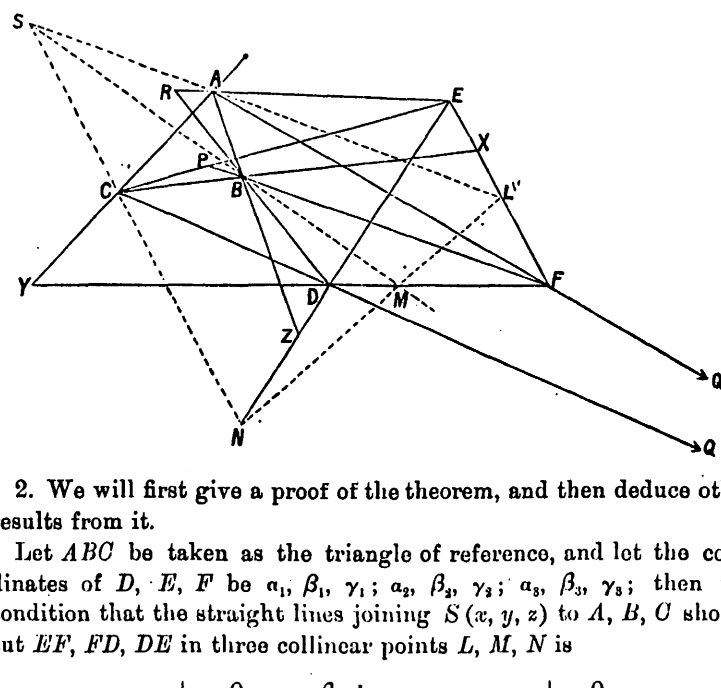
On the Degeneration of a Cubic Curve. By H. M. TAYLOR.
Received and communicated February 11th, 1897.

1. Grassmann seems to have been the first to give the following theorem relating to the generation of a cubic curve. The locus of a point, such that the straight lines joining it to the angular point of one triangle ABC cut the sides of a second triangle DEF, taken in order, in three collinear points, is a cubic curve passing through the angular points of both triangles.



2. We will first give a proof of the theorem, and then deduce other results from it.

Let ABC be taken as the triangle of reference, and let the coordinates of D, E, F be $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$; $\alpha_3, \beta_3, \gamma_3$; then the condition that the straight lines joining $S(x, y, z)$ to A, B, C should cut EF, FD, DE in three collinear points L, M, N is

$$\begin{vmatrix} 0, & \beta_1 x - \alpha_1 y, & \gamma_1 x - \alpha_1 z \\ \alpha_2 y - \beta_2 x, & 0, & \gamma_2 y - \beta_2 z \\ \alpha_3 z - \gamma_3 x, & \beta_3 z - \gamma_3 y, & 0 \end{vmatrix} = 0,$$

or $$(\alpha_2 y - \beta_2 x)(\beta_3 z - \gamma_3 y)(\gamma_1 x - \alpha_1 z) + (\alpha_3 z - \gamma_3 x)(\beta_1 x - \alpha_1 y)(\gamma_2 y - \beta_2 z) = 0.$$

This is the equation of a cubic curve which passes through A, B, C: through D, E, F: through P, Q, R, the intersections of the pairs of lines BF, CE; CD, AF; AE, BD: and also through X, Y, Z, the intersections of the pairs of lines BC, EF; CA, FD; AB, DE.

3. If the triangles ABC, DEF be interchanged, the cubic locus will pass through the same twelve points, and is therefore the same cubic. The cubic will also be unaltered by the interchange of two corresponding vertices of the triangles ABC, DEF. For instance, the triangles ABF, DEC would give rise to the same cubic.

By considering the limiting case when S is close to A, we can obtain the tangent at the point A by the following geometrical construction. Let AB, DF intersect in M'; AC, DE in N'. Draw $M'N'$ cutting EF in L'; then AL' is the tangent to the cubic at A. A similar construction will give the tangents at any one of the other five points B, C, D, E, F.

4. The three sets of points A, E, R; A, Q, F; X, B, C lie on straight lines, and the points X, E, F lie on a straight line; therefore the remaining points A, A, B, C, Q, R lie on a conic; in other words, the conic $ABCQR$ touches the curve at the point A. Similarly, the conics $ABFRY$, $ACEQZ$, $AEFYZ$ touch the cubic at the point A. In the same way, it can be shown that at each of the points B, C, D, E, F four conics can be drawn to touch the curve, and to pass through four other points in the figure $ABCDEFPQRXYZ$.

5. Again, because each of the sets of points D, B, R; A, E, R; A, B, Z lie on straight lines, and the points D, E, Z lie on a straight line, therefore the six points A, A, B, B, R, R lie on a conic; in other words, a conic can be drawn having contact with the curve at the points A, B, R. There are twelve such conics having triple contact with the cubic.

Their points of contact can be found by writing down any three collinear points in the figure, *e.g.*, A, F, Q, and substituting for A or F its conjugate point D or C; thus conics will touch the cubic at D, F, Q and at A, C, Q.

Similarly, we can pick out A, B, Z; A, Y, C, two sets of three collinear points, which include a common point A but do not include any conjugate points. By writing D for A we obtain six points D, D, B, C, Y, Z, which lie on a conic. We thus determine the five points

D, B, C, Y, Z, which lie on one of the conics touching the cubic at D.

6. It can be seen from inspection of the equation of the cubic that in certain cases the cubic degenerates into a straight line and a conic, or into three straight lines.

For instance, if $\alpha_1 = 0$, the cubic reduces to BC and a conic through A, D, E, F.

If $\alpha_1 = 0$ and $\beta_2 = 0$, the cubic reduces to BC, CA, FZ.

If $\alpha_1 = 0$, $\beta_2 = 0$, and $\gamma_3 = 0$, the cubic reduces to the three sides of the triangle ABC, except in the case when

$$\gamma_1\alpha_2\beta_3 + \beta_1\gamma_2\alpha_3 = 0,$$

that is, when D, E, F are collinear: in which case the cubic becomes indeterminate.

7. It is at once seen that, if D, E, F be collinear, and S be any point in the straight line DEF, then L, M, N are collinear.

We see that in this case the straight line DEF is part of the locus, and therefore the remaining part must be a conic circumscribing the triangle ABC.

8. Let us inquire whether it is possible that the cubic should include the straight line

$$lx + my + nz = 0.$$

The equation of the cubic may be written

$$\gamma_3(\gamma_1\beta_2 - \beta_1\gamma_2)\, x^2y + \gamma_3(\alpha_1\gamma_2 - \alpha_2\gamma_1)\, xy^2 + \alpha_1(\alpha_2\gamma_3 - \alpha_3\gamma_2)\, y^2z$$
$$+ \alpha_1(\alpha_2\beta_3 - \alpha_3\beta_2)\, yz^2 + \beta_2(\alpha_1\beta_3 - \alpha_3\beta_1)\, z^2x + \beta_2(\beta_1\gamma_3 - \beta_3\gamma_1)\, zx^2$$
$$+ (\gamma_1\alpha_2\beta_3 + \beta_1\gamma_2\alpha_3 - 2\alpha_1\beta_2\gamma_3)\, xyz = 0.$$

If $lx + my + nz = 0$ is part of the cubic, the remaining part must be

$$xy\,\frac{\gamma_3(\gamma_1\beta_2 - \beta_1\gamma_2)}{l} + yz\,\frac{\alpha_1(\alpha_2\gamma_3 - \alpha_3\gamma_2)}{m} + zx\,\frac{\beta_2(\alpha_1\beta_3 - \alpha_3\beta_1)}{n} = 0,$$

and we must have the conditions

$$\frac{m}{l}\,\gamma_3(\gamma_1\beta_2 - \beta_1\gamma_2) = \gamma_3(\alpha_1\gamma_2 - \alpha_2\gamma_1),$$

$$\frac{n}{m}\,\alpha_1(\alpha_2\gamma_3 - \alpha_3\gamma_2) = \alpha_1(\beta_2\alpha_3 - \beta_3\alpha_2),$$

$$\frac{l}{n}\,\beta_2(\alpha_1\beta_3 - \alpha_3\beta_1) = \beta_2(\beta_1\gamma_3 - \beta_3\gamma_1),$$

and $$\frac{n}{l}\gamma_3(\gamma_1\beta_2-\gamma_2\beta_1)+\frac{l}{m}\alpha_1(\alpha_2\gamma_3-\alpha_3\gamma_2)+\frac{m}{n}\beta_2(\alpha_1\beta_3-\alpha_3\beta_1)$$
$$=\gamma_1\alpha_2\beta_3-2\alpha_1\beta_2\gamma_3+\beta_1\gamma_2\alpha_3.$$

The ratio $m:n$ becomes indeterminate when $\alpha_1=0$, and also when
$$\alpha_2\gamma_3-\alpha_3\gamma_2=\alpha_3\beta_2-\alpha_2\beta_3=0.$$

The first case has already been discussed.

In the second case E, F are coincident.

If none of the ratios $l:m:n$ be indeterminate, then
$$(\gamma_1\beta_2-\gamma_2\beta_1)(\alpha_2\gamma_3-\alpha_3\gamma_2)(\alpha_1\beta_3-\alpha_3\beta_1)$$
$$=(\alpha_1\gamma_2-\alpha_2\gamma_1)(\beta_2\alpha_3-\beta_3\alpha_2)(\beta_1\gamma_3-\beta_3\gamma_1).$$

This equation is equivalent to the condition that the points X, Y, Z should be collinear.

This will be the case when the points D, E, F are collinear, and also when the straight lines AD, BE, CF are concurrent: as is apparent also from the algebraical identity
$$(\gamma_1\beta_2-\beta_1\gamma_2)(\alpha_2\gamma_3-\alpha_3\gamma_2)(\alpha_1\beta_3-\alpha_3\beta_1)$$
$$-(\alpha_1\gamma_2-\alpha_2\gamma_1)(\beta_2\alpha_3-\beta_3\alpha_2)(\beta_1\gamma_3-\beta_3\gamma_1)$$
$$\equiv(\alpha_2\beta_3\gamma_1-\alpha_3\beta_1\gamma_2)\left\{\alpha_1(\beta_2\gamma_3-\beta_3\gamma_2)+\alpha_2(\beta_3\gamma_1-\beta_1\gamma_3)+\alpha_3(\beta_1\gamma_2-\beta_2\gamma_1)\right\}.$$

9. The coordinates of P, Q, R are given by the following equations:—
$$P;\quad \frac{x_1}{\alpha_2\alpha_3}=\frac{y_1}{\beta_2\alpha_3}=\frac{z_1}{\alpha_2\gamma_3},$$
$$Q;\quad \frac{x_2}{\alpha_1\beta_3}=\frac{y_2}{\beta_1\beta_3}=\frac{z_2}{\beta_1\gamma_3},$$
$$R;\quad \frac{x_3}{\alpha_1\gamma_2}=\frac{y_3}{\beta_2\gamma_1}=\frac{z_3}{\gamma_1\gamma_2}.$$

If we substitute the coordinates of P, Q, R for the coordinates of D, E, F in the equation of the cubic
$$(\alpha_2y-\beta_2x)(\beta_3z-\gamma_3y)(\gamma_1x-\alpha_1z)+(\alpha_3z-\gamma_3x)(\beta_1x-\alpha_1y)(\gamma_2y-\beta_2z)=0,$$
we obtain the equation
$$\beta_3(\alpha_1y-\beta_1x)\gamma_1(\beta_2z-\gamma_2y)\alpha_2(\gamma_3x-\alpha_3z)$$
$$+\gamma_2(\alpha_1z-\gamma_1x)\alpha_3(\beta_2x-\alpha_2y)\beta_1(\gamma_3y-\beta_3z)=0.$$

This proves that in general the substitution of P, Q, R for D, E, F

gives rise to a second cubic; but that the cubic is unaltered by this substitution in the case when $\alpha_2\beta_3\gamma_1 = \alpha_3\beta_1\gamma_2$, that is, when AD, BE, CF are concurrent.

10. In discussing the case when AD, BE, CF are concurrent it is more convenient to use a modified system of coordinates instead of the ordinary triangular coordinates.

We will take the coordinates of a point, multiples of the triangular coordinates of the point, such that the equations of AD, BE, CF will be $y = z$, $z = x$, $x = y$ respectively. Then we may take the coordinates of D, E, F, thus:

$$\alpha, 1, 1; \quad 1, \beta, 1; \quad 1, 1, \gamma.$$

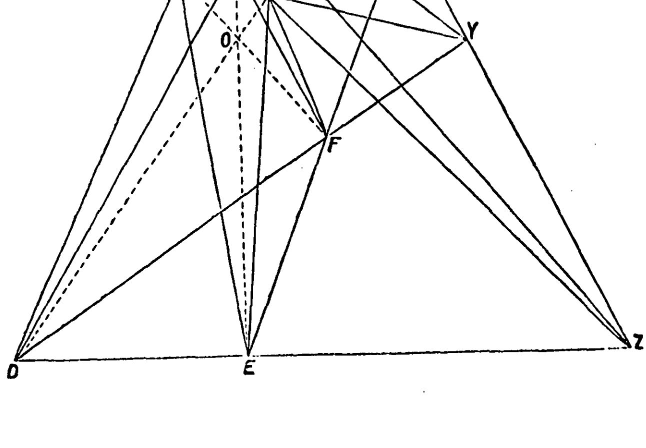
The equation of the cubic will then be

$$(x-\alpha y)(y-\beta z)(z-\gamma x) + (x-\alpha z)(y-\beta x)(z-\gamma y) = 0,$$

or $$\gamma(\beta-1)\,x^2y+\alpha(\gamma-1)\,y^2z+\beta(\alpha-1)\,z^2x+\beta(\gamma-1)\,x^2z$$
$$+\gamma(\alpha-1)\,y^2x+\alpha(\beta-1)\,z^2y+2(1-\alpha\beta\gamma)\,xyz = 0.$$

11. The equations of EF, FD, DE are now as follows:—

$$EF: (\beta\gamma-1)\,x+(1-\gamma)\,y+(1-\beta)\,z = 0,$$
$$FD: (1-\gamma)\,x+(\gamma\alpha-1)\,y+(1-\alpha)\,z = 0,$$
$$DE: (1-\beta)\,x+(1-\alpha)\,y+(\alpha\beta-1)\,z = 0.$$



In this case X, Y, Z are collinear, and the equation of XYZ is

$$\frac{x}{\alpha-1}+\frac{y}{\beta-1}+\frac{z}{\gamma-1}=0.$$

12. The equations of the tangents to the cubic at the angular points of the triangle ABC are

$$\gamma(\beta-1)\,y+\beta(\gamma-1)\,z=0,$$

$$\alpha(\gamma-1)\,z+\gamma(\alpha-1)\,x=0,$$

$$\beta(\alpha-1)\,x+\alpha(\beta-1)\,y=0.$$

We see therefore that a conic can be described touching the cubic at A, B, C.

Its equation is

$$\frac{\alpha}{\alpha-1}yz+\frac{\beta}{\beta-1}zx+\frac{\gamma}{\gamma-1}xy=0.$$

As the cubic is unaltered by the interchange of any two of the three triangles ABC, DEF, PQR, we conclude that conics can be described to touch the cubic at the angular points of the triangles DEF, PQR.

13. Since the triangles ABC, DEF are compolar, the points X, Y, Z are collinear, also the triangles ABF, DEC are compolar; therefore P, Q, Z are collinear. Similarly, X, Q, R and P, Y, R are collinear. Therefore the triangle PQR is coaxial and also compolar with each of the triangles ABC, DEF. It follows, therefore, in this case that Grassmann's cubic is the same for each of the six possible permutations of the triangles ABC, DEF, PQR.

14. The triangle XYR is compolar with each of the compolar triangles ABF, DEC, and they have a common axis PQZ. Hence conics can be described to touch the curve at the angular points of each of the triangles ABF, DEC, XYR. In the same way, it can be shown that, if we take the 64 combinations of letters which it is possible to obtain by taking one letter out of each of the three sets $ADPX$; $BEQY$; $CFRZ$, each of 48 will give the angular points of a triangle through which a conic can be drawn having triple contact with the cubic, and the remaining 16 combinations will give sets of collinear points.

15. From the general theory of cubic curves, we see that the tangents to the curve at A, B, Z, three points on a straight line, meet

the curve again in A', B', Z', three points on a straight line, also the three tangents at A, B, C, the points of contact of a conic having triple contact with the curve, meet the curve again in A', B', C', three points on a straight line.

It follows therefore that the points Z', C' are coincident. Similarly, it can be shown that the tangents at F and R pass through the same point C'. Similarly, the tangents at A, D, P, X all pass through A', and the tangents at B, E, Q, Y all pass through B'.

16. It appears from what has gone before that, if $lx+my+nz=0$ be part of the cubic,

$$l(\alpha-1)=m(\beta-1)=n(\gamma-1),$$

and therefore the equation of the straight line may be written

$$\frac{x}{\alpha-1}+\frac{y}{\beta-1}+\frac{z}{\gamma-1}=0,$$

which is the equation of XYZ. If this line be part of the cubic, it is seen, by comparing the coefficients of x^2y, y^2z, z^2x in the cubic, that the remaining part of the cubic must be the conic

$$\frac{\alpha}{\alpha-1}yz+\frac{\beta}{\beta-1}zx+\frac{\gamma}{\gamma-1}xy=0.$$

Multiplication of the left-hand sides of the last two equations gives at once the coefficients of the terms xy^2, yz^2, zx^2, the same as those of the cubic; but, in order that the cubic may admit of the two factors, α, β, γ must satisfy the condition

$$\frac{\alpha(\beta-1)(\gamma-1)}{\alpha-1}+\frac{\beta(\gamma-1)(\alpha-1)}{\beta-1}+\frac{\gamma(\alpha-1)(\beta-1)}{\gamma-1}=2(1-\alpha\beta\gamma),$$

or $$\alpha(\beta-1)^2(\gamma-1)^2+\beta(\gamma-1)^2(\alpha-1)^2+\gamma(\alpha-1)^2(\beta-1)^2$$
$$-2(1-\alpha\beta\gamma)(\alpha-1)(\beta-1)(\gamma-1)=0.$$

Let $p=\alpha\beta\gamma,\quad q=\beta\gamma+\gamma\alpha+\alpha\beta,\quad s=\alpha+\beta+\gamma.$

Now $$\Sigma\alpha(\beta-1)^2(\gamma-1)^2$$
$$=\Sigma\alpha(\beta^2-2\beta+1)(\gamma^2-2\gamma+1)$$
$$=\Sigma\alpha\{\beta\gamma(\beta\gamma-2\beta-2\gamma+4)+\beta^2-2\beta+\gamma^2-2\gamma+1\}$$
$$=pq-4ps+12p+sq-3p-4q+s$$
$$=pq-4ps+9p+sq-4q+s.$$

Therefore $\Sigma\alpha(\beta-1)^2(\gamma-1)^2-2(1-\alpha\beta\gamma)(\alpha-1)(\beta-1)(\gamma-1)$

$$=pq-4ps+9p+sq-4q+s+2p(p-q+s-1)-2(p-q+s-1)$$
$$=2p^2+p(-q-2s+5)+sq-2q-s+2$$
$$=(p-s+2)(2p-q+1).$$

Therefore the cubic consists of the straight line

$$\frac{x}{\alpha-1}+\frac{y}{\beta-1}+\frac{z}{\gamma-1}=0$$

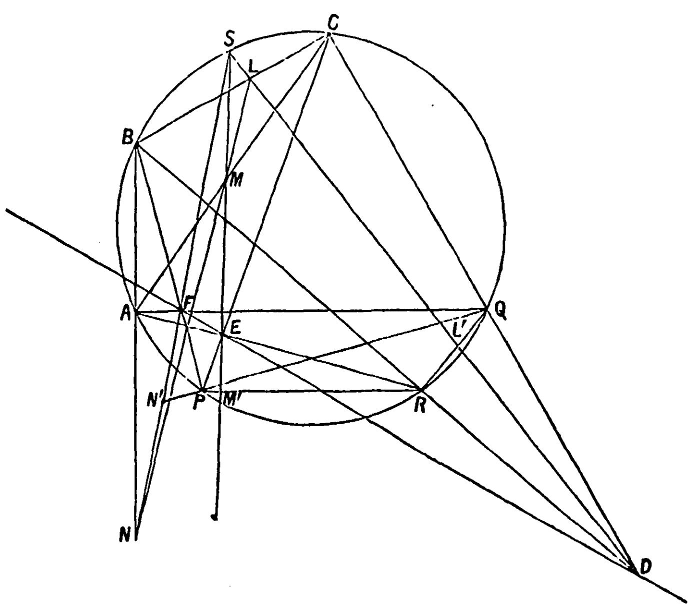
and the conic $$\frac{\alpha}{\alpha-1}yz+\frac{\beta}{\beta-1}zx+\frac{\gamma}{\gamma-1}xy=0,$$

if either $$\alpha\beta\gamma-\alpha-\beta-\gamma+2=0,$$

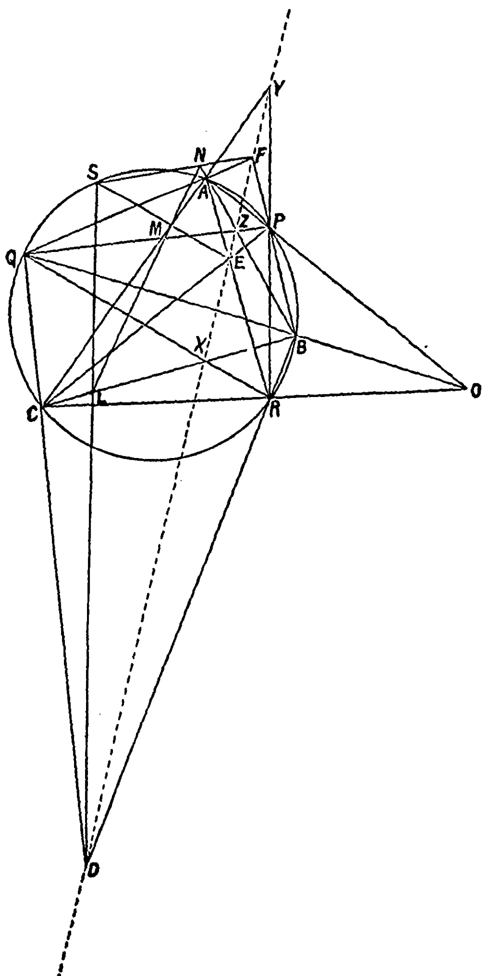
or $$\frac{1}{\alpha\beta\gamma}-\frac{1}{\alpha}-\frac{1}{\beta}-\frac{1}{\gamma}+2=0,$$

that is to say, either if the points D, E, F are collinear, or if the points A, B, C, D, E, F lie on a conic. In the first case, the line is DEF and the conic $ABCPQR$; in the second case, the line is PQR and the conic $ABCDEF$.

17. We have proved that Grassmann's cubic degenerates, in the case when D, E, F are collinear, into a straight line DEF and the conic $ABCPQR$.



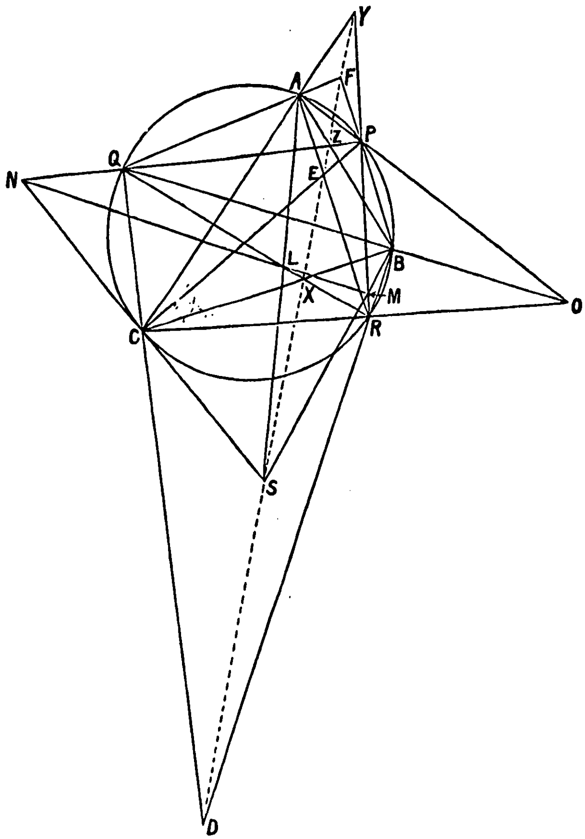
18. We have also proved that, in the case when AD, BE, CF are concurrent, the cubic is unchanged by the interchange of any two of the three triangles ABC, DEF, PQR.



19. The following theorem is a corollary from § 17 and Pascal's theorem:—

If BR, CQ; CP, AR; AQ, BP, pairs of opposite sides of a hexagon inscribed in a conic, intersect in D, E, F respectively, and if S be any point on the conic, then SD, SE, SF intersect the sides of each of the triangles ABC, PQR in three collinear points.

Professor Elliott has remarked that this theorem may also be obtained by projection from an extension of Simson's line theorem given in the *Pitt Press Euclid* (Book III., p. 272, Ex. 1).



20. The following theorems are corollaries from §§ 17, 18.

If ABC, PQR be a pair of compolar triangles, whose pole is O, inscribed in a conic, and S be any point on the conic, then the straight lines joining S to the angular points of either of the triangles ABC, PQR intersect the sides of the other triangle in three points lying on a straight line.

It can be proved by the help of Pascal's theorem that this straight line passes through O. (Townsend's *Modern Geometry*, Vol. II., p. 207).

Further: if ABC, PQR be a pair of compolar triangles inscribed in a conic, and D, E, F the points of intersection of BR, CQ; CP, AR; AQ, BP, and S any point on DEF, then the three straight lines joining S to the angular points of either of the triangles ABC, PQR intersect the sides of the other triangle in three collinear points.

Professor Elliott remarks that this result can be proved by comparing J. W. Russell, *An Elementary Treatise on Pure Geometry* (1893), p. 182, Ex. 6, and J. W. Russell, *l.c.*, p. 215, Ex. 6.

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*The Calculus of Equivalent Statements. (Sixth Paper.)** *By* HUGH MACCOLL, B.A. Communicated June 10th, 1897, by Mr. TUCKER. Received, in revised form, September 4th, 1897.

It will be observed that the three-dimensional system of logic which I proposed in my last paper is closely connected with the theory of probability; so much so that the three classes of statements, ϵ, η, θ may be respectively defined as statements whose chances of being true are 1, 0, x, in which x is some proper fraction less than 1 and greater than 0. But it is clear that we might develop other analogous three-dimensional systems on similar, yet not identical, principles. For example, κ might denote all statements *known to be true*, or capable of being proved true; λ all statements *known to be false*, or capable of being proved false; and μ all statements which, through insufficiency of data, *can neither be proved true nor proved false*. Or, again, the symbol u might denote all *functional* statements which, like $\alpha\beta+\alpha'+\beta'$, are *true for all values of their constituents*; v all functional statements which, like $\alpha\beta(\alpha'+\beta')$, are *false for all values of their constituents*; and w all functional statements which, like $\alpha:\beta$ or $\alpha\beta+\gamma$, *may be true or false* according to the values we give to their elementary constituents. By combining the

* Though I call this the *Sixth* Paper, and the one preceding it was called the *Fifth*, I may mention that my paper in the *Proceedings* on "The Limits of Multiple Integrals" (printed after my *Fourth*) belongs to the same subject, though it has a different title. It is a continuation on purely mathematical lines of my *First* Paper.