On the Geometrical Meaning of a Form of the Orthogonal Transformation.

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The orthogonal transformation in space of three dimensions has been put by Lipschitz in a publication entitled *Untersuchungen über die Summen von Quadraten*, published at Bonn, in 1886, into the form

$$\left.\begin{aligned} x+\nu y-\mu z &= X-\nu Y+\mu Z \\ -\nu x+y+\lambda z &= \nu X+Y-\lambda Z \\ \mu x-\lambda y+z &= -\mu X+\lambda Y+Z \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where the new axes X, Y, Z are derived by a right-handed rotation through an angle θ from the old axes x, y, z, the constants λ, μ, ν being defined thus:

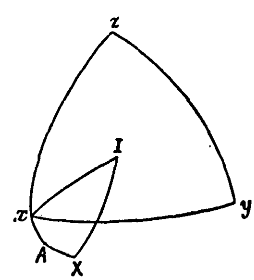
$$\lambda = \tan\tfrac{1}{2}\theta\cos\xi,\quad \mu = \tan\tfrac{1}{2}\theta\cos\eta,\quad \nu = \tan\tfrac{1}{2}\theta\cos\zeta \quad\ldots\ldots(2),$$

where ξ, η, ζ are the direction angles of the axis of rotation.

The object of this note is to point out the geometrical meaning of the equations (1).

Draw a sphere whose centre is at the origin O, cutting the axes of x, y, z at x, y, z; the axes of X, Y, Z at X, Y, Z; the axis of rotation at I.

Draw great circle arcs perpendicular to Ix, IX at x and X, respectively, meeting at A.



Then the first of equations (1) is obtained by projecting the coordinates of any point in the two systems along OA.

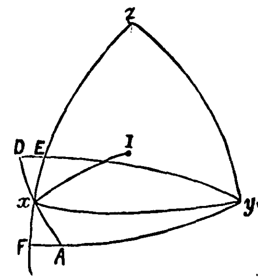
Let the direction angles of OA with regard to Ox, Oy, Oz be α, β, $\pi-\gamma$; so that in the annexed figure α, β, γ are all acute angles.

To find the direction angles of OA with regard to OX, OY, OZ imagine the figure rotated about I until IX coincides with Ix; then

A moves to a point D, so that

$$xD = xA.$$

Let yA cut zx at F, and yD cut zx at E. Then the spherical triangles xED, xFA are equal in all respects. Therefore



$$ED = AF,$$

$$yD + yA = \pi;$$

therefore $yD = \pi - \beta.$

In like manner

$$zD + zA = \pi;$$

therefore $zD = \gamma.$

Hence the direction angles of OD with regard to Ox, Oy, Oz are α, $\pi-\beta$, γ.

Hence the direction angles of OA with regard to OX, OY, OZ are α, $\pi-\beta$, γ.

The next step is to express α, β, γ in terms of θ, ξ, η, ζ.

$$\tan \tfrac{1}{2}\theta \sin \xi = \tan AIx \sin xI = \tan xA = \tan \alpha;$$

therefore
$$\cos \alpha = (1 + \sin^2 \xi \tan^2 \tfrac{1}{2}\theta)^{-\frac{1}{2}},$$

$$\begin{aligned}
\cos \beta &= \cos Ay \\
&= \cos Ax \cos xy + \sin Ax \sin xy \cos Axy \\
&= \sin \alpha \sin Ixy \\
&= \frac{\sin \alpha}{\sin \xi} \sin \xi \sin Ixy \\
&= \frac{\sin \alpha}{\sin \xi} \sin \left(\frac{\pi}{2} - Iz\right) \\
&= \cos \alpha \frac{\tan \alpha}{\sin \xi} \cos \zeta \\
&= \cos \alpha \tan \tfrac{1}{2}\theta \cos \zeta,
\end{aligned}$$

$$\begin{aligned}
\cos \gamma &= -\cos(\pi - \gamma) = -\cos Az \\
&= -(\cos Ax \cos xz + \sin Ax \sin xz \cos Axz) \\
&= -\sin \alpha \cos \left(\frac{\pi}{2} + Ixz\right) \\
&= \sin \alpha \sin Ixz
\end{aligned}$$

$$= \frac{\sin a}{\sin \xi} \sin \xi \sin Ixz$$

$$= \frac{\sin a}{\sin \xi} \sin \left(\frac{\pi}{2} - Iy \right)$$

$$= \frac{\sin a}{\sin \xi} \cos \eta$$

$$= \cos a \tan \tfrac{1}{2}\theta \cos \eta \,;$$

therefore $\cos \alpha : \cos \beta : \cos \gamma = 1 : \tan \frac{1}{2}\theta \cos \zeta : \tan \frac{1}{2}\theta \cos \eta$

$$= 1 : \nu : \mu.$$

Now, projecting the coordinates x, y, z, and then X, Y, Z of any point P along OA, it follows that

$$x \cos a + y \cos \beta + z \cos (\pi - \gamma) = X \cos a + Y \cos (\pi - \beta) + Z \cos \gamma;$$

therefore $$x + \nu y - \mu z = X - \nu Y + \mu Z,$$

which is the first of equations (1).

The second and third equations can be obtained in like manner.

A Property of Skew Determinants. By M. J. M. Hill, M.A., D.Sc., F.R.S., Professor of Mathematics at University College, London. Received and read May 9th, 1895.

It has been shown by Professor Cayley that the orthogonal transformation could be expressed thus

$$\left.\begin{array}{l} x_1 = a_{1,1} y_1 + a_{1,2} y_2 + \ldots + a_{1,n} y_n \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ x_n = a_{n,1} y_1 + a_{n,2} y_2 + \ldots + a_{n,n} y_n \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $$a_{r,r} = \frac{2\beta_{r,r} - \Delta}{\Delta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$a_{r,s} = \frac{2\beta_{r,s}}{\Delta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$