

Note on the Groups of Subtraction and Division, and on the Hyperbolic Functions

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We must, however, distinguish two separate cases corresponding to $n = 2p + 1$, $n = 2p$.

Let n be an odd number.

The ratio $k = \frac{n-1}{2} = p$ is an integer: that is to say the two generating straight lines BM and DM return to their initial mutual position after a semi-rotation of BM ; in the successive semi-rotation the locus is repeated.

The intersections of the curve with the circle, in addition to that corresponding to the point D , are n in number and are distributed in the vertices of the regular inscribed n -gon, one vertex being at M .

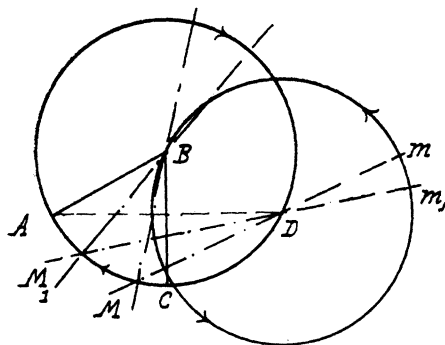


FIG. 2.

Let us then consider a point M_1 on the circumference such that the arc $MM_1 = \frac{2\pi}{n}$; it will suffice to show that this point must belong to the locus. In fact, the ratio of the angle MDm_1 measured in the sense of its generation, to the corresponding angle M_1BM is $\frac{n-1}{2}$, as follows from the value of MDm_1 , the supplementary angle to M_1DM measured by $\frac{1}{2}M_1M = \frac{\pi}{n}$, is

$$\frac{n-1}{n}\pi \text{ or } \frac{n-1}{2} \text{ times } \frac{2\pi}{n} = M_1BM. \quad \text{Q.E.D.}$$

The same demonstration can be applied to the other $n-2$ vertices.

CAMILLO MANZITTI.

(To be continued.)

NOTE ON THE GROUPS OF SUBTRACTION AND DIVISION, AND ON THE HYPERBOLIC FUNCTIONS.

THE objects of the present note are to exhibit an interesting property of some groups of subtraction and division, and to present a method of using such groups in the study of some elementary properties of the hyperbolic

Q 2

functions. The four finite groups of subtraction and division which transform every rational number into a rational number are of orders 4, 6, 8, and 12 respectively.* With respect to each of these groups all the numbers, with the exception of two more than the order of the group, are divided into sets of conjugates, each set containing as many distinct numbers as the order of the group. The property to which we referred in the first sentence is that each of these sets contains two and only two negative numbers whenever the set is composed of real numbers and the number from which we subtract is positive. In the special case, when the symmetric group of order 6 is obtained by subtracting from unity and by dividing unity, this theorem is clearly illustrated by the known fact that all such sets of six conjugate numbers may be obtained by assigning successively different values to x , $0 < x < \frac{\pi}{4}$, in the following functions :

$$\sin^2 x, \sec^2 x, -\cot^2 x, \cos^2 x, \csc^2 x, -\tan^2 x.$$

Thus if $\lambda = \sin^2 x$, the six values found by operating upon λ with the operations of the group are :

$$\begin{aligned} \lambda = \sin^2 x, \quad 1 - \lambda = \cos^2 x, \quad \frac{1}{\lambda} = \csc^2 x, \\ \frac{1}{1 - \lambda} = \sec^2 x, \quad \frac{\lambda}{\lambda - 1} = -\tan^2 x, \quad \frac{\lambda - 1}{\lambda} = -\cot^2 x. \end{aligned}$$

Hence one property of this group, which is also presented by the six anharmonic ratios of four points, is to associate four positive numbers with two negative ones, and *vice versa*. That this property is preserved when instead of subtracting from unity and dividing unity we subtract from x_1 and divide x_2 , x_1 and x_2 being any two positive numbers satisfying the condition $x_1^2 = x_2$, follows directly from the fact that the resulting six operations may be represented by

$$n, \quad \frac{x_1^2}{x_1 - n}, \quad x_1 - \frac{x_1^2}{n}, \quad x_1 - n, \quad \frac{x_1^2}{n}, \quad \frac{x_1 n}{n - x_1}.$$

Two, and only two, of these numbers are always negative when n is a positive number.

From this point of view we may regard the six anharmonic ratios of four points as constituting a very special case of six related distinct numbers which always include four positive and two negative ones. In the case of the first of the four groups mentioned above, viz. the group of order 4 generated by the operations of subtracting from 0 and dividing an arbitrary number, it is clear that two of the four distinct conjugates are always positive, while the other two are negative. Moreover, it is not difficult to verify that only two negative numbers occur in each complete set of distinct conjugates under each of the remaining two groups in question wherever x_1 is positive, since the operations of these groups may be represented as follows :

$$\begin{aligned} n, \quad x_1 - n, \quad \frac{x_1^2}{2n}, \quad \frac{x_1^2}{2x_1 - 2n}, \quad \frac{2x_1 n - x_1^2}{2n}, \quad \frac{x_1^2 - 2x_1 n}{2x_1 - 2n}, \quad \frac{x_1^2 - x_1 n}{x_1 - 2n}, \quad \frac{x_1 n}{2n - x_1}; \\ n, \quad x_1 - n, \quad \frac{x_1^2}{3n}, \quad \frac{x_1^2}{3x_1 - 3n}, \quad \frac{3x_1 n - x_1^2}{3n}, \quad \frac{x_1 n}{3n - x_1}, \quad \frac{2x_1 n - x_1^2}{3n - x_1}, \quad \frac{3x_1 n - x_1^2}{6n - 3x_1}, \\ \frac{3x_1 n - 2x_1^2}{6n - 3x_1}, \quad \frac{2x_1 n - x_1^2}{3n - 2x_1}, \quad \frac{x_1 n - x_1^2}{3n - 2x_1}, \quad \frac{2x_1^2 - 3x_1 n}{3x_1 - 3n}. \end{aligned}$$

When x_1 and n are both negative, $x_1 - n = -(|x_1| - |n|)$, and hence it results that a change in the sign of x_1 without changing x_2 changes the signs

* *Quarterly Journal of Mathematics*, vol. 37 (1905), p. 80.

of all the numbers of a system of conjugates under the group generated by $x_1 - n$ and $\frac{x_2}{n}$, but does not affect their absolute values. From this it follows that a system of distinct conjugates, or a non-degenerate system, under one of the groups in question, contains only two positive numbers when x_1 is negative. For example, the numbers which are conjugate under the group generated by subtracting from $x_1 = -1$ and by dividing $x_2 = 1$ may be obtained by assigning successively different values to x , $0 < x < \frac{\pi}{4}$, in the following six functions :

$$-\sin^2 x, -\sec^2 x, \cot^2 x, -\cos^2 x, -\csc^2 x, \tan^2 x.$$

The only set of two numbers which forms a complete set of conjugates under this group is the pair of imaginary cube roots of unity, and the only two sets of conjugates consisting of three numbers only are

$$1, -2, -\frac{1}{2} \text{ and } 0, -1, \infty.$$

All other numbers are transformed into six distinct conjugates by the operations of this group, two of these conjugates being positive, while the remaining four are negative. The more general result which has been proved may be stated as follows: *Any complete non-degenerate set of conjugates under one of the four finite groups of subtraction and division which transform every rational number into a rational number, contains either only two negative numbers or only two positive numbers as the number from which we subtract is positive or negative.*

From the well-known equations

$$1 - \operatorname{sech}^2 x = \tanh^2 x, \quad \frac{1}{\operatorname{sech}^2 x} = \cosh^2 x,$$

it follows that the squares of the hyperbolic functions may be obtained from any one of them by means of the group generated by the operations (s_1, s_2) of subtracting from unity and dividing unity. The (1, 1) correspondence between the operations of obtaining the hyperbolic and the trigonometric functions from one of them may perhaps become clearer from the following array :

1	$s_1 s_2$	$s_2 s_1$	s_1	s_2	$s_1 s_2 s_1$
$\operatorname{sech}^2 x$	$\coth^2 x$	$-\sinh^2 x$	$\tanh^2 x$	$\cosh^2 x$	$-\operatorname{csch}^2 x$
$\sin^2 x$	$\sec^2 x$	$-\cot^2 x$	$\cos^2 x$	$\csc^2 x$	$-\tan^2 x$
ρ	$\frac{1}{1-\rho}$	$\frac{\rho-1}{\rho}$	$1-\rho$	$\frac{1}{\rho}$	$\frac{\rho}{\rho-1}$
1	$l_1 l_2$	$l_2 l_1$	l_1	l_2	$l_1 l_2 l_1$
1	$(a_1 a_3 a_2)$	$(a_1 a_2 a_3)$	$(a_2 a_3)$	$(a_1 a_3)$	$(a_1 a_2)$

The fourth row of this array represents the group formed by the six anharmonic ratios of four points, while the fifth and sixth rows represent the group of movements of an equilateral triangle, and the symmetric group of degree three respectively, l_1 and l_2 representing rotations through π any two lines of symmetry of the triangle. The last two rows are added mainly to emphasise the known fact that the same group properties present themselves in such different subjects, each of these rows being simply isomorphic with the first. The first and fourth rows seem to exhibit most clearly how the squares of the hyperbolic functions, and hence these functions themselves, may be obtained from any one of them. In this array they are derived from $\operatorname{sech}^2 x$, but they could just as readily have been derived from any other one of the set.

The second and third rows, in connection with the first, exhibit a method of expressing all these hyperbolic functions in terms of an arbitrary one of them in exactly the same manner as the trigonometric functions are

expressed in terms of one of them. If the functions in these rows are supposed to imply the operation by means of which each of them has been derived from the first one in the row, these lines may be regarded as representing the symmetric group simply isomorphic with that represented by each of the other rows. From the table above it is evident that *the values of the squares of the six hyperbolic functions of x , as well as of the six trigonometric functions of x , are for any value of x conjugate numbers with respect to the group of subtracting from unity and dividing unity provided that a properly chosen pair of them is taken with the negative sign.*

It is of some interest to observe that $\coth^2 x$ and $-\sinh^2 x$ may be derived from $\operatorname{sech}^2 x$ by an operation of period three under $\{s_1, s_2\}$, while each of the remaining three functions in this row may be derived from $\operatorname{sech}^2 x$ by an operation of period two under this group. That is, the operation under $\{s_1, s_2\}$ which transforms $\operatorname{sech}^2 x$ into $\tanh^2 x$, for instance, must also transform $\tanh^2 x$ into $\operatorname{sech}^2 x$. Similar remarks apply to $\cosh^2 x$ and to $-\operatorname{csch}^2 x$ but not to $\coth^2 x$ or to $-\sinh^2 x$. With respect to the cyclic subgroup of order three $(1, s_1 s_2, s_2 s_1)$, the first three functions of the second row form a complete set of conjugates. Thus we see that *the six squares of the hyperbolic functions as given in the table are divided into two sets of three each. Any one of either set may be transformed into one of the same set by means of an operation in the cyclic subgroup $(1, s_1 s_2, s_2 s_1)$. Any one of the first set may be transformed into any one of the second set by s_1, s_2 or $s_1 s_2 s_1$.* Each of the last three operators is of period two and interchanges these two sets. From what precedes, it is clear that the hyperbolic functions, the trigonometric functions, the anharmonic ratios, and any other similarly related numbers may be obtained from each other by the same fundamental operations.

Since the hyperbolic functions have the period $2\pi i$, the two operations $\frac{\pi i}{2} - a$, $\pi i - a$ may be associated with those of taking the complement and the supplement of an angle. The octic group generated by these two operations may therefore be employed in the study of the hyperbolic functions in practically the same way as it was employed in the study of the trigonometric functions in an Article published in this Journal, volume III, No. 59. The present note may be regarded as a continuation of this Article. Similar remarks hold as regards the other groups of subtraction which have been used to find values of trigonometric functions. Cf. *Annals of Mathematics*, volume 8 (1907), p. 97.

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DECIMALS.

MR. GRANT, in the last issue of this *Gazette*, having made several references to "Arithmetical Types and Examples," perhaps a few remarks in reply will not be altogether out of place.

1. *Multiplication.* The only difference between the two methods first given is in the position of the various digits of the multiplicand.

We are told that the first rule (that "there are the same number of figures to the right of the decimal point in the first partial product as there are in the multiplicand," and that therefore the first figure set down must be under the right-hand figure of the multiplicand) is *mechanical*; whereas the rule of "putting each first figure of a partial product underneath the figure to which that partial product is due" is *sound and fundamental*. It is not obvious to the writer that one is more mechanical than the other.

The explanation immediately afterwards given by Mr. Grant of course applies to both methods of working. The reason for the position of the first figure of the first partial product in the first method is contained in the note to Ex. 1, p. 69, in the statement that "we are multiplying by 8 *units*,"