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The Theory of Cauchy's Principal Values. (Second Paper : *The use of Principal Values in some of the Double Limit Problems of the Integral Calculus.*) *By* G. H. HARDY. Read and received June 13th, 1901.

Principal Values depending on a Parameter.

1. If $f(x, \alpha)$ is a function of the two variables x, α, which for certain values of α possesses a convergent integral from $x = a$ to $x = A$,

$$I(\alpha) = \int_a^A f(x, \alpha)\, dx$$

is a function of α defined for those values of α. We may suppose a, A independent of α ; for, if they depended on α, we could make the substitution

$$x = a + (A - a)\, y,$$

and so obtain an integral with the constant limits 0, 1.

We suppose further that the values of α for which $I(\alpha)$ is defined are infinite in number, and form a *closed set* S ; and that α_0 is a limiting point of the set. Then the general double limit problem of the integral calculus is : *To determine the relations between*

$$I(\alpha_0) = \int_a^A f(x, \alpha_0)\, dx$$

and the limits of indetermination of $I(\alpha)$ *for* $\alpha = \alpha_0$.

It is not difficult to show that we may without loss of generality suppose that the parameter is either *a positive integer which tends steadily to* ∞, *or a continuous variable which tends steadily to any given value.*

These problems—problems such as that of the integration of an infinite series term by term, or of differentiation under the integral sign—are well known and have been very frequently discussed. It has, however, generally been assumed that all the integrals which occur in connection with them are *unconditionally* convergent. In this paper I shall begin a discussion of some of the corresponding problems which arise when we are considering integrals which are only *conditionally* convergent, the *principal values,* in fact, the elementary theory of which formed the subject of my first paper.* I shall begin with the case in which the parameter is integral.

Principal Values and Infinite Series.

2. Let
$$S(x) = \sum_0^\infty u_n(x)$$
be a series whose terms are functions of x, convergent, at any rate in general—*i.e.*, with the possible exception of a closed enumerable set of points—for values of x in an interval (a, A). Then $S(x)$ is *integrable term by term over* (a, A) if
$$\int_a^A S(x)\,dx = \sum_0^\infty \int_a^A u_n(x)\,dx, \tag{1}$$
or
$$\int_a^A \lim_{N=\infty} \sum_0^N u_n(x)\,dx = \lim_{N=\infty} \int_a^A \sum_0^N u_n(x)\,dx.$$

The conditions under which this equation is true have been discussed by many writers. We may refer especially to Dini, *Grundlagen*, pp. 512–530, and Osgood, "On Non-uniform Convergence, &c.," *American Jour. of Math.*, Vol. XIX.

The question with which we are concerned at present is: Under what circumstances is (1) true when some or all of the integrals which appear in it are only principal values?

3. Let us suppose, in the first place, that the interval (a, A) is finite, and that $S(x)$ is integrable term by term across any part of (a, A) which does not include a single point α $(a < \alpha < A)$. Then, however small be the positive quantity ϵ,
$$\left(\int_a^{\alpha-\epsilon} + \int_{\alpha+\epsilon}^A\right) \sum_0^\infty u_n\,dx = \sum_0^\infty \left(\int_a^{\alpha-\epsilon} + \int_{\alpha+\epsilon}^A\right) u_n\,dx. \tag{1}$$

* "The Elementary Theory of Cauchy's Principal Values," *Proc. L.M.S.*, Vol. XXXIV., pp. 16–40.

Let us suppose further that

$$P\int_a^A u_n(x)\,dx$$

is convergent for every value of n, and that

$$\sum_0^\infty P\int_a^A u_n(x)\,dx$$

is convergent. Then the right-hand side is

$$\sum_0^\infty P\int_a^A u_n\,dx - \sum_0^\infty P\int_{\alpha-\epsilon}^{\alpha+\epsilon} u_n\,dx.$$

If, finally, we suppose that the last term tends to zero with ϵ, the left-hand side of (1) will also tend to a limit, which is by definition

$$P\int_a^A \sum_0^\infty u_n(x)\,dx;$$

and

$$P\int_a^A \sum_0^\infty u_n\,dx = \sum_0^\infty P\int_a^A u_n\,dx. \qquad (2)$$

This equation is certainly true, then, if (i.) Σu_n is integrable term by term over any part of (a, A) not including α,

$$\text{(ii.)}\quad F(x) = \Sigma P\int_a^x u_n\,dx$$

is a continuous function of x except at α, and

$$\text{(iii.)}\quad \lim_{\epsilon=0}\left\{F(\alpha-\epsilon)-F(\alpha+\epsilon)\right\} = 0.$$

4. We may distinguish three cases: (i.) that in which no one of the individual terms u_n becomes infinite at $x = \alpha$, (ii.) that in which a finite number of them become infinite, and (iii.) that in which an infinite number of them do so. The first and last are the only cases of importance, as in the second case we can consider the terms which become infinite separately.

5. (i.), (ii.). In this case the symbol of the principal value on the right of (2) of § 3 is unnecessary, *i.e.*,

$$P\int_a^A \sum_0^\infty u_n\,dx = \sum_0^\infty \int_a^A u_n\,dx.$$

And we may state the conditions of § 3 as follows: that Σu_n is integrable term by term over any part of (a, A) which does not include a, and

$$F(x) = \sum_0^\infty \int_a^x u_n\,dx$$

is a continuous function of x except at a, and

$$\lim_{\epsilon=0} \{F(a-\epsilon)-F(a+\epsilon)\} = 0.$$

6. Suppose, for instance,

$$u_n = -\frac{x^n}{a^{n+1}} \quad (0 \leqq x < a)$$

$$= \frac{a^n}{x^{n+1}} \quad (a < x \leqq 1);$$

the value of u_n for $x = a$ is immaterial. Then

$$S(x) = \frac{1}{x-a} \quad (0 \leqq x \leqq 1),$$

except for $x = a$.* Also, if $0 \leqq x < a$,

$$F(x) = -\sum_0^\infty \frac{x^{n+1}}{(n+1)\,a^{n+1}} = \log\left(1-\frac{x}{a}\right);$$

while, if $a < x \leqq 1$,

$$F(x) = -\left[\frac{x}{a}\right]_0^a + \left[\log x\right]_a^x - \sum_1^\infty \left\{\left[\frac{x^{n+1}}{(n+1)\,a^{n+1}}\right]_0^a + \left[\frac{a^n}{nx^n}\right]_a^x\right\}$$

$$= \log\left(\frac{x}{a}-1\right).$$

Thus $F(x)$ is continuous except at a, and

$$F(a-\epsilon)-F(a+\epsilon) = 0.$$

Also

$$\int_0^1 u_n\,dx = \frac{1-a^n}{n} - \frac{1}{n+1} \quad (n > 0),$$

and

$$P\int_0^1 \frac{dx}{x-a} = \log\frac{1}{a} - 1 + \sum_1^\infty \int_0^1 u_n\,dx$$

$$= \log\left(\frac{1}{a}-1\right).$$

7. (iii.) The simplest case in which an infinite number of the terms u_n become infinite is that in which they all become infinite owing to the occurrence in all of them of the same factor

$$\Omega_\nu\,(x-a).$$

Let us suppose that $u_n = \Omega_\nu\,(x-a)\,v_n$,

where v_n is a function of x which, whatever be n, has a continuous

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* Here a is the a of §§ 3–5.

derivate for all values of x in question. Then, by a lemma proved in my first paper,

$$P\int_{a-\epsilon}^{a+\epsilon} u_n\,dx = \left[v_n'\right]_{a+\mu}\int_{a-\epsilon}^{a+\epsilon}(x-a)\,\Omega_\nu(x-a)\,dx,$$

where $-\epsilon \leq \mu \leq \epsilon$. In particular, if

$$\Omega_\nu(x-a) = \frac{1}{x-a},$$

$$P\int_{a-\epsilon}^{a+\epsilon} u_n\,dx = 2\epsilon\left[v_n'\right]_{a+\mu}.$$

Suppose now that $$|v_n'| < V_n$$

for all values of x and n in question, V_n being independent of x and ΣV_n convergent. Then the last condition of § 3 will certainly be fulfilled.

8. Let, for instance,

$$u_0 = 1, \quad u_n = \frac{2p^n \cos nx}{\cos x - \cos a} \quad (n > 0),$$

where $0 < a < \pi$, $|p| < 1$. Then

$$S(x) = \frac{1-p^2}{(\cos x - \cos a)(1-2p\cos x + p^2)},$$

and, if we may use equation (2) of § 3,

$$(1-p^2)\,P\int_0^\pi \frac{dx}{(\cos x - \cos a)(1-2p\cos x + p^2)} = 2\sum_1^\infty p^n P\int_0^\pi \frac{\cos nx\,dx}{\cos x - \cos a},$$

since $$P\int_0^\pi \frac{dx}{\cos x - \cos a} = 0.$$

But the left-hand is

$$\frac{1-p^2}{1-2p\cos a + p^2}\left\{2p\int_0^\pi \frac{dx}{1-2p\cos x + p^2} + P\int_0^\pi \frac{dx}{\cos x - \cos a}\right\} = \frac{2\pi p}{1+p^2-2p\cos a}$$

$$= \frac{2\pi}{\sin a}\sum_1^\infty p^n \sin na.$$

And so $$P\int_0^\pi \frac{\cos nx\,dx}{\cos x - \cos a} = \frac{\pi \sin na}{\sin a}.$$

To justify the use of § 3 we have only to observe that

$$v_n = 2p^n \cos nx\,\frac{x-a}{\cos x - \cos a},$$

and $$|v_n'|_{a+\mu} < K|p|^n,$$

where K is some quantity independent of x and of n.

9. We have so far supposed that (a, A) is finite and contains but one point a across which the series ceases to be integrable term by term in the ordinary way. No new point arises if (a, A) contains

any other finite number of such points a. This is so even if A be infinite; for all the points a can be included in a finite interval (a, A_1), and integration over (A_1, ∞) is merely an ordinary case of integration term by term.

10. Thus, for instance,

$$P\int_0^\infty \frac{1}{1-2p\cos x+p^2}\,\frac{dx}{x^2-a^2} \quad (a>0,\ |p|<1)$$

$$=\frac{1}{1-p^2}\left[P\int_0^\infty \frac{dx}{x^2-a^2}+2\sum_1^\infty p^n P\int_0^\infty \frac{\cos nx}{x^2-a^2}\,dx\right]$$

$$=-\frac{\pi}{a(1-p^2)}\sum_1^\infty p^n \sin na=-\frac{\pi}{a(1-p^2)}\,\frac{p\sin a}{1-2p\cos a+p^2}.$$

This integral is given by De Haan (*Tables*, 193, 1).

11. Let us suppose now that there are infinitely many points a. I shall confine myself at present to the simplest case; I suppose $A=\infty$, the points a (a_i) isolated,

$$a<a_1<a_2<\ldots \quad (\lim_{i=\infty} a_i=\infty),$$

and

$$a_{i+1}-a_i>H \quad (i=1, 2, \ldots),$$

where H is a positive quantity independent of i. That is to say, I suppose that the principal values with which we are dealing are of the type covered by the earlier definitions of my first paper. There is no particular difficulty in applying similar considerations to the more general cases dealt with by the later definitions.

I suppose, moreover, that the conditions of § 3 are satisfied over any finite interval (a, A_1), provided $A_1 \neq a_i$—that is to say, that Σu_n is integrable, term by term, over any part of such an interval which does not include any point a_i; that

$$F(x)=\sum_0^\infty P\int_a^x u_n\,dx$$

is a continuous function of x, except at the points a_i; and that

$$\lim_{\epsilon=0}\{F(a_i-\epsilon)-F(a_i+\epsilon)\}=0 \quad (i=1, 2, \ldots).$$

Then, if δ be any small positive quantity, and

$$|x-a_i|>\delta \quad (i=1, 2, \ldots),$$

$$F(x)-F(a)=P\int_a^x \Sigma u_n\,dx. \tag{1}$$

Now let us suppose that

$$P\int_a^\infty \Sigma u_n\,dx$$

is convergent; that is, that when x tends to ∞ through any system of values satisfying the above conditions the right side of (1) tends to a finite limit independent of the particular system chosen, and therefore independent of δ. Then $F(x)$ tends to a limit, and, if

$$\lim F(x) = \Sigma P\int_a^\infty u_n\,dx,$$

it will follow that

$$P\int_a^\infty \Sigma u_n\,dx = \Sigma P\int_a^\infty u_n\,dx. \qquad (2)$$

12. Let us apply this formula to the evaluation of

$$P\int_0^\infty \cos ax \cot ax \frac{x\,dx}{1+x^2} \quad (0<a<2\alpha),$$

which, after my previous paper, we know to be determinate.

Since $$\cot \alpha x = \frac{1}{\alpha x}+2\sum_1^\infty \frac{\alpha x}{\alpha^2x^2-n^2\pi^2},$$

$$P\int_0^\infty \cos ax \cot \alpha x \frac{x\,dx}{1+x^2} = \frac{1}{\alpha}\int_0^\infty \frac{\cos ax\,dx}{1+x^2}+2\alpha\sum_1^\infty P\int_0^\infty \frac{x^2\cos ax\,dx}{(1+x^2)(\alpha^2x^2-n^2\pi^2)},$$

if we may use the formula. Now

$$P\int_0^\infty \frac{x^2\cos ax\,dx}{(1+x^2)(\alpha^2x^2-n^2\pi^2)} = \frac{1}{\alpha^2+n^2\pi^2}\left\{\int_0^\infty \frac{\cos ax\,dx}{1+x^2}+n^2\pi^2P\int_0^\infty \frac{\cos ax\,dx}{\alpha^2x^2-n^2\pi^2}\right\}$$

$$= \tfrac{1}{2}\frac{\pi}{\alpha^2+n^2\pi^2}\left\{e^{-a}-\frac{n\pi}{\alpha}\sin\frac{na\pi}{\alpha}\right\}.$$

Also $$\sum_1^\infty \frac{1}{\alpha^2+n^2\pi^2} = \frac{1}{2\alpha}\left(\coth\alpha-\frac{1}{\alpha}\right),$$

$$\sum_1^\infty \frac{\frac{n\pi}{\alpha}\sin\frac{na\pi}{\alpha}}{\alpha^2+n^2\pi^2} = \frac{1}{2\alpha}\frac{\sinh(\alpha-a)}{\sinh\alpha} \quad (0<a<2\alpha).$$

Hence $$P\int_0^\infty \cos ax \cot \alpha x\frac{x\,dx}{1+x^2} = \tfrac{1}{2}\pi\left\{e^{-a}\coth\alpha-\frac{\sinh(\alpha-a)}{\sinh\alpha}\right\} = \frac{\pi\cosh a}{e^{2\alpha}-1}.$$

This result may be found in other ways. See the *Quarterly Journal*, 1900, p. 126.

We have still to justify our use of the formula of § 11. This, as might be anticipated, requires an argument of some little complexity. In the first place it is clear, after §§ 3–8, that (1) of § 11 holds; and so what we have to prove is that we can so choose ξ that

$$\sum_1 P\int_x^\infty \frac{x^2\cos ax\,dx}{(1+x^2)(\alpha^2x^2-n^2\pi^2)}$$

is numerically less than any assigned positive quantity σ, for all values of $x>\xi$, and such that

$$|x-a_i|>\delta \quad (i=1, 2, \ldots);$$

and that however small be δ.

Now this quantity is

$$\Sigma \frac{1}{a^2+n^2\pi^2}\int_x^\infty \frac{\cos ax\,dx}{1+x^2}+\Sigma \frac{n^2\pi^2}{a^2+n^2\pi^2}P\int_x^\infty \frac{\cos ax\,dx}{a^2x^2-n^2\pi^2},$$

and evidently we need only trouble about the second part.

Now suppose either $$(N-\tfrac{1}{2})\frac{\pi}{a}<x<\frac{N\pi}{a}-\delta,$$

or $$\frac{N\pi}{a}+\delta<x<(N+\tfrac{1}{2})\frac{\pi}{a},$$

say the former. Then, if $n<N$,

$$\left|\frac{n^2\pi^2}{a^2+n^2\pi^2}\int_{(N-\frac{1}{2})\pi/a}^{x}\right| < \int_{(N-\frac{1}{2})\pi/a}^{(N+\frac{1}{2})\pi/a}\frac{dx}{a^2x^2-n^2\pi^2}$$

$$< \frac{1}{2an\pi}\log\left\{\frac{N+\frac{1}{2}-n}{N-\frac{1}{2}-n}\,\frac{N-\frac{1}{2}+n}{N+\frac{1}{2}+n}\right\};$$

if $n>N$, $$< \frac{1}{2an\pi}\log\left\{\frac{n-N-\frac{1}{2}}{n-N+\frac{1}{2}}\,\frac{N-\frac{1}{2}+n}{N+\frac{1}{2}+n}\right\};$$

and, if $n=N$, $$<\frac{1}{a\delta}\,\frac{1}{(2N-\frac{1}{2})\pi}\,\frac{\pi}{2a}.$$

This last quantity can be made as small as we please by choice of N. Again,

$$\sum_1^{N-1}\frac{1}{n}\log\left\{\frac{N+\frac{1}{2}-n}{N-\frac{1}{2}-n}\,\frac{N-\frac{1}{2}+n}{N+\frac{1}{2}+n}\right\}=\sum_1^{N_1}+\sum_{N_1+1}^{N-1}\quad [N_1=E(\sqrt{N})]$$

$$<\log\left\{\frac{N-\frac{1}{2}}{N-\frac{1}{2}-N_1}\,\frac{N+\frac{1}{2}+N_1}{N+\frac{1}{2}}\right\}$$

$$+\frac{1}{N_1}\log\left\{\frac{N-\frac{1}{2}-N_1}{\frac{1}{2}}\,\frac{2N-\frac{1}{2}}{N+\frac{1}{2}+N_1}\right\},$$

which can be made as small as we please by choice of N. Also

$$\sum_{N+1}^{\infty}\frac{1}{n}\log\left\{\frac{n-N+\frac{1}{2}}{n-N-\frac{1}{2}}\,\frac{N+\frac{1}{2}+n}{N-\frac{1}{2}+n}\right\}=\sum_{N+1}^{2N}+\sum_{2N}^{\infty}$$

$$<\frac{1}{N+1}\log\left\{\frac{N+\frac{1}{2}}{\frac{1}{2}}\,\frac{3N+\frac{1}{2}}{2N+\frac{1}{2}}\right\}+\sum_{2N}^{\infty},$$

which too can be made as small as we please by choice of N. Hence, finally,

$$\left|\Sigma\frac{n^2\pi^2}{a^2+n^2\pi^2}\int_{(N-\frac{1}{2})\pi/a}^{x}\right|$$

can be made as small as we please by choice of N. And so we need only consider

$$\Sigma\frac{n^2\pi^2}{a^2+n^2\pi^2}P\int_{(N-\frac{1}{2})\pi/a}\frac{\cos ax\,dx}{a^2x^2-n^2\pi^2}.$$

We consider first the terms for which $n<N$: in them no P is needed. As x increases from $(N-\frac{1}{2})\frac{\pi}{a}$, $\cos ax$ oscillates, and $\frac{1}{a^2x^2-n^2\pi^2}$ steadily decreases. And so, if $(m-\frac{1}{2})\frac{\pi}{a}$ is the first odd multiple of $\frac{\pi}{2a}$ which is $\geqq (N-\frac{1}{2})\frac{\pi}{a}$, and $(N+p-\frac{1}{2})\frac{\pi}{a}$

is the first odd multiple of $\frac{\pi}{2a}$ which is $\geqq (m+\frac{1}{2})\frac{\pi}{a}$,

$$\left|\frac{n^2\pi^2}{a^2+n^2\pi^2}\int_{(N-\frac{1}{2})\pi/a}^{\infty}\right| < \int_{(N-\frac{1}{2})\pi/a}^{(N+p-\frac{1}{2})\pi/a}\frac{dx}{a^2x^2-n^2\pi^2}$$

$$< \frac{1}{2an\pi}\log\left\{\frac{N+p-\frac{1}{2}-n}{N-\frac{1}{2}-n}\,\frac{N-\frac{1}{2}+n}{N+p-\frac{1}{2}+n}\right\};$$

and it follows by a slight modification of our previous argument that

$$\left|\sum_1^{N-1}\frac{n^2\pi^2}{a^2+n^2\pi^2}\int_{(N-\frac{1}{2})\pi/a}^{\infty}\right|$$

can be made as small as we please by choice of N.

There remain the terms for which $n \geqq N$. We may write them in the form

$$\tfrac{1}{2}\sum_N^{\infty}\frac{n\pi}{a^2+n^2\pi^2}\left(\int_{(N-\frac{1}{2})\pi/a}^{\infty}\frac{\cos ax}{ax+n\pi}\,dx + P\int_{(N-\frac{1}{2})\pi/a}^{\infty}\frac{\cos ax}{ax-n\pi}\,dx\right). \tag{1}$$

The first series is

$$\tfrac{1}{2}\sum_N^{\infty}\frac{n\pi}{a^2+n^2\pi^2}\sum_N^{\infty}\int_{(i-\frac{1}{2})\pi/a}^{(i+\frac{1}{2})\pi/a}\frac{\cos ax}{ax+n\pi}\,dx = \tfrac{1}{2}\sum_N^{\infty}\frac{n\pi}{a^2+n^2\pi^2}\sum_N^{\infty}\int_{-\pi/2a}^{\pi/2a}\frac{\cos a\left(x+\frac{i\pi}{a}\right)}{ax+(i+n)\pi}\,dx.$$

Now we may sum the series

$$\sum_N^{\infty}\frac{\begin{matrix}\cos\\ \sin\end{matrix}\,\frac{ai\pi}{a}}{ax+(i+n)\pi}$$

under the integral sign. And it is equal, by Abel's lemma, to

$$\sum_N^{\infty}\left\{\frac{1}{ax+(i+n)\pi}-\frac{1}{ax+(i+1+n)\pi}\right\}S_i,$$

where

$$S_i = \sum_N^{i}\begin{matrix}\cos\\ \sin\end{matrix}\,\frac{a\kappa\pi}{a}.$$

This is in absolute value $< \dfrac{S}{ax+(N+n)\pi}$,

where S is the numerically greatest of the moduli of the sums S_i; and therefore $< \frac{S}{n\pi}$. And the first series in (1) is therefore numerically less than

$$\frac{\pi S}{a}\sum_N^{\infty}\frac{1}{a^2+n^2\pi^2},$$

and can be made as small as we please by choice of N.

The second series in (1) is

$$\tfrac{1}{2}\sum_{n=N}^{\infty}\frac{n\pi}{a^2+n^2\pi^2}\sum_{i=N}^{\infty}P\int_{-\pi/2a}^{\pi/2a}\frac{\cos a\left(x+\frac{i\pi}{a}\right)}{ax+(i-n)\pi}\,dx.$$

We separate the terms which correspond to one value of n into two classes, from $i = N$ to $i = 2n-N$, and from $i = 2n-N+1$ to $i = \infty$. By an argument similar to that which we used when we were considering the terms for which $n < N$, we can show (i.) that the series

$$\tfrac{1}{2}\sum_N^{\infty}\left(\sum_{2n-N+1}^{\infty}\right)$$

is convergent, and (ii.) that it is numerically less than a certain constant multiple of

$$\sum_N^\infty \frac{n}{a^2+n^2\pi^2}\,\frac{1}{n-N+\frac{1}{2}},$$

which is $$< \frac{1}{\pi^2}\sum_N^\infty \frac{1}{n(n-N+\frac{1}{2})} < \frac{1}{N\pi^2}\left\{2+\sum_{N+1}^\infty\left(\frac{1}{n-N}-\frac{1}{n}\right)\right\}$$
$$< \frac{1}{N\pi^2}\left\{2+\sum_1^N \frac{1}{n}\right\};$$

and this can be made as small as we please by choice of N.

It only remains to consider

$$\tfrac{1}{2}\sum_N^\infty \frac{n\pi}{a^2+n^2\pi^2}\left[\sum_N^{2n-N} P\int_{-\pi/2a}^{\pi/2a} \frac{\cos a\left(x+\dfrac{i\pi}{a}\right)}{ax+(i-n)\pi}\,dx\right].$$

That this series is convergent follows from what precedes. Also the inner sum is

$$\left[P\int_{(N-\frac{1}{2})\pi/a}^{(2n-N+\frac{1}{2})\pi/a}\frac{\cos ax}{ax-n\pi}\,dx = P\int_{(N-n-\frac{1}{2})\pi/a}^{(n-N+\frac{1}{2})\pi/a}\frac{\cos a(x+n\pi)}{ax}\,dx\right.$$
$$= \int_0^{(n-N+\frac{1}{2})\pi/a}\frac{\cos a\left(x+\dfrac{n\pi}{a}\right)-\cos a\left(x-\dfrac{n\pi}{a}\right)}{ax}\,dx$$
$$= -2\sin\frac{na\pi}{a}\int_0^{(n-N+\frac{1}{2})\pi/a}\frac{\sin ax}{ax}\,dx$$
$$= -2\sin\frac{na\pi}{a}\sum_0^{n-N} v_k,$$

where $$v_k = \int_{(k-\frac{1}{2})\pi/a}^{(k+\frac{1}{2})\pi/a}\frac{\sin ax}{ax}\,dx \quad (k>0),$$
$$v_0 = \int_0^{\pi/2a}\frac{\sin ax}{ax}\,dx.$$

Now, let $$u_n = -\frac{n\pi\sin\dfrac{na\pi}{a}}{a^2+n^2\pi^2}.$$

Then $$\sum_N^\infty u_n\sum_0^{n-N} v_k = u_N v_0 + u_{N+1}(v_0+v_1)+\ldots = \sum_0^\infty v_k\sum_{N+k}^\infty u_n$$

(we assume for the moment that this transformation is legitimate). Now v_k, as it is easy to see, decreases like $\frac{1}{k}$ as k increases. And

$$\sum_{N+k}^\infty u_n = \sum_{N+k}^\infty\left\{\frac{(n+1)\pi}{a^2+(n+1)^2\pi^2}-\frac{n\pi}{a^2+n^2\pi^2}\right\}S_n,$$

where $$S_n = \sum_{N+k}^n \sin\frac{na\pi}{a},$$

and is therefore numerically less than a constant multiple of

$$\frac{(N+k)\pi}{a^2+(N+k)^2\pi^2};$$

and, *a fortiori*, numerically less than a constant multiple of $\frac{1}{k}$. Hence $v_k\sum_{N+k}^\infty u_n$

is numerically less than a constant multiple of $\frac{1}{k^2}$. Moreover, when k is fixed, it decreases indefinitely as N increases. It follows that we can make

$$\sum_0^\infty v_k \sum_{N+k}^\infty u_n$$

as small as we please by choice of N.—*October*, 1901.]

It only remains to show that our assumption as to the transformation of $\sum_N^\infty u_n \sum_0^{n-N} v_k$ was justified. I pass over the proof of this, as it is not difficult, and presents no point of special interest in connexion with my present subject. I conclude, finally, that *the series*

$$\sum_1^\infty \frac{n^2\pi^2}{a^2+n^2\pi^2} P\int_{(N-\frac{1}{2})\pi/a}^\infty \frac{\cos ax\, dx}{a^2x^2-n^2\pi^2}$$

can be made as small as we please by choice of N. It follows that the use I made at the beginning of this section of the formula of § 11 was legitimate.

It was really by this method that Legendre and Lacroix "verified" Cauchy's formulæ

$$P\int_0^\infty \frac{\cos ax}{\cos bx}\frac{dx}{1+x^2} = \tfrac{1}{2}\pi\frac{\cosh a}{\cosh b} \quad (0 \leq a \leq b), \ \ldots;$$

(see their *Rapport* on his "Mémoire sur les Intégrales définies," Cauchy, *Œuvres*, Vol. I.). The preceding analysis will be sufficient to show how little they had appreciated the difficulties which it involves.*

Similarly

$$P\int_0^\infty \cos ax \cot ax \frac{x\,dx}{1-x^2} = \frac{1}{a}\cdot P\int_0^\infty \frac{\cos ax}{1-x^2}dx + 2a\sum_1^\infty P\int_0^\infty \frac{x^2\cos ax\,dx}{(1-x^2)(a^2x^2-n^2\pi^2)}.$$

Now
$$P\int_0^\infty \frac{x^2\cos ax\,dx}{(1-x^2)(a^2x^2-n^2\pi^2)} = \frac{1}{a^2-n^2\pi^2}\left\{P\int_0^\infty \frac{\cos ax\,dx}{1-x^2} + n^2\pi^2 P\int_0^\infty \frac{\cos ax\,dx}{a^2x^2-n^2\pi^2}\right\}$$
$$= \tfrac{1}{2}\frac{\pi}{a^2-n^2\pi^2}\left\{\sin a - \frac{n\pi}{a}\sin\frac{n a\pi}{a}\right\}.$$

Also
$$\sum_1^\infty \frac{1}{a^2-n^2\pi^2} = \frac{1}{2a}\left(\cot a - \frac{1}{a}\right),$$
$$\sum_1^\infty \frac{\frac{n\pi}{a}\sin\frac{n a\pi}{a}}{a^2-n^2\pi^2} = -\frac{1}{2a}\cdot\frac{\sin(a-a)}{\sin a} \quad (0 < a < 2a),$$

and so
$$P\int_0^\infty \cos ax \cot ax \frac{x\,dx}{1-x^2} = \tfrac{1}{2}\pi\left\{\sin a \cot a + \frac{\sin(a-a)}{\sin a}\right\} = \tfrac{1}{2}\pi\cos a.$$

Principal Values containing a Continuous Parameter.

13. We shall suppose now that the parameter a is *continuous*, and, in the first place, that the range of integration (a, A) and the range of variation of the parameter a are finite.

We suppose, moreover, that the infinities of $f(x, a)$ across which

* [Though I have no doubt it might be simplified to some extent.—*Nov.* 6, 1901.]

$\int f dx$ is not unconditionally convergent lie (for the values of x and a in question) on a finite number of continuous curves $x = X_i(a)$, which do not meet, and have at every point a definite direction nowhere parallel to x.

Uniform and Regular Convergence.

14. The principal value

$$P\int_a^A f(x, a)\, dx \tag{1}$$

will be said to be *uniformly convergent in* (β, γ) if (i.) it is convergent for every value of a in (β, γ); and (ii.) we can find a pair of positive quantities δ_0, ϵ_0 corresponding to any assigned positive quantity σ, such that

$$\left|\int_x^{x+\epsilon} f\, dx\right| < \sigma$$

for all values of a in (β, γ), every $\epsilon \leqq \epsilon_0$, and every value of x such that $a \leqq x$, $x+\epsilon \leqq A$, and x, $x+\epsilon$ differ by at least δ_0 from any of X_i; and

$$\left| P\int_{X_i-\delta}^{X_i+\delta} f\, dx \right| < \sigma$$

for all values of a in (β, γ), and every $\delta \leqq \delta_0$.

I may remark (i.) that the possibility of any of the curves $x = X_i(a)$ meeting $x = a$ or $x = A$ is excluded by the first condition, and (ii.) that the second presupposes the first.

15. Thus, if $f(x, a) = \Omega_\nu(x-a)\,\Theta(x, a)$,

where Θ is a function whose derivate $\dfrac{\partial\Theta}{\partial x}$ is a continuous function of both variables,

$$P\int_a^A f(x, a)\, dx$$

is uniformly convergent in $(a+\xi, A-\xi')$, if $0<\xi<\xi+\xi'<A-a$.

For, in the first place, condition (i.) is satisfied. Again

$$P\int_{a-\delta}^{a+\delta} = \Theta'_x(a+\mu, a)\int_{a-\delta}^{a+\delta} (x-a)\,\Omega_\nu(x-a)\, dx;$$

and, however small be σ, we can choose δ_0 so that the modulus of this is $<\sigma$ for all values of a in question, and every $\delta \leqq \delta_0$.

Moreover, if $a \leq x < x+\epsilon \leq a-\delta_0$ (or $a+\delta_0 \leq x < x+\epsilon \leq A$),

$$\int_x^{x+\epsilon} \Omega_\nu(x-a)\,\Theta\,dx = \int_u^{u+\epsilon} \Omega_\nu(u)\,\Theta(u+a,\,a)\,du$$

$$< K\int_u^{u+\epsilon} |\,\Omega_\nu(u)\,|\,du \text{ (where } K \text{ is a constant)}$$

$$< \frac{K}{\delta_0}\int_u^{u+\epsilon} u\,|\,\Omega_\nu(u)\,|\,du:$$

and, however small be σ, δ_0, we can choose ϵ_0 so that the modulus of this is $<\sigma$ for all values of u and a in question, and every $\epsilon \leq \epsilon_0$. Hence condition (ii.) is satisfied.

Again, if $\qquad f(x,\,a) = \Omega_\nu\{x-X(a)\}\,\Theta(x,\,a),$

where Θ is a function satisfying the same conditions as before, and $X(a)$ is a function of a with a continuous and positive derivate, (1) will be uniformly convergent in $(\beta,\,\gamma)$ if

$$a < X(\beta) < X(\gamma) < A.$$

This follows at once if we put $X(a) = \beta$, and treat f as a function of β.

16. Thus (i.) $P\int_a^A \frac{dx}{x-a}$, $P\int_a^A \frac{l(x-a)}{x-a}\,dx$ are uniformly convergent in $(a+\xi,\,A-\xi')$ if $0<\xi<\xi+\xi'<A-a$.

(ii.) $P\int_0^\pi \frac{dx}{\sin(x-a)}$, $P\int_0^\pi \frac{l\sin(x-a)}{\sin(x-a)}\,dx$ are uniformly convergent in $(n\pi+\xi,\,n\pi+\pi-\xi')$, $n=0,\,1,\,\ldots$, if $0<\xi<\xi+\xi'<\pi$.

(iii.) $P\int_0^{2\pi} \frac{\cos ax\,dx}{\cos x-\cos a}$ is uniformly convergent in $(n\pi+\xi,\,n\pi+\pi-\xi')$, $n=0,\,1,\,\ldots$, if $0<\xi<\xi+\xi'<\pi$.

(iv.) $P\int_0^{2\pi} \frac{\cos ax\,dx}{\cos ax-\cos\theta}$ is uniformly convergent in

$$\left(\frac{2n\pi-\theta}{2\pi}+\xi,\ \frac{2n\pi+\theta}{2\pi}-\xi'\right) \quad (n=1,\,2,\,\ldots),$$

and in

$$\left(\frac{2n\pi+\theta}{2\pi}+\xi_1,\ \frac{2n\pi+2\pi-\theta}{2\pi}-\xi_1'\right) \quad (n=0,\,1,\,\ldots),$$

if $0<\theta<\pi$, $0<\xi<\xi+\xi'<\theta$, $0<\xi_1<\xi_1+\xi_1'<\pi-\theta$.

Again, if $\qquad f(x) = \Omega_\nu(x-X)\,\Theta(x),$

where Θ is a function which has a continuous derivate $\frac{d\Theta}{dx}$, and $a < X < A$,

$$P\int_a^A \sin \alpha x f(x)\,dx,\quad P\int_a^A \cos \alpha x f(x)\,dx$$

are uniformly convergent in any finite interval (β, γ).

17. The principal value

$$P\int_a^A f(x, \alpha)\,dx$$

will be said to be *regularly convergent in* (β, γ) if (i.) it is uniformly convergent in any part of (β, γ) which does not include any one of a finite number of points $\alpha'_1, \ldots, \alpha'_r$, for which it ceases to be determinate; and (ii.) we can find positive quantities ξ, ξ' and $p_i < p_0$ corresponding to any assigned positive quantity p_0, such that

$$P\int_{a+p_i}^{A-p_i} f(x, \alpha)\,dx$$

is uniformly convergent in

$$(\alpha'_i - \xi,\ \alpha'_i + \xi') \quad (i = 1, 2, \ldots, r).$$

If $\alpha'_1 = \beta$, or $\alpha'_r = \gamma$, it is sufficient that $P\int_{a+p_i}^{A-p_i}$ be uniformly convergent in $(\beta, \beta + \xi')$ or $(\gamma - \xi, \gamma)$.

18. This case arises when the conditions for *uniform* convergence are violated owing to some of the curves $x = X_i(\alpha)$ meeting $x = a$ or $x = A$. Then $\alpha'_1, \ldots, \alpha'_r$ are roots of the equations

$$a = X_i(\alpha), \quad A = X_i(\alpha).$$

Thus (i.) $P\int_a^A \frac{dx}{x-\alpha}$, $P\int_a^A \frac{l(x-\alpha)}{x-\alpha}\,dx$ are regularly convergent in any finite interval of values of α. The exceptional values of α are a, A; if $\alpha < a$, or $\alpha > A$, the integrals are unconditionally convergent.

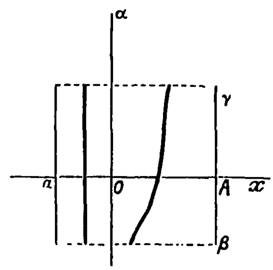
(ii.) $P\int_0^\pi \frac{dx}{\sin(x-\alpha)}$, $P\int_0^\pi \frac{l\sin(x-\alpha)}{\sin(x-\alpha)}\,dx$ are regularly convergent in any finite interval of values of α. The exceptional values of α are 0, $n\pi$.

(iii.) $P\int_0^{2\pi} \frac{\cos \alpha x\,dx}{\cos x - \cos \alpha}$ is regularly convergent in $(2n\pi - \pi + \xi,\ 2n\pi + \pi - \xi')$, if $0 < \xi < \xi + \xi' < 2\pi$; but not in any interval which includes any of the points $(2n+1)\pi$.

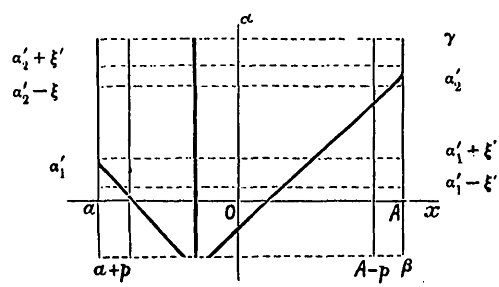
(iv.) $P\int_0^{2\pi} \frac{\cos \alpha x\,dx}{\cos \alpha x - \cos \theta}$ $(0 < \theta < \pi)$ is regularly convergent in any finite interval of values of α. The exceptional values of α are $\frac{2n\pi \pm \theta}{2\pi}$.

A glance at the figures may make the examples of this paragraph and § 16 clearer. The thick lines are the curves $x = X_i(\alpha)$. In (iv.) they are the rectangular hyperbolas

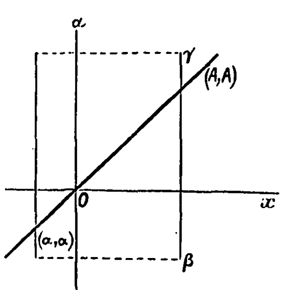
$$\alpha x = \frac{2n\pi \pm \theta}{2\pi}.$$

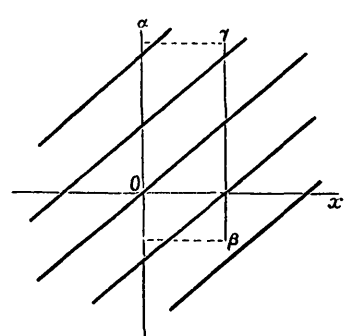


Uniform convergence, finite range.

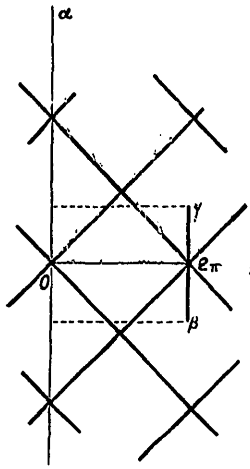


Regular convergence, finite range.

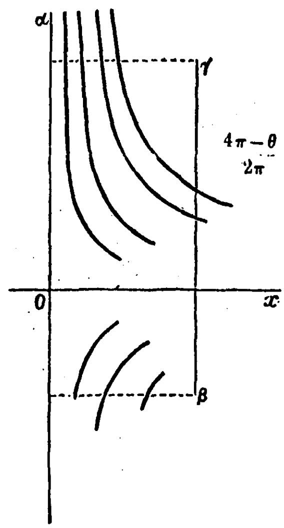




(i.) $f(x) = \frac{1}{x-\alpha},\ \frac{l(x-\alpha)}{x-\alpha}.$ (ii.) $f(x) = \frac{1}{\sin(x-\alpha)},\ \frac{l\sin(x-\alpha)}{\sin(x-\alpha)}.$



(iii.) $f(x) = \dfrac{\cos ax}{\cos x - \cos a}$.



(iv.) $f(x) = \dfrac{\cos ax}{\cos ax - \cos \theta}$.

Infinite Limits.

19. The principal value

$$P\int_a^\infty f(x, \alpha)\, dx$$

will be said to be *uniformly convergent in* (β, γ) if (i.) it is convergent for every value of α in (β, γ); and (ii.) we can find a quantity A, corresponding to any assigned positive quantity σ, such that

$$P\int_a^A f(x, \alpha)\, dx$$

is uniformly convergent in (β, γ), and

$$\left| P\int_A^\infty f(x, \alpha)\, dx \right| < \sigma$$

for all values of α in (β, γ).*

* [It is to be observed that we do not demand that condition (ii.) should be satisfied for all values of A greater than a certain finite value, as we do in the corresponding condition for the uniform convergence of an ordinary integral. A similar remark applies to the definition of *regular* convergence in § 21.—*October*, 1901.]

20. Thus (i.) $P\int_0^\infty \frac{1}{\cos x - \cos \alpha} \frac{dx}{\theta^2 + x^2}$ is uniformly convergent in $(\xi, \pi - \xi')$ if $0 < \xi < \xi + \xi' < \pi$.

(ii.) $P\int_0^\infty \frac{1}{\cos(x - \alpha)} \frac{dx}{\theta^2 + x^2}$ is uniformly convergent in $\left(\frac{\pi}{2} + \xi, \frac{3\pi}{2} - \xi'\right)$ if $0 < \xi < \xi + \xi' < \pi$.

Consider (i.), for instance. In the first place, $P\int_0^{2N\pi}$ is uniformly convergent in $(\xi, \pi - \xi')$. Also

$$P\int_{2N\pi}^\infty = \sum_N^\infty P\int_0^{2\pi} \frac{1}{\cos x - \cos \alpha} \frac{dx}{\theta^2 + (x + 2i\pi)^2}$$

$$= \frac{1}{2\sin \alpha} \sum_N^\infty P\int_0^{2\pi} \left\{\cot \tfrac{1}{2}(\alpha - x) + \cot \tfrac{1}{2}(\alpha + x)\right\} \frac{dx}{\theta^2 + (x + 2i\pi)^2}.$$

Now
$$P\int_0^{2\pi} \cot \tfrac{1}{2}(\alpha - x) \frac{dx}{\theta^2 + (x + 2i\pi)^2} = P\int_0^{2\alpha} + \int_{2\alpha}^{2\pi}.$$

The second term is numerically less than

$$\frac{K}{\theta^2 + (2i\pi)^2},$$

where K is a suitably chosen constant. And the first

$$= 2\alpha \left[\cot \tfrac{1}{2}(\alpha - x) \frac{x - \alpha}{\theta^2 + (x + 2i\pi)^2}\right]'_{\alpha + \mu},$$

where $-\alpha \leq \mu \leq \alpha$. This, too, is numerically $< \frac{K'}{\theta^2 + (2i\pi)^2}$. Finally, $\frac{1}{\sin \alpha}$ is less than the greater of $\operatorname{cosec} \xi$, $\operatorname{cosec} \xi'$; and $\sum_N^\infty \frac{1}{\theta^2 + (2i\pi)^2}$ can be made as small as we please by choice of N. Hence $P\int_{2N\pi}^\infty$ can be made $< \sigma$, by choice of N, for all values of α in $(\xi, \pi - \xi')$.

The uniform convergence of (ii.) may be proved in the same way. And by a slight modification of some of the arguments of my first paper we can prove general theorems as to the uniform convergence of principal values of the forms

$$P\int_0^\infty \frac{1}{\cos x - \cos \alpha} \phi(x)\, dx, \quad P\int_0^\infty \frac{1}{\cos(x - \alpha)} \phi(x)\, dx, \ \ldots,$$

in suitably chosen intervals. But I shall not delay over this at present.

21. The principal value

$$P\int_a^\infty f(x, \alpha)\, dx \qquad (1)$$

will be said to be *regularly convergent in* (β, γ) if (i.) it is convergent for every value of α in (β, γ);

$$\text{(ii.)} \quad P\int_a^A f(x, \alpha)\, dx$$

is regularly convergent in (β, γ) for every finite value of $A > a$; and (iii.) we can find (1) a value of A, (2) a division of (β, γ) into two finite sets of intervals θ, η, and (3) a set of positive quantities p_i, each corresponding to an interval η_i, and each less than some fixed quantity p_0, corresponding to any assigned positive quantity σ, and such that

$$\left| P \int_A^\infty f(x, a)\, dx \right| < \sigma$$

for all values of a in θ, and

$$\left| P \int_{A-p_i}^\infty f(x, a)\, dx \right| < \sigma$$

for all values of a in η_i.

22. We may remark that, if $a_1^A, \ldots, a_r^A$ are the exceptional values of a (§ 17) which correspond to any value of A, the intervals η will be intervals of the type $(a_i^A - \xi, a_i^A + \xi')$. The number r may increase beyond all limit with A.

The intervals θ, η_i are all to be understood as including their extremities; so that, at the point of division of θ, η_i, *both* the conditions

$$\left| P\int_A^\infty \right| < \sigma, \quad \left| P\int_{A-p_i}^\infty \right| < \sigma$$

are satisfied.

If $P\int_a^A$ is regularly, but not uniformly, convergent in (β, γ), it ceases to converge at all for certain values of a. But $P\int_a^\infty$ can only be regularly convergent in (β, γ) if it converges for all values of a in (β, γ).

23. Theorem.—*If* $\psi(x, a, \alpha)$ *is a function whose derivate* $\dfrac{\partial\psi}{\partial x}$ *is continuous and of constant sign for all positive values of* x, *and all values of* a, α *in question, and*

$$\lim_{x=\infty} \psi(x) = 0,$$

the principal values

$$P\int_0^\infty \frac{\sin ax}{\sin \alpha x}\psi(x)\, dx, \quad P\int_0^\infty \frac{\cos ax}{\cos \alpha x}\psi(x)\, dx \quad (\alpha > 0)$$

will be convergent, so long as $\dfrac{a}{\alpha}$ *is not an odd integer. They will be uniformly convergent in any interval* (b, c) *of values of* a *throughout which this condition is satisfied, if* $\lim \psi(x) = 0$ *uniformly for all these* *values of a; and they will be regularly convergent in any interval* (β, γ) *of values of* α *throughout which it is satisfied, if* $\lim \psi(x) = 0$ *uniformly for all these values of* α.

The first part of this theorem was proved in my first paper. The second part requires only a very slight modification of the proof there given of the first.

There remains the third part. We consider the first of the two principal values; and we suppose, *e.g.*,

$$0 < \beta < \gamma < a.$$

In the first place
$$P\int_0^A \frac{\sin ax}{\sin \alpha x}\psi(x)\,dx$$

is regularly convergent for any finite value of A. Also

$$P\int_A^\infty = P\int_{(N-\frac{1}{2})\pi/\alpha}^\infty - \int_{(N-\frac{1}{2})\pi/\alpha}^A$$

or
$$P\int_{(N-\frac{1}{2})\pi/\alpha}^\infty + \int_A^{(N-\frac{1}{2})\pi/\alpha},$$

if $(N-\frac{1}{2})\dfrac{\pi}{\alpha}$ be that odd multiple of $\dfrac{\pi}{2\alpha}$ between which and A lies no multiple of $\dfrac{\pi}{\alpha}$. Now

$$I\int_{(N-\frac{1}{2})\pi/\alpha}^\infty = \frac{1}{\alpha}P\int_{(N-\frac{1}{2})\pi}^\infty \frac{\sin\dfrac{au}{\alpha}}{\sin u}\psi\left(\frac{u}{\alpha}\right)du;$$

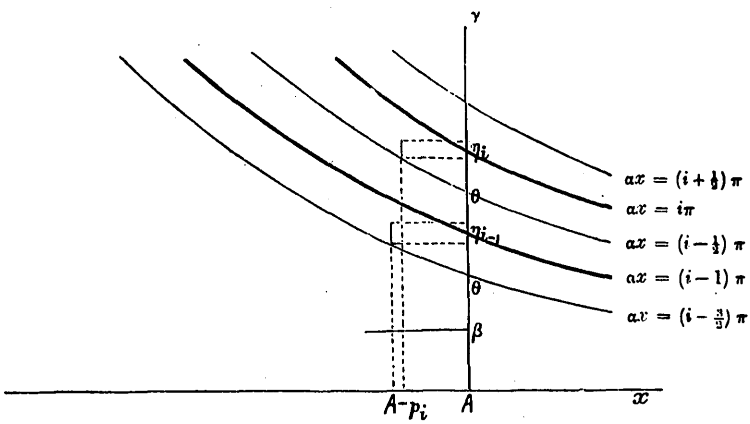
and by the second part of our theorem we can make this as small as we please by choice of N, for all values of α in question. It remains to consider $\int_{(N-\frac{1}{2})\pi/\alpha}^A$ or $\int_A^{(N-\frac{1}{2})\pi/\alpha}$.

Suppose
$$\beta < \frac{k\pi}{A} < \ldots < \frac{(k+l)\pi}{A} < \gamma.$$

The values α_i^A are $\dfrac{i\pi}{A}$, $i = k, \ldots, k+l$; and the intervals η_i are of the type

$$\left(\frac{i\pi}{A} - \xi,\quad \frac{i\pi}{A} + \xi'\right).$$

We take $N = k$, $\xi = \xi' = \dfrac{\pi}{8A}$ (see the figure). And $A - p_i =$ the value of x where $\alpha = (i-\frac{1}{8})\frac{\pi}{A}$ meets $\alpha x = (i-\frac{1}{2})\pi$; *i.e.*, $p_i = \frac{\frac{3}{8}A}{i-\frac{1}{8}}$.



Regular convergence, infinite range.

As $A < \frac{k\pi}{\beta}$, and $i \geqq k$, p_i is certainly less than $p_0 = \frac{\pi}{2\beta}$. Also, if α lies in η_i, and x between $(i-\frac{1}{2})\frac{\pi}{\alpha}$ and $A-p_i$, αx lies between $(i-\frac{1}{2})\pi$ and

$$(A-p_i)(i+\tfrac{1}{8})\frac{\pi}{A} = \frac{(i-\frac{1}{2})(i+\frac{1}{8})}{i-\frac{1}{8}}\pi;$$

so that

$$\tfrac{1}{2}\pi \geqq i\pi - \alpha x \geqq \frac{\frac{1}{4}i+\frac{1}{16}}{i-\frac{1}{8}}\pi > \tfrac{1}{4}\pi.$$

Hence

$$\left|\int_{(i-\frac{1}{2})\pi/\alpha}^{A-p_i}\right| < \sqrt{2}\,\psi'\left\{A-p_i-(i-\tfrac{1}{2})\frac{\pi}{\alpha}\right\},$$

where ψ' is the greatest value of $|\psi(x)|$ in the range of integration. The least value of x in the range is $\geqq \frac{i-\frac{1}{2}}{i+\frac{1}{8}}A$, and $\psi(x)$ tends to zero for $x = \infty$, uniformly for all values of α. And

$$A-p_i-(i-\tfrac{1}{2})\frac{\pi}{\alpha} \leqq \left(\frac{1}{i-\frac{1}{8}}-\frac{1}{i+\frac{1}{8}}\right)(i-\tfrac{1}{2})A < \frac{A}{4i},$$

which does not increase indefinitely with A and i. Hence we can choose A so great that

$$\left|P\int_{A-p_i}^{\infty}\right| < \sigma$$

throughout the intervals η_i.

The intervals θ are of the type

$$(i-\tfrac{7}{8})\frac{\pi}{A}, \quad (i-\tfrac{1}{8})\frac{\pi}{A};$$

and if a lies in this interval, and x between A and $(i-\frac{1}{2})\frac{\pi}{a}$, ax lies between $(i-\frac{7}{8})\pi$ and $(i-\frac{1}{8})\pi$; so that

$$\left|\frac{1}{\sin ax}\right| \leq \operatorname{cosec}\tfrac{1}{8}\pi.$$

Hence $\left|\int_A^{(i-\frac{1}{2})\pi/a}\right|$ or $\left|\int_{(i-\frac{1}{2})\pi/a}^{A}\right| < \operatorname{cosec}\frac{1}{8}\pi\,\psi'\left|(i-\frac{1}{2})\frac{\pi}{a}-A\right|$,

where ψ' is again the greatest value of $|\psi(x)|$ in the range of integration. And it follows, as before, that we can choose A so great that

$$\left|P\int_A^{\infty}\right| < \sigma$$

throughout the intervals θ. Hence

$$P\int_0^{\infty}\frac{\sin ax}{\sin ax}\psi(x)\,dx$$

is regularly convergent.

We may, for instance, suppose

$$\psi(x) = x^{-\mu} \quad (0<\mu<1), \quad \frac{1}{x^2\pm\theta^2}, \quad e^{-\lambda x} \quad (\lambda>0), \quad \ldots.$$

This theorem may be extended in various ways. We may suppose, *e.g.*, that $\frac{\partial\psi}{\partial x}$ is of constant sign only after some finite value of x independent of a and a; or we may substitute for

$$\frac{\sin ax}{\sin ax}, \quad \frac{\cos ax}{\cos ax}$$

such factors as $\sin ax \tan ax$, $\sin ax \cot ax$, ...

In these two cases the exceptional values of $\frac{a}{a}$ will be the *even* integral values.

It is to be observed that no difficulty arises with these exceptional values, if $\int^{\infty}\psi(x)\,dx$ is convergent. Thus, if in

$$P\int_0^{\infty}\frac{\sin ax}{\sin ax}\psi(x)\,dx$$

we make $a = \alpha$, 3α, we obtain

$$\int_0^\infty \psi(x)\,dx, \quad \int_0^\infty (3-4\sin^2 \alpha x)\,\psi(x)\,dx;$$

and the latter of these converges or diverges with the former.

24. It is to be observed that a very simple transformation may change regular into uniform convergence, or *vice versa*. Thus the substitution $\alpha x = y$ transforms the principal values of the theorem into

$$\frac{1}{\alpha} P \int_0^\infty \frac{\sin \frac{a}{\alpha} y}{\sin y} \psi\left(\frac{y}{\alpha}\right) dy, \quad \frac{1}{\alpha} P \int_0^\infty \frac{\cos \frac{a}{\alpha} y}{\cos y} \psi\left(\frac{y}{\alpha}\right) dy,$$

which are uniformly convergent in the interval of values of α in question.

Continuity of Principal Values.

25. THEOREM 1.—*If $f(x, \alpha)$ is a continuous function of both variables in any finite part of the rectangle*

$$(a, A, \beta, \gamma)$$

which does not include any point of any of the curves $x = X_r(\alpha)$, and

$$P \int_a^A f(x, \alpha)\,dx$$

is uniformly convergent in (β, γ), it will be a continuous function of α in (β, γ).

This is true whether A be finite or infinite.

In the first place, suppose A finite. We may without loss of generality suppose that there is only one curve $x = X(\alpha)$; for we can reduce any case to this by dividing the range of integration and the interval (β, γ) into a finite number of parts.

We draw two auxiliary curves

$$x = X(\alpha) \pm \delta;$$

in the region R_δ exterior to these curves $f(x, \alpha)$ is a continuous function of both variables, and therefore a uniformly continuous function of α. Let α_0 be any value of α in (β, γ); and suppose, *e.g.*, that $X'(a_0)>0$. Then, if h be a small positive quantity, and $a_0+h<\gamma$,

$$P\int_a^A f(x, a_0+h)\,dx - P\int_a^A f(x, a_0)\,dx$$
$$= \left(\int_a^{X(a_0)-\delta} + \int_{X(a_0+h)+\delta}^A\right)\left[f(x, a_0+h)-f(x, a_0)\right]dx$$
$$+\int_{X(a_0)-\delta}^{X(a_0+h)-\delta} f(x, a_0+h)\,dx - \int_{X(a_0)+\delta}^{X(a_0+h)+\delta} f(x, a_0)\,dx$$
$$+P\int_{X(a_0+h)-\delta}^{X(a_0+h)+\delta} f(x, a_0+h)\,dx - P\int_{X(a_0)-\delta}^{X(a_0)+\delta} f(x, a_0)\,dx.$$

Now let σ be any positive quantity. We can choose δ so small that

$$\left| P\int_{X(a)-\delta}^{X(a)+\delta} \right| < \tfrac{1}{6}\sigma$$

for all values of a in (β, γ), and ϵ_0 so small that

$$\left| \int_x^{x+\epsilon} \right| < \tfrac{1}{6}\sigma$$

for all values of $\epsilon \leqq \epsilon_0$ and all values of x, a such that x, $x+\epsilon$ fall within R_δ. Then we can choose h' so small that

$$X(a_0+h)-X(a_0) \leqq \epsilon_0,$$

and
$$|f(x, a_0+h)-f(x, a_0)| < \frac{\sigma}{3(A-a)},$$

for all values of x in either of the intervals

$$a,\ X(a_0)-\delta;\quad X(a_0+h)+\delta,\ A;$$

and all values of $h \leqq h'$. And then

$$\left| P\int_a^A f(x, a_0+h)\,dx - P\int_a^A f(x, a_0)\,dx \right| < \sigma$$

for all values of $h \leqq h'$. A similar proof applies to negative values of h, if $a_0>\beta$. Hence $P\int_a^A$ is continuous at a_0.

In the second place, let us suppose that the upper limit is ∞. We can choose A so that $P\int_a^A$ is uniformly convergent in (β, γ), and

$$\left| P\int_A^\infty \right| < \tfrac{1}{3}\sigma$$

for all values of a in (β, γ). And, since, by the first part of the theorem, $P\int_a^A$ is continuous in (β, γ), we can choose h' so small that

$$\left| P\int_a^A f(x, a_0+h)\,dx - P\int_a^A f(x, a_0)\,dx \right| < \tfrac{1}{3}\sigma$$

if $h \leqq h'$. Hence

$$\left| P\int_a^\infty f(x, a_0+h)\,dx - P\int_a^\infty f(x, a_0)\,dx \right| < \sigma$$

if $h \leq h'$. A similar proof applies to negative values of h. Hence $P\int_a^\infty$ is continuous at a_0.

26. Thus the principal values

$$P\int_0^A \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{A-a}{A+a}, \qquad \text{(i.)}$$

$$P\int_0^\infty \frac{dx}{x^2-a^2} = 0, \qquad \text{(ii.)}$$

$$P\int_0^\infty \frac{\cos ax}{x^2-a^2}\,dx = -\frac{\pi}{2a}\sin aa \qquad \text{(iii.)}$$

are continuous in (β, γ) if β, γ be any positive quantities. As a approaches zero they tend to the finite limits

$$-\frac{1}{A}, \quad 0, \quad -\frac{a\pi}{2};$$

but they are meaningless for $a = 0$. And

$$P\int_0^\pi \frac{\cos nx\,dx}{\cos x - \cos a} = \pi\frac{\sin na}{\sin a} \qquad \text{(iv.)}$$

if n be a positive integer; and this is continuous in $(\xi, \pi-\xi')$ if $0 < \xi < \xi + \xi' < \pi$; and tends, as a approaches 0 or π, to the finite limits

$$n\pi, \quad (-)^{n-1}n\pi,$$

but is meaningless for $a = 0$ or π.

Again, the principal value

$$P\int_0^\infty \frac{\log(1+2a\cos ax+a^2)}{1-x^2}\,dx = \pi\tan^{-1}\frac{a\sin a}{1+a\cos a} \qquad \text{(v.)}$$

is uniformly convergent in $(-1, 1)$; and, for $a = 1, -1$ becomes

$$P\int_0^\infty \frac{\log 4\cos^2\frac{1}{2}ax}{1-x^2} = \tfrac{1}{2}\pi a,$$

$$P\int_0^\infty \frac{\log 4\sin^2\frac{1}{2}ax}{1-x^2} = \tfrac{1}{2}\pi(a-\pi).$$

27. An interesting case is that in which an unconditionally convergent integral changes continuously into a principal value for some special value of a parameter.

Let us consider, for instance, the integral

$$\int_0^{2\pi} \frac{\sin x}{1+2a\cos x+a^2} f(x)\,dx, \qquad \text{(i.)}$$

where $f(x)$ is a function which has a continuous derivate for all values of x in question, and a is positive and <1. Then

$$\int_x^{x+\epsilon} = \left[-\frac{1}{2a}\log(1+2a\cos x+a^2)f(x)\right]_x^{x+\epsilon} + \frac{1}{2a}\int_x^{x+\epsilon}\log(1+2a\cos x+a^2)f'(x)\,dx.$$

Now $$\frac{\partial}{\partial a}\log\frac{(1+a)^2}{1+2a\cos x+a^2} = 2\,\frac{1-a}{1+a}\,\frac{1-\cos x}{1+2a\cos x+a^2} \geqq 0$$

if $0 \leqq a < 1$; and so

$$0 < \log\frac{(1+a)^2}{1+2a\cos x+a^2} < \log\sec^2\tfrac{1}{2}x.$$

Hence $$\left|\frac{1}{2a}\int_x^{x+\epsilon}\log(1+2a\cos x+x^2)f'(x)\,dx\right|$$
$$< \frac{\log(1+a)}{a}\int_x^{x+\epsilon}|f'(x)|\,dx + \frac{1}{2a}\int_x^{x+\epsilon}\log\sec^2\tfrac{1}{2}x\,|f'(x)|\,dx;$$

and this can be made as small as we please, by choice of ϵ, for all values of a and x in question.

Again, $$\left|\left[-\frac{1}{2a}\log(1+2a\cos x+a^2)f(x)\right]_x^{x+\epsilon}\right|$$

can be made as small as we please, by choice of ϵ, for all values of a in $(0, 1)$, and all values of x in $(0, \pi-\delta)$ or $(\pi+\delta, 2\pi)$, where δ is any positive quantity $<\pi$, however small. But, if $x=\pi$, this condition cannot be satisfied. If, for instance, $f(x)=1$,

$$\frac{1}{2a}\left\{\log(1-a)^2-\log(1-2a\cos\epsilon+a^2)\right\} = \frac{1}{2a}\log\left\{1+\frac{4a\sin^2\frac{1}{2}\epsilon}{(1-a)^2}\right\},$$

and the value of ϵ which we have to take decreases beyond all limit as a approaches unity. And, in fact, since

$$\lim_{a=1}\frac{\sin x}{1+2a\cos x+a^2} = \tfrac{1}{2}\tan\tfrac{1}{2}x,$$

the integral (i.) is not convergent when $a=1$. However,

$$P\int_0^{2\pi}\tfrac{1}{2}\tan\tfrac{1}{2}x\,f(x)\,dx$$

is convergent. Moreover,

$$\int_{\pi-\delta}^{\pi+\delta}\frac{\sin x}{1+2a\cos x+a^2}f(x)\,dx$$
$$= -\frac{1}{2a}\left\{f(\pi+\delta)-f(\pi-\delta)\right\}\log(1-2a\cos\delta+a^2) + \frac{1}{2a}\int_{\pi-\delta}^{\pi+\delta}\log(1+2a\cos x+a^2)f'(x)\,dx.$$

As before, the second term can be made as small as we please, by choice of δ, for all values of a in question. And so can the first, as it is equal to

$$-\frac{1}{2a}\log\left\{(1-a)^2+4a\sin^2\tfrac{1}{2}\delta\right\}2\delta f'(\pi+\theta\delta) \quad (-1\leqq\theta\leqq 1).$$

We can choose δ, then, so small that

$$\left|\int_{\pi-\delta}^{\pi+\delta}\frac{\sin x}{1+2a\cos x+a^2}f(x)\,dx\right| < \sigma$$

for all values of a between 0 and 1, and

$$\left| P\int_{\pi-\delta}^{\pi+\delta} \tfrac{1}{2}\tan\tfrac{1}{2}x\, f(x)\, dx \right| < \sigma.$$

Hence
$$P\int_0^{2\pi} \frac{\sin x}{1+2a\cos x+a^2} f(x)\, dx$$

is uniformly convergent in (0, 1), and therefore a continuous function of a in that interval. Consequently,

$$P\int_0^{2\pi} \tfrac{1}{2}\tan\tfrac{1}{2}x\, f(x)\, dx = \lim_{a=1} \int_0^{2\pi} \frac{\sin x}{1+2a\cos x+a^2} f(x)\, dx. \tag{1}$$

Similarly,
$$P\int_{-\pi}^{\pi} \tfrac{1}{2}\cot\tfrac{1}{2}x\, f(x)\, dx = \lim_{a=1} \int_{-\pi}^{\pi} \frac{\sin x}{1-2a\cos x+a^2} f(x)\, dx, \tag{2}$$

$$P\int_0^{2\pi} \tfrac{1}{2}\sec\tfrac{1}{2}x\, f(x)\, dx = \lim_{a=1} \int_0^{2\pi} \frac{(1+a)\sqrt{a}\cos\tfrac{1}{2}x}{1+2a\cos x+a^2} f(x)\, dx, \tag{3}$$

$$P\int_{-\pi}^{\pi} \tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}x\, f(x)\, dx = \lim_{a=1} \int_{-\pi}^{\pi} \frac{(1+a)\sqrt{a}\sin\tfrac{1}{2}x}{1-2a\cos x+a^2} f(x)\, dx. \tag{4}$$

In (1) and (3) we may substitute a, A for 0, 2π as limits, provided $-\pi < a < \pi$, $\pi < A < 3\pi$. Similarly for (2) and (4).

28. We may expand the functions under the integral signs on the right in powers of a, and integrate term by term. Thus from (1) we deduce

$$P\int_0^{2\pi} \tfrac{1}{2}\tan\tfrac{1}{2}x\, f(x)\, dx = \lim_{a=1} \sum_1^{\infty} (-)^{n-1} a^n \int_0^{2\pi} \sin nx\, f(x)\, dx. \tag{1}$$

This is equal to
$$\sum_1^{\infty} (-)^{n-1} \int_0^{2\pi} \sin nx\, f(x)\, dx,$$

if the latter series is convergent. Thus, if $f(x)$ can be expanded as a Fourier series,

$$a_0 + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx) \quad (0 < x < 2\pi),$$

$$P\int_0^{2\pi} \tfrac{1}{2}\tan\tfrac{1}{2}x\, f(x)\, dx = \pi \lim_{a=1} \sum_1^{\infty} (-)^{n-1} a_n a^n = \sum_1^{\infty} (-)^{n-1} a_n,$$

if this is convergent.

For instance,
$$\sum_1^{\infty} \frac{\sin nx}{n} = \tfrac{1}{2}(\pi - x) \quad (0 < x < 2\pi);$$

and therefore
$$\int_0^{2\pi} \tfrac{1}{2}\tan\tfrac{1}{2}x \,.\, \tfrac{1}{2}(\pi - x)\, dx = \pi \sum_1^{\infty} (-)^{n-1} \frac{1}{n} = \pi\log 2,$$

$$P\int_0^{\pi} \phi\tan\phi\, d\phi = -\pi\log 2.$$

Similarly, we deduce from (2), (3), and (4) of § 27

$$P\int_{-\pi}^{\pi} \tfrac{1}{2}\cot\tfrac{1}{2}x\, f(x)\, dx = \lim_{a=1} \sum_1^{\infty} a^n \int_{-\pi}^{\pi} \sin nx\, f(x)\, dx, \tag{2}$$

$$P\int_0^{2\pi} \tfrac{1}{2}\sec\tfrac{1}{2}x\, f(x)\, dx = \lim_{a=1} \sum_0^{\infty} (-)^n a^{n+\frac{1}{2}} \int_0^{2\pi} \cos(n+\tfrac{1}{2})\, x\, f(x)\, dx, \tag{3}$$

$$P\int_{-\pi}^{\pi} \tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}x\, f(x)\, dx = \lim_{a=1} \sum_0^{\infty} a^{n+\frac{1}{2}} \int_{-\pi}^{\pi} \sin(n+\tfrac{1}{2})\, x\, f(x)\, dx. \tag{4}$$

In each of the series on the right we may put $a = 1$ if the resulting series are convergent.

29. In the same way we can establish four more general formulæ of which

$$P\int_0^{2\pi} \tfrac{1}{2}\tan\tfrac{1}{2}(x-\theta) f(x)\,dx = \lim_{a=1} \sum_1^{\infty} (-)^{n-1} a^n \int_0^{2\pi} \sin n(x-\theta) f(x)\,dx$$

is typical. If, for instance,

$$f(x) = \cos px,\ \sin px\ (p \text{ an integer}),\quad x = 2\phi;\quad \theta = 2\psi,$$

we obtain
$$P\int_0^{\pi} \tan(\phi-\psi)\cos 2p\phi\,d\phi = (-)^p \pi \sin 2p\psi,$$

$$P\int_0^{\pi} \tan(\phi-\psi)\sin 2p\phi\,d\phi = (-)^{p-1} \pi \cos 2p\psi.$$

30. Again, $P\int_0^{\infty} \tfrac{1}{2}\tan\tfrac{1}{2}x\, f(x)\,dx = \lim_{a=1} \sum_1^{\infty} (-)^{n-1} a^n \int_0^{\infty} \sin nx\, f(x)\,dx,$ (1)

if
$$P\int_0^{\infty} \frac{\sin x}{1+2a\cos x + a^2} f(x)\,dx$$

be uniformly convergent in (0, 1). Now it follows from what precedes that, if $f'(x)$ is continuous, $P\int_0^{2n\pi}$ is uniformly convergent for any value of n. Hence $P\int_0^{\infty}$ will be so if we can so choose n that

$$\left| P\int_{2n\pi}^{\infty} \frac{\sin x}{1+2a\cos x + a^2} f(x)\,dx \right| < \sigma \qquad (\alpha)$$

for all values of a in (0, 1).

Let us suppose, in the first place, that $f(x)$ is positive and tends steadily to zero for $x = \infty$. Then

$$\int_{2n\pi}^{\infty} \frac{\sin x}{1+2a\cos x+a^2} f(x)\,dx = \frac{1}{2a}\log(1+a)^2 f(2n\pi) + \frac{1}{2a}\int_{2n\pi}^{\infty} \log(1+2a\cos x+a^2) f'(x)\,dx$$

$$= \frac{1}{2a}\int_{2n\pi}^{\infty} \log\frac{1+2a\cos x+a^2}{(1+a)^2} f'(x)\,dx.$$

Also
$$0 < \log\frac{1+2a\cos x+a^2}{(1+a)^2} f'(x) < \log\cos^2\tfrac{1}{2}x\, f'(x),$$

and
$$\int_{2n\pi}^{\infty} \log\cos^2\tfrac{1}{2}x\, f'(x)\,dx$$

is convergent. Hence condition (α) can be satisfied.

If, *e.g.*,
$$f(x) = \frac{1}{x},$$

$$P\int_0^{\infty} \tfrac{1}{2}\tan\tfrac{1}{2}x \frac{dx}{x} = \lim_{a=1} \sum_1^{\infty} (-)^{n-1} a^n \int_0^{\infty} \sin nx \frac{dx}{x} = \lim_{a=1} \frac{\pi}{2(1+a)},$$

$$P\int_0^{\infty} \frac{\tan x}{x}\,dx = \tfrac{1}{2}\pi.$$

Similarly, $P\int_0^\infty \tfrac{1}{2}\sec\tfrac{1}{2}x\, f(x)\, dx = \lim_{a=1} \sum_0^\infty (-)^n a^{n+\frac{1}{2}} \int_0^\infty \cos(n+\tfrac{1}{2})\, x\, f(x)\, dx$

$$= \sum_0^\infty (-)^n \int_0^\infty \cos(n+\tfrac{1}{2})\, x\, f(x)\, dx,$$

if this series be convergent.

Legendre determined

$$P\int_0^\infty \frac{x\tan ax\, dx}{x^2+m^2}, \quad P\int_0^\infty \frac{x\cot ax\, dx}{x^2+m^2}, \quad P\int_0^\infty \frac{x\operatorname{cosec} ax\, dx}{x^2+m^2}$$

(considering them as ordinary integrals) by assuming that they were the limiting values of

$$\int_0^\infty \frac{\sin 2ax}{1+2a\cos 2ax+a^2}\,\frac{x\,dx}{x^2+m^2} = \tfrac{1}{2}\,\frac{\pi}{e^{2am}+a},$$

$$\int_0^\infty \frac{\sin 2ax}{1-2a\cos 2ax+a^2}\,\frac{x\,dx}{x^2+m^2} = \tfrac{1}{2}\,\frac{\pi}{e^{2am}-a},$$

$$\int_0^\infty \frac{\sin ax}{1-2a\cos 2ax+a^2}\,\frac{x\,dx}{x^2+m^2} = \frac{\pi}{2(1+a)}\,\frac{e^{am}}{e^{2am}-a}$$

for $a = 1$. We can see now that his assumption was correct.

31. Suppose that, in (2) of § 30,

$$f(x) = x^{-a} \quad (0 < a < 1).$$

Then
$$\int_0^\infty \cos(n+\tfrac{1}{2})\, x\,\frac{dx}{x^a} = \frac{\pi}{2\Gamma(a)\cos\frac{1}{2}a\pi}\,\frac{1}{(n+\frac{1}{2})^{1-a}}.$$

Also
$$\tfrac{1}{2}\sec\tfrac{1}{2}x = -2\pi\sum_0^\infty (-)^n \frac{2n+1}{x^2-(2n+1)^2\pi^2};$$

and
$$P\int_0^\infty \tfrac{1}{2}\sec\tfrac{1}{2}x\,\frac{dx}{x} = -2\pi\sum_0^\infty (-)^n (2n+1)\, P\int_0^\infty \frac{1}{x^2-(2n+1)^2\pi^2}\,\frac{dx}{x^a}$$

$$= -2\pi^{-a}\sum_0^\infty \frac{(-)^n}{(2n+1)^a}\, P\int_0^\infty \frac{1}{x^2-1}\,\frac{dx}{x^a}$$

$$= \pi^{1-a}\tan\tfrac{1}{2}a\pi \sum_0^\infty \frac{(-)^n}{(2n+1)^a},$$

by §§ 3–12. This series is therefore equal to

$$\frac{\pi}{2\Gamma(a)\cos\frac{1}{2}a\pi}\sum_0^\infty \frac{(-)^n}{(n+\frac{1}{2})^{1-a}}.$$

That is to say, the function $\psi(a) = \sum_0^\infty \frac{(-)^n}{(2n+1)^a} \quad (0<a<1)$

satisfies the functional equation

$$\psi(1-a) = \left(\frac{2}{\pi}\right)^a \sin\tfrac{1}{2}a\pi\,\Gamma(a)\,\psi(a).$$

This is a well known relation first proved by Schlömilch, and closely connected with the theory of Riemann's ζ-function.

32. The formulæ of § 30 may be generalized, as those of § 28 were in § 29. There will be no difficulty in proving, for example, that

$$P\int_0^\infty \frac{1}{\cos\delta x-\cos\phi}\, f(x)\, dx = \frac{2}{\sin\phi}\sum_0^\infty \sin n\phi \int_0^\infty \cos n\delta x\, f(x)\, dx,$$

if this series is convergent.

If, for instance, $$f(x) = \frac{1}{\cosh \frac{1}{2}\pi x},$$

$$\int_0^{x} \cos n\delta x\, f(x)\, dx = \frac{1}{\cosh n\delta}.$$

But

$$P\int_0^{\infty} \frac{1}{\cos\delta x - \cos\phi}\,\frac{dx}{\cosh\frac{1}{2}\pi x} = \frac{4}{\pi}\sum_0^{\infty}(-)^m(2m+1)\,P\int_0^{\infty}\frac{1}{\cos\delta x-\cos\phi}\,\frac{dx}{x^2+(2m+1)^2}$$

$$= 2\sum_0^{\infty}\frac{(-)^m}{\cosh(2m+1)\delta-\cos\phi}.$$

Hence $$\sum_0^{\infty}\frac{(-)^m}{\cosh(2m+1)\delta-\cos\phi} = \frac{1}{\sin\phi}\sum_0^{\infty}\frac{\sin n\phi}{\cosh n\delta}.$$

This becomes obvious if $\phi = \frac{1}{2}\pi$. It is really a formula in elliptic functions; for, if we write q for $e^{-\delta}$, it takes the form

$$\sum_0^{\infty}\frac{(-)^m q^{2m+1}}{1-2q^{2m+1}\cos\phi+q^{4m+2}} = \frac{1}{\sin\phi}\sum_0^{\infty}\frac{q^n\sin n\phi}{1+q^{2n}}.$$

If we integrate this from $\phi = 0$ to $\phi = \pi$, observing that $\int_0^{\pi}\frac{\sin n\phi}{\sin\phi}\,d\phi$ is π or 0, according as n is odd or even, we obtain the well known formula

$$\frac{q}{1+q^2}+\frac{q^3}{1+q^6}+\ldots = \frac{q}{1-q^2}-\frac{q^3}{1-q^6}+\ldots$$

(Jacobi, *Fundamenta Nova*, XL. 5).

If we had taken $\phi = \frac{1}{2}\pi$, $(x) = \dfrac{1}{\cosh x-\cos\theta}$,
we should have found that

$$\frac{1}{\cosh\delta\theta}+\sum_0^{\infty}\left\{\frac{1}{\cosh(2n\pi+\theta)\delta}-\frac{1}{\cosh(2n\pi-\theta)\delta}\right\} = \frac{2}{\sin\theta}\sum_0^{\infty}(-)^m\frac{\sinh(2m+1)(\pi-\theta)\delta}{\sinh(2m+1)\pi\delta}.$$

This too becomes obvious if $\theta = \frac{1}{2}\pi$. It is not difficult to obtain general formulæ which include these as particular cases; but my present purpose is only to show how the methods of the preceding sections can be applied to obtain results of interest in different branches of analysis.

33. The equations of § 30 also hold (except for certain exceptional values of a) if

$$f(x) = {\cos \atop \sin} ax\,\psi(x),$$

where $\psi(x)$ is a function whose first two derivates are continuous and of constant sign after a certain value of x, and

$$\lim_{x=\infty}\psi(x) = 0.$$

For $$\int_{2n\pi}^{\infty}\frac{\sin x}{1+2a\cos x+a^2}f(x)\,dx = \sum_n^{\infty}\int_{2i\pi}^{2(i+1)\pi} = \int_0^{2\pi}\frac{\sin x}{1+2a\cos x+a^2}F(x)\,dx,$$

where $$F(x) = \sum_n^{\infty}{\cos\atop\sin}a(x+2i\pi)\,\psi(x+2i\pi).$$

Now, provided a be not an integer, the series

$$\Sigma{\cos\atop\sin}2ia\pi\,\psi(x+2i\pi),\quad \Sigma{\cos\atop\sin}2ia\pi\,\psi'(x+2i\pi)$$

are uniformly convergent in $(0, 2\pi)$. It follows (i.) that, whatever be the value of n, $F(x)$ and $F'(x)$ are continuous in $(0, 2\pi)$; (ii.) that we can choose n so great that the moduli of $F(x)$, $F'(x)$ are as small as we please for all values of x in $(0, 2\pi)$.

Moreover, $$\int_0^{2\pi} \frac{\sin x}{1+2a\cos x+a^2} F(x)\,dx = -\frac{1}{2a}\int_0^{2\pi} \log\frac{(1+a)^2}{1+2a\cos x+a^2} F'(x)\,dx,$$

and $$0 < \log\frac{(1+a)^2}{1+2a\cos x+a^2} < \log\sec^2\tfrac{1}{2}x.$$

Hence $$\left|\int_{2n\pi}^{\infty} \frac{\sin x}{1+2a\cos x+a^2} f(x)\,dx\right| < \frac{1}{2a}\int_0^{2\pi} \log\sec^2\tfrac{1}{2}x \,|F'(x)|\,dx,$$

and can therefore be made as small as we please, by choice of n, for all values of a in $(0, 1)$. Hence $$P\int_0^{\infty} \frac{\sin x}{1+2a\cos x+a^2} f(x)\,dx$$

is uniformly convergent in $(0, 1)$.

Suppose, *e.g.*, that $$f(x) = \frac{x\cos ax}{x^2+\theta^2} \quad (0<a<1).$$

Then $$P\int_0^{\infty} \tan\tfrac{1}{2}x\cos ax \frac{x\,dx}{x^2+\theta^2} = 2\lim_{a=1}\sum_1^{\infty}(-)^{n-1}a^n\int_0^{\infty}\sin nx\cos ax\frac{x\,dx}{x^2+\theta^2}$$
$$= \tfrac{1}{2}\pi\sum_1^{\infty}(-)^{n-1}\left\{e^{-(n-a)\theta}+e^{-(n+a)\theta}\right\} = \frac{\pi\cosh a\theta}{e^{\theta}+1}.$$

Similarly, $$P\int_0^{\infty}\cot\tfrac{1}{2}x\cos ax\frac{x\,dx}{x^2+\theta^2} = \frac{\pi\cosh a\theta}{e^{\theta}-1}.$$

This agrees with the result found in another way in § 12. A third proof will be found in the *Quarterly Journal*, No. 125, 1900, p. 120.

It is not difficult to prove that $$P\int_0^{\infty} \frac{\sin x}{1+2a\cos x+a^2} f(x)\,dx$$

is still continuous in $(0, 1)$ if the conditions of § 30 are satisfied, except that $f(x)$ has a finite number of infinities X' none of which are odd multiples of $\tfrac{1}{2}\pi$. In this case the integral is not unconditionally convergent for any value of a.

Thus $$P\int_0^{\infty}\tan\tfrac{1}{2}x\cos ax\frac{x\,dx}{x^2-\theta^2} = 2\lim_{a=1}\sum_1^{\infty}(-)^{n-1}a^n P\int_0^{\infty}\sin nx\cos ax\frac{x\,dx}{x^2-\theta}$$
$$= \tfrac{1}{2}\pi\lim_{a=1}\sum_1^{\infty}(-)^{n-1}a^n\left\{\cos(n-a)\theta+\cos(n+a)\theta\right\}$$
$$= \pi\cos a\theta\lim_{a=1}\frac{\cos\theta+a}{1+2a\cos\theta+a^2} = \tfrac{1}{2}\pi\cos a\theta.$$

Similarly, $$P\int_0^{\infty}\cot\tfrac{1}{2}x\cos ax\frac{x\,dx}{x^2-\theta^2} = -\tfrac{1}{2}\pi\cos a\theta.$$

This, again, agrees with § 12, and with the paper in the *Quarterly Journal* referred to above.

Discontinuous Principal Values.

34. We shall now consider some examples in which the conditions of § 25 are not satisfied.

(i.) If $0 < \alpha < a$,

$$P\int_0^\infty \frac{\sin \alpha x}{\cos ax}\,\frac{x\,dx}{1+x^2} = \tfrac{1}{2}\pi\,\frac{\sinh \alpha}{\cosh a}, \tag{1}$$

$$P\int_0 \frac{\cos \alpha x}{\sin ax}\,\frac{x\,dx}{1+x^2} = \tfrac{1}{2}\pi\,\frac{\cosh \alpha}{\sinh a}. \tag{2}$$

These principal values are discontinuous for $\alpha = a$. For, if we put $\alpha = a$ in the first, for instance, we obtain

$$P\int_0^\infty \frac{x\tan ax}{1+x^2}\,dx = \tfrac{1}{2}\pi \tanh a,$$

which is incorrect, the proper value being

$$\frac{\pi}{e^{2a}+1}.$$

Hence (1), which is, after § 23, uniformly convergent in $(0, \alpha' < 1)$ cannot be uniformly convergent in $(0, 1)$.

Now, if ϵ be a small positive quantity,

$$P\int_0^\infty \frac{\sin(a-\epsilon)x}{\cos ax}\,\frac{x\,dx}{1+x^2} = P\int_0^\infty \tan ax \cos \epsilon x\,\frac{x\,dx}{1+x^2} - \int_0^\infty \frac{x\sin \epsilon x}{1+x^2}\,dx.$$

The latter integral is, as is well known, discontinuous for $\epsilon = 0$, being $\doteq \tfrac{1}{2}\pi e^{-\epsilon}$ if $\epsilon > 0$. And it is easy to see that it is not uniformly convergent in an interval including $\epsilon = 0$. For $\dfrac{x}{1+x^2}$ decreases steadily after $x = 1$. Hence, for sufficiently small values of ϵ,

$$\begin{aligned}\int_{2\pi/\epsilon}^\infty \frac{x\sin \epsilon x}{1+x^2}\,dx = \sum_2^\infty \int_{i\pi/\epsilon}^{(i+1)\pi/\epsilon} &> \int_{2\pi/\epsilon}^{4\pi/\epsilon} > \int_{2\pi}^{4\pi} \frac{u\sin u}{\epsilon^2+u^2}\,du \\ &> \int_{2\pi}^{3\pi}\left(\frac{u}{\epsilon^2+u^2} - \frac{u+\pi}{\epsilon^2+(u+\pi)^2}\right)\sin u\,du \\ &> \int_{2\pi}^{3\pi} \frac{\pi(u^2+u\pi-\epsilon^2)}{(\epsilon^2+u^2)\{\epsilon^2+(u+\pi)^2\}}\sin u\,du \\ &> \tfrac{1}{2}\pi^3 \int_{2\pi}^{3\pi} \frac{\sin u}{(u+\pi)^4}\,du,\end{aligned}$$

a positive quantity independent of ϵ. And, however great be A, we can choose ϵ so that $\frac{2\pi}{\epsilon} = A$; and then $\int_A^\infty \frac{x \sin \epsilon x}{1+x^2}\,dx$ is greater than this positive quantity, so that the integral is not uniformly convergent.

On the other hand, $P\int_0^\infty \tan ax \cos \epsilon x \frac{x\,dx}{1+x^2}$ is continuous for $\epsilon = 0$.

This does not follow at once from anything which precedes, but is not difficult to prove directly. For, in the first place,

$$P\int_0^\infty \tan ax \cos \epsilon x \left(\frac{1}{x} - \frac{x}{1+x^2}\right) dx = P\int_0^\infty \frac{\tan ax \cos \epsilon x}{x(1+x^2)}\,dx$$

is uniformly convergent in any finite interval of values of ϵ, and therefore continuous. This follows from the remark at the end of § 23, since $\int^\infty \frac{dx}{x(1+x^2)}$ is convergent. Moreover,

$$P\int_0^\infty \tan ax \cos \epsilon x \frac{dx}{x}$$

is continuous for $\epsilon = 0$. For, if $\frac{\epsilon}{a} = \delta$, it is

$$P\int_0^\infty \tan x \cos \delta x \frac{dx}{x} = \sum_0^\infty P\int_{i\pi}^{(i+1)\pi} = \sum_0^\infty P\int_0^\pi \tan x \cos \delta (x+i\pi) \frac{dx}{x+i\pi}$$

$$= \sum_0^\infty \int_0^{\frac{1}{2}\pi} \tan x \left\{\frac{\cos \delta (x+i\pi)}{x+i\pi} - \frac{\cos \delta (i\pi+\pi-x)}{i\pi+\pi-x}\right\} dx.$$

Now the series

$$\sum_1^\infty \left\{\frac{\cos \delta (x+i\pi)}{x+i\pi} - \frac{\cos \delta (i\pi+\pi-x)}{i\pi+\pi-x}\right\}$$

is uniformly convergent in $(0, \frac{1}{2}\pi)$ for any small value of $\delta > 0$. Hence we may sum under the sign of integration.

Moreover, it is not difficult to show that

$$\sum_n^\infty \left\{\frac{\cos \delta (x+i\pi)}{x+i\pi} - \frac{\cos \delta (i\pi+\pi-x)}{i\pi+\pi-x}\right\}$$

$$= \frac{1}{x+n\pi} + \sum_{n+1}^\infty \left\{\frac{1}{x+i\pi} + \frac{1}{x-i\pi}\right\} - \int_0^\delta \sin n\pi\delta \frac{\sin(\frac{1}{2}\pi - x)\,\delta}{\sin \frac{1}{2}\pi\delta}\,d\delta. \quad (1)$$

The last term is $$\frac{1}{n}\int_0^{n\delta}\sin\pi t\,\frac{\sin\left(\frac{1}{2}\pi-x\right)\dfrac{t}{n}}{\sin\dfrac{\pi t}{2n}}\,dt. \tag{a}$$

Now, if $0<u<\delta$, and δ is sufficiently small,

$$\frac{\sin\left(\frac{1}{2}\pi-x\right)u}{\sin\frac{1}{2}\pi u}\quad (0<x<\tfrac{1}{2}\pi)$$

is positive, and increases steadily as u increases from 0 to δ, and so lies between

$$\frac{\frac{1}{2}\pi-x}{\frac{1}{2}\pi}\text{ and }\frac{\sin\left(\frac{1}{2}\pi-x\right)\delta}{\sin\frac{1}{2}\pi\delta},$$

which differ by a quantity which vanishes with δ. And, if l is the greatest integer contained in $n\delta$, and

$$n\delta = l+\rho,$$

the modulus of (a) is less than

$$\frac{2}{n}\left|\int_{l-1}^{l}\sin\pi t\,\frac{\sin\left(\frac{1}{2}\pi-x\right)\dfrac{t}{n}}{\sin\frac{1}{2}\pi\dfrac{t}{n}}\,dt\right|+\frac{1}{n}\left|\int_l^{l+\rho}\right|<\frac{C}{n},$$

where C is a quantity independent of n, δ, and x. Hence (1) can be made as small as we please, by choice of n, for all values of δ and x in question.

Hence $\sum_1^\infty$ is uniformly convergent in the domain

$$x=(0,\ \tfrac{1}{2}\pi),\quad \delta=(0,\ \delta_0),$$

where δ_0 is any small positive quantity. It follows that

$$P\int_0^\infty \tan x\cos\delta x\,\frac{dx}{x}=\int_0^{\frac{1}{2}\pi}\tan x\sum_0^\infty\left\{\frac{\cos\delta\left(x+i\pi\right)}{x+i\pi}-\frac{\cos\delta\left(i\pi+\pi-x\right)}{i\pi+\pi-x}\right\}dx$$

is a continuous function of δ for $\delta=0$. And, in fact, we find on summing under the integral sign, since

$$\sum_0^\infty\left\{\frac{\cos\delta\left(x+i\pi\right)}{x+i\pi}-\frac{\cos\delta\left(i\pi+\pi-x\right)}{i\pi+\pi-x}\right\}=\cot x,$$

that $$P\int_0^\infty \tan x\cos\delta x\,\frac{dx}{x}=\tfrac{1}{2}\pi.$$

This principal value is therefore independent of δ, and changes continuously, for $\delta = 0$, into

$$P\int_0^\infty \tan x \frac{dx}{x} = \tfrac{1}{2}\pi,$$

as the preceding analysis shows that it should.

Hence
$$P\int_0^\infty \frac{\sin ax}{\cos ax}\frac{x\,dx}{1+x^2}$$

has a discontinuity of magnitude $\frac{1}{2}\pi$ to the left of $\alpha = a$.

Cauchy noticed the corresponding discontinuity of

$$P\int_0^\infty \frac{\cos \alpha x}{\sin ax}\frac{x\,dx}{1+x^2};$$

which, if $\alpha = a-\epsilon$, is

$$P\int_0^\infty \cot ax \cos \epsilon x \frac{x\,dx}{1+x^2} - \int_0^\infty \frac{x \sin \epsilon x}{1+x^2}dx.$$

But his discussion of it cannot be considered satisfactory. For he assumes that the first term is continuous for $\epsilon = 0$. And, moreover, he is content to accept the discontinuity of the second as a fact, without in any way attempting to explain it.

35. (ii.) It is easy to prove that, if

$$\alpha > 1, \quad a > 0, \quad c > 0,$$

$$\int_0^\infty \frac{\alpha \cos ax - \cos (a-c)x}{1-2\alpha \cos cx + \alpha^2}\frac{dx}{1+x^2} = \tfrac{1}{2}\pi \frac{e^{-a}}{\alpha - e^{-c}}, \tag{1}$$

$$P\int_0^\infty \frac{\alpha \cos ax - \cos (a-c)x}{1-2\alpha \cos cx + \alpha^2}\frac{dx}{1-x^2} = \tfrac{1}{2}\pi \frac{\alpha \sin a - \sin (a-c)}{1-2\alpha \cos c + \alpha^2}. \tag{2}$$

We might expect, after our investigations in §§ 27–30, to be able to put $\alpha = 1$ in these formulæ, provided we introduce the sign of the principal value before (1). But this gives

$$P\int_0^\infty \frac{\sin (a-\frac{1}{2}c)x}{\sin \frac{1}{2}cx}\frac{dx}{1+x^2} = -\pi\frac{e^{-a}}{1-e^{-c}},$$

$$P\int_0^\infty \frac{\sin (a-\frac{1}{2}c)x}{\sin \frac{1}{2}cx}\frac{dx}{1-x^2} = \tfrac{1}{2}\pi\frac{\cos (a-\frac{1}{2}c)}{\sin \frac{1}{2}c};$$

both of which are incorrect.

The explanation of this is very simple. For let us consider the simplest case, in which $a = c$. Then

$$\int_0^\infty \frac{\alpha \cos cx - 1}{1-2\alpha \cos cx + \alpha^2}\frac{dx}{1+x^2} = \tfrac{1}{2}\pi\frac{e^{-c}}{\alpha - e^{-c}} \quad (\alpha > 1),$$

and the limit of this for $\alpha = 1$ is $\dfrac{\pi}{2(e^c-1)}$, whereas its value for $\alpha = 1$ is $-\frac{1}{2}\pi$.

The fact is that
$$\int_a^A \frac{\alpha \cos cx - 1}{1-2\alpha \cos cx + \alpha^2}\phi(x)\,dx \tag{i.}$$

is discontinuous for $\alpha = 1$ if (a, A) include any of the points

$$\frac{2n\pi}{c}.$$

And it is easy to see that it is not uniformly convergent. Suppose, *e.g.*, $c = 1$, $a = 0$, $\phi(x) = 1$. Then

$$\int_0^{\xi} \frac{\alpha \cos x - 1}{1 - 2\alpha \cos x + \alpha^2} dx = \tan^{-1} \frac{\sin \xi}{\alpha - \cos \xi}.$$

Now
$$\tan^{-1} \frac{\sin \xi}{\alpha - \cos \xi} < \sigma$$

involves
$$\sin \xi < \tan \sigma (\alpha - 1 + \tfrac{1}{2} \sin^2 \xi \ldots);$$

and, however small be ξ, we can choose a value of α so nearly equal to 1 that this inequality is not satisfied.

The integral (i.) is, in fact, substantially Poisson's integral, which is so important in the theory of trigonometrical series.

[36. (iii.) If $\phi(x)$ is a function of x whose derivate $\phi'(x)$ is continuous, the principal value

$$\Phi(\alpha) = P\int_a^A \log\left(1 - \frac{\alpha}{x}\right)^2 \frac{\phi(x)}{x - \beta} dx \quad (a < \alpha < A,\ a < \beta < A)$$

is continuous for $\alpha = \beta$. For

$$\Phi(\beta) - \Phi(\beta - \epsilon) = P\int_a^A \log\left(\frac{x - \beta}{x - \beta + \epsilon}\right)^2 \frac{\phi(x)}{x - \beta} dx.$$

Now we may replace a, A by $\beta - \rho$, $\beta + \rho$, where ρ is any small fixed positive quantity; for the limits of $\int_a^{\beta - \rho}$, $\int_{\beta + \rho}^A$ for $\epsilon = 0$ are evidently both zero. And

$$P\int_{\beta - \rho}^{\beta + \rho} = P\int_{-\rho}^{\rho} \log\left(\frac{u}{u + \epsilon}\right)^2 \phi(u + \beta) \frac{du}{u}$$
$$= \phi(\beta)\, P\int_{-\rho}^{\rho} \log\left(\frac{u}{u + \epsilon}\right)^2 \frac{du}{u} + \int_{-\rho}^{\rho} \log\left(\frac{u}{u + \epsilon}\right)^2 \phi'(u_1 + \beta)\, du,$$

where $-\rho \leqq u_1 \leqq \rho$. It is easy to see that the last integral tends to zero with ϵ.

But the first is

$$\phi(\beta) \int_0^{\rho} \log\left(\frac{u - \epsilon}{u + \epsilon}\right)^2 \frac{du}{u} = \phi(\beta) \int_0^{\rho/\epsilon} \log\left(\frac{t - 1}{t + 1}\right)^2 \frac{dt}{t};$$

and the limit of this for $\epsilon = 0$ is

$$\phi(\beta) \int_0^{\infty} \log\left(\frac{t - 1}{t + 1}\right)^2 \frac{dt}{t} = -\tfrac{1}{2}\pi^2 \phi(\beta).$$

Hence
$$\Phi(\beta - 0) - \Phi(\beta) = \tfrac{1}{2}\pi^2 \phi(\beta);$$

and, similarly,
$$\Phi(\beta) - \Phi(\beta + 0) = \tfrac{1}{2}\pi^2 \phi(\beta).$$

We shall frequently meet with discontinuities of this kind when we come to consider the differentiation and integration of principal values.—*November 8th*, 1901.]

Continuity of Principal Values (continued).

37. THEOREM 2.—*If $f(x, \alpha)$ be a continuous function of both variables in any finite part of the rectangle*

$$(a, \alpha, \beta, \gamma)$$

which does not include any point of any of the curves $x = X_i(\alpha)$; and

$$P\int_a^\infty f(x, \alpha)\, dx$$

be regularly convergent in (β, γ), it will be a continuous function of α in (β, γ).

For, if σ be any assigned positive quantity, we can determine (§ 21) a value of A, a division of (β, γ) into two sets of finite intervals θ, η_i, and a set of positive quantities p_i, such that

$$\left| P\int_A^\infty \right| < \tfrac{1}{3}\sigma$$

in the intervals θ and $\qquad \left| P\int_{A-p}^\infty \right| < \tfrac{1}{3}\sigma$

in the intervals η_i.

And, if α_0 be any value of α in (β, γ), we can choose h' so small that α_0 and α_0+h' lie in the same sub-interval. Suppose, for instance, that they lie in η_i. Then

$$P\int_a^{A-p_i}$$

is uniformly convergent in η_i. And the conclusion follows as in § 25.

38. Thus
$$P\int_0^\infty \frac{\tan \alpha x}{x}\, dx \tag{1}$$

is regularly convergent in (β, γ) if $0<\beta<\gamma$, and therefore continuous. It is, in fact, $= \tfrac{1}{2}\pi$ (§ 30). But it is not regularly convergent in $(0, \gamma)$. For $\alpha = 0$ all the curves $x = X_i(\alpha)$ recede to infinity. And it is easy to show, by an argument similar to that used in § 34 in the case of the integral

$$\int_0^\infty \frac{x \sin \alpha x}{1+x^2}\, dx,$$

that, however great be A, we can always determine a positive quantity τ and a value of a such that

$$\left| P \int_A^{\infty} \right| > \tau,$$

and, moreover,

$$\left| P \int_{A-p}^{\infty} \right| > \tau,$$

for all values of p less than any fixed quantity p_0.

It is obvious that (1) is, as a matter of fact, discontinuous for $a = 0$.

On the Exponential Theorem for a Simply Transitive Continuous Group, and the Calculation of the Finite Equations from the Constants of Structure. By H. F. Baker.

Communicated February 14th, 1901. Received, in revised form, November 28th, 1901.

The present note was originally presented to the London Mathematical Society in February, 1901, in connexion with Mr. Campbell's paper, Vol. XXXIII., p. 285, and had then the purposes of suggesting the methodical use of a certain notation—that of the theory of matrices—and of showing how Mr. Campbell's results follow from Schur's determination of the infinitesimal transformations of a group of given structure (§ 4). Incidentally the theorem (§ 2) here called the exponential theorem was then obtained, and it was stated that it would lead to a method of finding the finite transformations of a group of given structure. The present form of the note differs from the original form by the addition of a verification of this statement, with examples (§§ 3, 5, and the latter part of § 4), and a considerable abbreviation of some parts of the paper whose novelty was stated to consist only in the methods employed.

1. The following notation is employed.

The differential equations satisfied by the functions f in the equations

$$x_i = f_i(x^0, a)$$

of a finite continuous group of n variables $x_1, \ldots, x_n$ and r parameters