except this part near the positive half of the real axis. We thus again arrive at the theorem of § 26.

If now the contour C_2 of § 17 include the circle of radius l round $-\theta$, we see that, when $|\arg(-x)| < \pi$ and $|x| > 1$, $f_\beta(x;\theta)$ is equal to the integral along the contour C_2. The integral along the straight parts of the latter contour tends to zero as these move away to infinity.

Under the assigned limitations we therefore have

$$f_\beta(x;\theta) = -\sum_{n=1}^{\infty} \frac{\chi(-n+\theta)}{x^n(\theta-n)^\beta} - \frac{1}{2\pi\iota}\int_{C'} \frac{\pi(-x)^{-\theta-y}\chi(-y)}{\sin\pi(\theta+y)(-y)^\beta}\,dy. \tag{1}$$

The contour C' embraces the real axis and encloses within its bulb the circle of convergence of $\chi(-y)$.

35. The equality is true *when β is an integer*. In this case C' may be replaced by a contour just outside the circle of convergence of $\chi(-y)$.

In either case we see that the final integral in the equality (1) is equal to $(-x)^{l'-\theta} J(x)$, where, if $l' > l$, $|J(x)|$ tends to zero as $|x|$ tends to infinity. We thus get a superior limit to the asymptotic value of $f_\beta(x;\theta)$ when $|x|$ is very large and β is or is not an integer.

The problem of obtaining a complete asymptotic expansion for $f_\beta(x;\theta)$ when $|x|$ is large evidently depends upon the consideration of the singularities of $\chi(y)$ within its circle of convergence. The reader will compare the similar property of $F_\beta(x;\theta)$ when $R(x) < 0$, which was established in my original memoir.

[*Note added May*, 1906.—

The first author to consider simple cases of the series which we investigate in the present memoir was William Spence, whose *Essay on the Theory of the Various Orders of Logarithmic Transcendents* was published in 1809. Spence considered series of the type $\sum_{n=0}^{\infty} n^{-r}x^n$, where r is an integer, and by a process of induction obtained the continuation of such series when $|x| > 1$.* This essay, so rare, that no copy is to be found in the library of the University of Cambridge, seems to have been almost entirely forgotten. Such series were also considered by Lambert, Legendre, Abel, and Kummer, among others.

Abel considered such series in two papers. The first paper,† "Sommation de la Série $\sum_{n=0} \phi(n)x^n, \cdot \ldots$, $\phi(n)$ étant une fonction algébrique

* *Loc. cit.*, p. 45.

† *Œuvres Complètes*, 1881, T. II., pp. 14–18.

rationnelle de n," has several points of interest. Abel considers explicitly (p. 16) the function $g_\beta(x;\theta)$ where β is a positive integer; and obtains (p. 18) a rudimentary form of the formula*

$$G(x;\theta)=\Gamma(\theta)(-x)^{-\theta}+e^x\sum_{n=0}^{\infty}\frac{(1-\theta)\dots(n-\theta)}{x^{n+1}}.$$

In the second paper† "Note sur la fonction $\psi(x)=\sum_{n=1}^{\infty}x^n/n^2$," he attains anew several of Spence's results. Both papers were first published posthumously by Holmboe in 1839, and are evidently mere sketches.

But the modern theory of such series is largely due to Leau,‡ whose work, closely associated with the investigations of Hadamard,§ Borel,|| and Fabry,¶ led to the investigation of series of the type $\Sigma g(1/m)x^m$ where $g(t)$ is holomorphic at the origin.

Then came Le Roy,** to whom appear to be due the theorems (1) and (3) of § 2.

My own developments were largely completed before I saw Hardy's†† paper. Subsequently my attention has been called to Lindelöf's‡‡ monograph, and to another paper by Hardy.§§ In the former will be found series such as occur in theorem (2) of § 2, and in the latter an equation similar to (A) of § 7. Nothing in the second part of the present paper appears to have been anticipated. But so rapid is the development of the subject that it is difficult to assign priority to respective authors, and almost impossible to state that any investigation is new in all its details.]

* See the author's paper, *Quarterly Journal of Mathematics*, Vol. XXXVII., p. 294.

† *Loc. cit.*, pp. 189–193.

‡ *Liouville*, Sér. 5, T. V. (1899).

§ *Ibid.*, Sér. 4, T. VIII. (1892).

|| *Acta Mathematica*, T. XXI. (1897); *Liouville*, Sér. 5, T. II. (1896).

¶ *Annales Scientifiques de l'Ecole Normale Supérieure*, Sér. 3, T. XIII. (1896); *Liouville*, Sér. 5, T. IV. (1898); *Acta Mathematica*, T. XXII. (1899).

** *Annales de la Faculté des Sciences de Toulouse*, Sér. 2, T. II. (1900).

†† *Proc. London Math. Soc.*, Ser. 2, Vol. 2, pp. 401–431.

‡‡ *Le Calcul des Résidus* (Paris: Gauthier-Villars, 1905).

§§ *Proc. London Math. Soc.*, Ser. 2, Vol. 3, pp. 381–389.