Generational Relations of an Abstract Simple Group of Order 4080. *By* L. E. Dickson.

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Introduction.

1. In an earlier paper in these *Proceedings*, Vol. xxxv., pp. 292–305, the writer investigated, for the case $p > 2$, the abstract group G simply isomorphic with the group Γ of all linear fractional transformations of determinant unity in the $GF[p^n]$. The present paper deals with the case $p = 2$, when the group Γ is of order $2^n(2^{2n}-1)$, and is simple if $n > 1$. Use is made of the transformations

$$(1)\quad T:\ z' = \frac{1}{z};\quad S_\lambda:\ z' = z+\lambda.$$

For $p = 2$, $n = 2$, the field is defined by the primitive irreducible congruence

$$i^2 \equiv i+1 \pmod 2.$$

The group Γ of order 60 is generated by

$$A = S_i T \quad\text{and}\quad B = S_{i+1};$$

a complete set of generational relations for G is*

$$(2)\quad A^5 = I,\quad B^2 = I,\quad (AB)^3 = I.$$

Setting $a = AB$, $\beta = B$, whence $A = a\beta$, $B = \beta$, we obtain a second set of generational relations for G:

$$(2')\quad a^3 = I,\quad \beta^2 = I,\quad (a\beta)^5 = I.$$

For $p = 2$, $n = 3$, the field is defined by the primitive irreducible congruence

$$i^3 \equiv i+1 \pmod 2.$$

The group Γ of order 504 is generated by

$$A = S_i T \quad\text{and}\quad B = S_{i+1};$$

* Dickson, *Linear Groups*, § 281, A being replaced by its inverse.

a complete set of generational relations for G is*

$$(3)\quad A^9 = I,\quad B^2 = I,\quad (AB)^3 = I,\quad (A^3BA^5B)^2 = I.$$

As generators of Γ may be taken $\alpha = TS_{i+1}$ and $\beta = S_i$; the resulting complete set of generational relations for G is†

$$(3')\quad \alpha^7 = I,\quad \beta^2 = I,\quad (\alpha\beta)^3 = I,\quad (\alpha^3\beta\alpha^5\beta\alpha^3\beta)^2 = I.$$

Since the sets of relations (2) and (3) are simpler than the respective sets (2′) and (3′), it is natural to seek as generators of Γ, for the $GF\,[2^4]$, two operators A and B of respective periods 17 and 2, rather than two operators α and β of respective periods 15 and 2. Some general results bearing on this point are reserved for a later communication to the Society.

For $p = 2$, $n = 4$, the field is defined by the primitive irreducible congruence

$$i^4 \equiv i+1 \pmod 2.$$

It will be shown that the group Γ of order 4080 is generated by

$$A = S_iT \quad \text{and} \quad B = S_{i+1},$$

and that a complete set of generational relations for the isomorphic group G_{4080} is

$$(4)\qquad A^{17} = I,\quad B^2 = I,\quad (AB)^3 = I,$$

$$(5)\quad (A^3BA^7B)^2 = I,\quad (A^4BA^{12}B)^2 = I,\quad (A^6BA^9B)^2 = I.$$

From these follow at once the relations

$$(5')\quad (A^5BA^{13}B)^2 = I,\quad (A^8BA^{11}B)^2 = I,\quad (A^{10}BA^{14}B)^2 = I.$$

From (4) alone follow

$$(5'')\quad (ABA^2B)^2 = I,\quad (A^{15}BA^{16}B)^2 = I.$$

Interchanging the two exponents in any one of the eight relations (5), (5′), (5″), we obtain a relation equivalent to it in virtue of (4). Hence‡ there exists a relation

$$(A^{\xi}BA^{\eta}B)^2 = I, \quad \text{for} \quad \xi = 1, 2, \ldots, 16.$$

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* Dickson, *Bulletin Amer. Math. Soc.*, January, 1903.

† Dickson, *ibid.*; Burnside and Fricke, *Math. Ann.*, Vol. LII.; de Séguier, *Journal de Mathématiques*, S. 5, t. VIII. (1902), p. 267.

‡ A like relation follows from (3), where now $\xi = 1, \ldots, 8$.

By means of Lagrange's interpolation formula, we find that

$$\eta \equiv 6\xi+9\xi^3+\xi^5+8\xi^7+9\xi^9+15\xi^{11}+12\xi^{13}+10\xi^{15} \pmod{17}.$$

In fact, this is the analytic representation of the substitution*

$$(1\ 2)(3\ 7)(4\ 12)(5\ 13)(6\ 9)(8\ 11)(10\ 14)(15\ 16).$$

The relation is thus less simple than in the case $p > 2$.

First Set of Generational Relations of G_{4080}.

2. The abstract G_{4080} is generated by the operators

$$T,\ S_\lambda\quad (\lambda = 0, i, i^2, \ldots, i^{14}, i^{15} = 1),\dagger$$

subject only to the generational relations‡

$$(6)\quad T^2 = I,\quad S_0 = I,\quad S_\lambda S_\mu = S_{\lambda+\mu}\quad (\lambda, \mu \text{ any marks}),$$

$$(7)\quad S_\lambda T S_\mu T S_{(\lambda-1)/(\lambda\mu-1)} T S_{-(\lambda\mu-1)} T S_{(\mu-1)/(\lambda\mu-1)}\, T = I$$

$$(\lambda, \mu \text{ any marks such that } \lambda\mu \neq 1).$$

Relation (7) will be designated by either of the symbols

$$(7')\quad (\lambda, \mu),\quad \left(\lambda, \mu, \frac{\lambda-1}{\lambda\mu-1}, -\lambda\mu+1, \frac{\mu-1}{\lambda\mu-1}\right).$$

If, in any particular relation (7), we permute the subscript cyclically, we obtain a relation, still of type (7), which follows at once from that particular relation and relations (6). The relations (7) with $\lambda = 0$ or 1, or with $\mu = 0$ or 1, all reduce in virtue of (6) to

$$(8)\quad (S_1 T)^3 = I.$$

If, of relations (7) with $\lambda \neq 0, 1$; $\mu \neq 0, 1$, we retain only one of the set arising by cyclic permutation of the subscripts, we obtain the following :—

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* *Annals of Math.*, Ser. 1, Vol. xi., p. 70.

† $i^4 \equiv i+1,\ i^5 \equiv i^2+i,\ i^6 \equiv i^3+i^2,\ i^7 \equiv i^3+i+1,\ i^8 \equiv i^2+1,\ i^9 \equiv i^3+i,$
$i^{10} \equiv i^2+i+1,\ i^{11} \equiv i^3+i^2+i,\ i^{12} \equiv i^3+i^2+i+1,\ i^{13} \equiv i^3+i^2+1,$
$i^{14} \equiv i^3+1;\quad i^{15} \equiv 1.$

‡ E. H. Moore. *Cf.* Dickson, *Linear Groups*, pp. 300-302.

$(i^2+i,\ i^2+i,\ i^2+i,\ i^2+i,\ i^2+i),$

$(i^2+i+1,\ i^2+i+1,\ i^2+i+1,\ i^2+i+1,\ i^2+i+1),$

$(i+1,\ i+1,\ i^3+1,\ i^2,\ i^3+1),$

$(i^2+1,\ i^2+1,\ i^3+i^2+1,\ i+1,\ i^3+i^2+1),$

$(i^3+1,\ i^3+1,\ i^3+i^2+i+1,\ i^3+i^2,\ i^3+i^2+i+1),$

$(i^3+i^2,\ i^3+i^2,\ i^2,\ i^3+i^2+i,\ i^2),$

$(i^3+i^2+i,\ i^3+i^2+i,\ i^3,\ i^3+i,\ i^3),$

$(i,\ i+1,\ i^3+i,\ i^2+i+1,\ i^3+i^2),$

$(i,\ i^2,\ i^2+i,\ i^3+1,\ i^3+i),$

$(i,\ i^2+i,\ i^3+i^2,\ i^3+i^2+1,\ i^3+i^2+i+1),$

$(i+1,\ i^2+i,\ i^3+i,\ i^3+i+1,\ i^3),$

$(i+1,\ i^2+i+1,\ i^3+i^2+1,\ i^3,\ i^2),$

$(i^2,\ i^3+1,\ i^3,\ i^2+i,\ i^3+i^2+i+1),$

$(i,\ i,\ i^3+i^2+i,\ i^2+1,\ i^3+i^2+i),$

$(i^2,\ i^2,\ i^3+i+1,\ i,\ i^3+i+1),$

$(i^3,\ i^3,\ i,\ i^3+i^2+1,\ i),$

$(i^3+i,\ i^3+i,\ i^2+1,\ i^3+1,\ i^2+1),$

$(i^3+i+1,\ i^3+i+1,\ i^3+i^2,\ i^3,\ i^3+i^2),$

$(i^3+i^2+1,\ i^3+i^2+1,\ i^3+i,\ i^3+i^2+i+1,\ i^3+i),$

$(i^3+i^2+i+1,\ i^3+i^2+i+1,\ i+1,\ i^3+i+1,\ i+1),$

$(i,\ i^2+1,\ i^3+i^2+i+1,\ i^3+i+1,\ i^2+i+1),$

$(i^2+i+1,\ i^3+i^2+i,\ i^3+i+1,\ i^3+i^2+1,\ i^3+1),$

$(i+1,\ i^2+1,\ i^2+i,\ i^3+i^2+i,\ i^3+i^2),$

$(i^2,\ i^2+i+1,\ i^3+i^2+i+1,\ i^3+i^2+i,\ i^3+i),$

$(i^3+1,\ i^3+i+1,\ i^2+i,\ i^3+i^2+1,\ i^3+i^2+i),$

$(i^2+1,\ i^2+i+1,\ i^3,\ i^3+1,\ i^3+i^2),$

together with the twelve derived from the last twelve by reversing the order of the subscripts.* The new twelve relations may be dropped, since they may be derived by inverting the first twelve and applying (6).

These relations may be separated into sets of three in various ways, such that from two of a set and relations (6) and (8) follows the third of that set. For example, $(i^2+1,\ i^2+1)$ may be written

$$S_{i^2+1}TS_{i^2+1}TS_{i^3+i^2+1}TS_i\,.\,S_1TS_{i^3+i^2+1}T = I.$$

We may replace $TS_{i^3+i^2+1}TS_i$ by $S_iTS_{i^3}TS_{i^3}T$, in view of $(i^3,\ i^3)$, and then replace TS_1T by S_1TS_1, in view of (8). There results the relation $(i^2+1,\ i^2+i+1)$. Again, from $(i^2+1,\ i^2+1)$ and $(i+1,\ i^2+1)$, we get

$$I = S_{i^2+1}TS_{i^2+1}TS_{i^3+i^2+1}\,.\,TS_{i+1}TS_{i^3+i^2}\,.\,S_1T$$

$$= S_{i^2+1}TS_{i^2+1}TS_{i^3+i^2+1}\,.\,S_{i^2+1}TS_{i^2+i}TS_{i^3+i^2+i}T\,.\,S_1T.$$

Replacing TS_1T by S_1TS_1 and then transforming by S_1, we get $(i^2,\ i^2+1)$. In this way we obtain the sets on the opposite page.

Hence relations (7) all follow from (6), (8), and

(9) $(i,\ i+1),\ (i+1,\ i+1),\ (i+1,\ i^2+1),\ (i,\ i^2+i),\ (i^2+i,\ i^2+i),$
$(i^2+i+1,\ i^2+i+1).$

3. The final relations (6) all follow from†

$$(10)\quad \begin{cases} S_0 = I, \quad S_\mu^2 = I, \quad S_{i+1}S_\mu = S_{i+1+\mu} \quad (\mu = 1,\ i,\ \ldots,\ i^{14}), \\ S_1S_{i^3} = S_{i^2+1}, \quad S_1S_{i^2} = S_{i^3+1}, \quad S_1S_{i^3+i^2} = S_{i^3+i^2+1}, \quad S_{i^2}S_{i^3} = S_{i^3+i^2}. \end{cases}$$

Indeed, inverting these relations, we get

$$S_\mu S_{i+1} = S_{i+1+\mu}, \quad S_{i^2}S_1 = S_{i^3+1}, \quad S_{i^3}S_1 = S_{i^2+1}, \quad \ldots.$$

In particular, $\qquad S_{i+1}S_1 = S_i, \quad S_1S_{i+1} = S_i.$

Hence $\qquad S_1S_i = S_iS_1 = S_{i+1},$

$$\left.\begin{aligned} S_iS_\tau &= S_{i+1}S_1S_\tau = S_{i+1}S_{\tau+1} = S_{\tau+i} \\ S_1S_{\tau+i} &= S_1S_iS_\tau = S_{i+1}S_\tau = S_{\tau+i+1} \end{aligned}\right\} \quad (\tau = i^2,\ i^3,\ i^3+i^2).$$

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* The number of relations is thirty-eight in agreement with *Linear Groups*, § 279. It was verified that in them every mark λ is followed directly by every mark μ for which $\lambda\mu \neq 1$, the first subscript being regarded as following the last.

† This section is not essential in the later parts of the paper.

(i, i^2), (i^2, i^3+i+1), (i^3+i^2+i, i^3+i^2+i);
(i, i^2), (i, i^2+1), (i^2+i+1, i^3+i^2+i);
(i, i^2), $(i+1, i^2+i+1)$, (i^3+1, i^3+i+1);
(i, i^2), $(i+1, i^2)$, (i^3+1, i^3+i+1);
(i, i^2), (i^2, i^2), (i^2, i^2+1);
(i, i^2+i), (i, i), (i^3+i^2, i^3+i^2);
(i, i^2+i), (i^3+1, i^3+i+1), (i^3+1, i^3+1);
(i, i^2+i), (i^3+1, i^3+i+1), $(i+1, i^2+i)$;
$(i+1, i^2+1)$, (i^2+1, i^3+i+1), (i^3+1, i^3+1);
$(i+1, i^2+i)$, $(i+1, i+1)$, (i^3+i, i^3+i);
$(i+1, i^2+i)$, (i^3+1, i^3+i+1), (i^3+i^2+i, i^3+i^2+i);
(i^2, i^2+1), (i^2+1, i^2+i+1), (i^3+i^2, i^3+i^2);
(i^2+1, i^2+i+1), (i^3, i^3), (i^2+1, i^2+1);
(i^2, i^2+i+1), (i^2+i+1, i^3+i^2+i), (i^3+1, i^2+i+1);

$(i, i+1)$, (i, i), (i, i^2+1);
$(i, i+1)$, (i^3, i^3), $(i+1, i^2+i)$;
$(i, i+1)$, $(i+1, i+1)$, $(i+1, i^2+i+1)$;
$(i, i+1)$, (i, i^2), $(i+1, i^2+1)$;
$(i, i+1)$, (i, i^2+i), $(i^3+i^2+i+1, i^3+i^2+i+1)$;
(i, i^2+1), (i, i^2+i), (i^3+i^2+1, i^3+i^2+1);
(i, i^2+1), $(i+1, i^2+1)$, (i^3+1, i^3+i+1);
(i, i^2+1), $(i+1, i^2+i+1)$, (i^2, i^2+1);
$(i+1, i^2+1)$, $(i+1, i^2+i+1)$, (i^2+i+1, i^3+i^2+i);
$(i+1, i^2+1)$, (i^2, i^2+1), (i^2+1, i^2+1);
$(i+1, i^2+i)$, $(i+1, i^2+i+1)$, (i^3+i+1, i^3+i+1);
(i^2, i^2+1), (i^3+i, i^3+i), (i^2, i^2+i+1);
(i^2, i^2+i+1), (i^2, i^2), $(i^3+i^2+i+1, i^3+i^2+i+1)$;
(i^2, i^2+i+1), (i^2+i+1, i^3+i^2+i), (i^3+i^2+1, i^3+i^2+1);
(i^2+1, i^2+i+1), (i^3+i+1, i^3+i+1), (i^2+i+1, i^3+i^2+i).

Hence $$S_1 S_\mu = S_{\mu+1},$$
for every μ. Then
$$S_i S_{i+1} = S_i S_1 S_i = S_{i+1} S_i = S_{i+i+1};$$
so that $$S_i S_\mu = S_{\mu+i},$$
for every μ. The other cases follow similarly.

The group G_{4080} *is generated by* T *and the* S_λ *subject to the relations* $T^2 = I$, (8), (9), *and* (10).

Second Set of Generational Relations of G_{4080}.

4. The commutative group formed by the sixteen operators
$$S_\lambda \quad (\lambda = 0, i, i^2, \ldots, i^{15} = 1),$$
subject to the final relations (6), is generated by four operators*
$$(11) \quad S_j \quad (j = 1, i+1, i^2+i, i^3),$$
subject only to the generational relations
$$(12) \quad S_j^2 = I, \quad S_j S_k = S_k S_j \quad (j, k = 1, i+1, i^2+i, i^3).$$

Indeed, from (6), follow (12) and
$$(13) \quad S_0 = I, \; S_i = S_1 S_{i+1}, \; S_{i^2} = S_1 S_{i+1} S_{i^2+i}, \; S_{i^2+1} = S_{i+1} S_{i^2+i}, \; \ldots,$$
which express every S_μ in terms of the generators (11). Inversely, from (12) and (13), the latter serving to define the S_μ other than (11), follow relations (6).

The group G_{4080} *is generated by* T, S_j $(j = 1, i+1, i^2+i, i^3)$, *subject to the relations* $T^2 = I$, (8), (9) *and* (12), *provided the operators* S_μ *in* (9) *are expressed in terms of the generators by means of relations* (13).

Third Set of Generational Relations of G_{4080}.

5. From relation (i, i), we get
$$S_i (TS_i)^3 = S_{i^3+i^2} TS_{i^2+1} TS_{i^3+i^2}.$$
But, from (i^2+1, i^2+i+1), we have
$$TS_{i^2+i^2} TS_{i^2+1} T = S_{i^2+1} TS_{i^2} TS_{i^2+i+1}.$$

* For the sake of the later application, these are chosen instead of the more symmetrical generators S_j $(j = 1, i, i^2, i^3)$.

Hence $$(TS_i)^4 = S_{i^3+1} TS_{i^3} TS_{i^3+i+1},$$
$$(TS_i)^8 = S_{i^3+1} TS_{i^2} TS_i TS_{i^3} TS_{i^3+i+1}.$$

But $$TS_{i^3} TS_i T = S_{i^3} TS_i TS_{i^3+i^2+1} \text{ by } (i^3, i^3).$$

Hence $$(TS_i)^8 = S_1 TS_i TS_{i^2+1} TS_{i^3+i+1}.$$

Writing the square of the second member and twice applying
$$TS_i TS_{i^2+1} T = S_{i^2+i+1} TS_{i^3+i+1} TS_{i^3+i^2+i+1},$$
which follows from (i, i^2+1), we get
$$(TS_i)^{16} = S_i S_{i^2} TS_{i^3+i+1} TS_i TS_{i^3+i+1} TS_{i^2}.$$

The second member equals $S_i T$ by (i^2, i^2). Hence*
$$(14) \quad (S_i T)^{17} = I.$$

6. Setting $$S_i T = A, \quad S_{i+1} = B,$$
we get $$B^2 = I, \quad BA = S_1 T,$$
from (6). Applying also (8) and (14), we obtain relations (4). For the concrete group Γ relations (5) hold, and
$$(15) \quad \begin{cases} S_1 = A^{10}BA^7BA^{10}, & S_{i^2+i} = A^9BA^3BA^9, \\ S_{i^3} = A^{-1}BA^8BA^9BA, & T = BA^{11}BA^7BA^9B. \end{cases}$$

To show that A and B, subject only to the relations (4) and (5), generate the group G_{4080}, it suffices to prove that the five operators T and S_j ($j = 1, i+1, i^2+i, i^3$), defined by (15) and $S_{i+1} = B$, satisfy the relations given in the theorem of § 4.

7. Applying (4) and (5_1), the first relation (5), we get
$$T^2 = BA^{11}.BA^7BA^3.BA^7BA^9B = BA^{11}.A^{-3}BA^{-7}B.BA^7BA^9B = I.$$

Evidently S_{i+1}, S_{i^2+i}, S_{i^3} are of period 2 by (4), while
$$S_1^2 = A^{10}.BA^7BA^3.BA^7BA^{10}$$
$$= A^{10}.A^{-3}BA^{-7}B.BA^7BA^{10} = I \quad [\text{by } (5_1)].$$

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* This may also be shown by means of the fractional group.
$$(S_i T)^8 : z' = \frac{(i^3+i^2+i)z+i^3+i+1}{(i^3+i+1)z+i^3+i+1}, \quad (S_i T)^9 : z' = \frac{(i^3+i+1)z+i^3+i+1}{(i^3+i+1)z+i^3+i^2+i}.$$
Hence $$(S_i T)^9 = (S_i T)^{-8}.$$
A third proof uses the canonical form of $S_i T$.

The condition for the identity $BAT = S_1$ is

$$BA.BA^{11}BA^7BA^9B.A^7BA^{10}BA^7 = I.$$

Replacing BA^9BA^7 by $A^{-6}BA^{-9}B.A$, in view of (5_3), and thrice replacing BAB by $A^{-1}BA^{-1}$, in view of (4), the condition reduces to

$$A^{-1}BA^9BA^6BA^9BA^7 = I,$$

a consequence of (5_3). Hence

$$BA = S_1T^{-1} = S_1T, \quad (S_1T)^3 = I.$$

To show that $\qquad BS_1 = S_1B,$

it suffices to prove that $\qquad AT = BATB.$

The condition is

$$BA.BA^{11}BA^7BA^9B.B.BA^8BA^{10}BA^6B.A^{-1} = I.$$

Replacing BAB by $A^{-1}BA^{-1}$, and transforming by A^2,

$$A^{-3}BA^{10}BA^7BA^9BA^8BA^{10}BA^6BA = I.$$

Replacing $A^{-3}BA^{10}B$ by BA^7BA^3, and transforming by BAB,

$$BA^6BA^9A\,BA^9BA^8BA^{10}BA^5 = I.$$

Replacing BA^6BA^9 by $A^8BA^{-6}B$, and BAB by $A^{-1}BA^{-1}$,

$$A^8BA^{-7}BA^8BA^8BA^{11}.A^{-1}BA^5 = I.$$

Replacing BA^8BA^{11} by A^6BA^9B, and then $BA^{-7}BA^{14}$ by A^3BA^7B, and $BA^{-1}B$ by ABA, we obtain an identity.

To show that $\qquad BS_{i^2+i} = S_{i^2+i}B,$

we transform $(BS_{i^2+i})^2$ by A^{-6} and twice replace A^6BA^9B by its inverse, and get

$$(BA^8BA^{-4}BA^3)^2 \equiv A^{-3}B\,(BA^3BA^7ABA^{-4})^2\,BA^3$$
$$\equiv A^{-3}B\,(A^{-7}BA^{-4}BA^{-5})^2\,BA^3,$$

by (5), and $\qquad BAB = A^{-1}BA^{-1}.$

Replacing $BA^{-13}BA^{-4}$ by its inverse, we obtain I.

To show that $\qquad BS_{i^2} = S_{i^2}B,$

we note that $(BS_{i^2})^2 = (ABA^9BA^9BA)^2$ [by (4)]

$$= ABA^9B.S_{i^2+i}.BA^9BA = ABA^9S_{i^2+i}A^9BA$$
$$= ABA^{18}BA^2BA^{18}BA = I \quad \text{[by (4)]}.$$

We next show that

$$(16) \quad S_{i^3} S_{i^3+i} = A^3 B A^9 B A^3.$$

The condition for this identity is

$$A^{-1} B A^8 B A^9 B A^{10} B A^2 B A^6 B A^8 B A^{-3} = I.$$

Replacing BA^2B by $A^{-1}BA^{-2}BA^{-1}$, in view of (4), and BA^9BA^9B by $A^{-2}BA^8BA^8BA^{-2}$, in view of $(BS_{i^3+i})^2 = I$, we get

$$A^{-1} B A^6 B A^8 B A^8 B A^{-4} B A^5 B A^8 B A^{-3} = I.$$

Replacing $BA^{-4}BA^5$ by $A^{12}BA^4B$, and transforming by A^7,

$$A^9 B A^6 B A^8 B A^3 B A^{12} B A^4 = I.$$

Replacing A^9BA^6B and $BA^{12}BA^4$ by their inverses, we get

$$BA^{-6}.BA^{-1}BA^{-1}B.A^5B = BA^{-6}.A.A^5B = I.$$

The second member of (16) is of period 2 in view of (5_3). Hence S_{i^3} is commutative with S_{i^3+i}. Next,

$$\begin{aligned} S_{i^3} S_1 &= A^{-1} B A^8 B A A^8 B A^{11} B A^7 B A^{10} \\ &= A^{-1} B A^8 B A . B A^6 B A^9 . A^7 B A^{10} \quad [\text{by } (5_3)] \\ &= A^{-1} B A^7 B A^6 B A^{11} \qquad [\text{by } (4_3)]; \end{aligned}$$

$$\begin{aligned} (S_{i^3} S_1)^2 &= A^{-1} B A^7 B A^6 B A^9 A B A^7 B A^6 B A^9 A^2 \\ &= A^{-1} B A^7 . A^8 B A^{-6} B . A B A^7 . A^8 B A^{-6} B . A^2 \\ &= A^{-1} B A^{-2} B A^{-7} B A^{-8} B A^{-6} B A^2 \quad [\text{by } (4_3)] \\ &= A^{-1} B A^{-2} . A^3 B A^7 B . B A^{-6} B A^2 \\ &= A^{-1} B A B A B A^2 = I \qquad [\text{by } (4)]. \end{aligned}$$

Hence S_{i^3} is commutative with S_1. Next,

$$\begin{aligned} S_{i^3+i} S_1 &= A^9 B A^2 . B A^2 B . A^7 B A^{10} \\ &= A^9 B A^2 . A^{-1} B A^{-2} B A^{-1} . A^7 B A^{10} \\ &= A^8 B A^{-3} B A^6 B A^{10}; \end{aligned}$$

$$\begin{aligned} (S_{i^3+i} S_1)^2 &= A^8 B A^{-3} B A^5 B A^{-4} B A^6 B A^{10} \\ &= A^8 B A^{-3} . A^4 B A^{-5} B . B A^6 B A^{10} \\ &= A^8 B A B A B A^{10} = A^8 A^{-1} A^{10} = I. \end{aligned}$$

Hence S_{i^3+i} is commutative with S_1. Hence relations (12) are all satisfied. Before considering relations (9), certain auxiliary results are derived.

8. Defining $S_i, S_{i^2}, \ldots$ by formulæ (13), we get

$$S_{i^2+1} = S_{i+1}S_{i^2+i} = BA^9BA^2BA^9$$

$$S_{i^2}T = S_{i^2+i}S_iT = A^9BA^2BA^{10},$$

$$S_{i^2+1}TS_{i^2} = BA^9BA^2BA^{-1}BA^{-2}BA^8 = BA^9BA^4BA^9,$$

upon twice replacing $BA^{-1}B$ by ABA. Also

$$S_{i^2}TS_{i^2+i} = (S_{i^2+1}TS_{i^2})^{-1}S_{i+1} = A^8BA^{-4}BA^8.$$

9. We may write $(i+1, i+1)$ in the form

$$S_{i+1}.TS_i.S_1T.S_{i^2}.S_1T.S_{i^2}T.S_{i^2}.S_1T = I.$$

Supplying the values of the factors, the relation becomes

$$BA^7BA^9BABAA^9BA^2BA^9BA^8BA^9BABA = I.$$

Twice replacing $BABA$ by $A^{-1}B$, we get

$$BA^7BA^8BA^9BA^2BA^9BA^8BA^8B = I.$$

Since B is commutative with

$$A^9BA^2BA^9 \equiv S_{i^2+i},$$

the relation reduces to an identity upon applying (4).

The left number of $(i, i+1)$ is

$$\begin{aligned}
&S_iT.S_{i+1}.TS_i.S_{i^2}.TS_{i^2}.S_{i+1}.TS_{i^2}S_{i^2}T \\
&= ABA^{-2}BA^8BA^9BA^8BA^{-2}BA^8B.TS_{i^2}S_{i^2}T \\
&= ABA^{-2}BA^8BA^9B.S_{i^2+i}^{-1}B.TS_{i^2}S_{i^2}T \\
&= ABA^{-2}BA^8BA^9S_{i^2+i}^{-1}.TS_{i^2}.S_{i^2}T \\
&= ABA^{-2}BA^6BA^8.A^7BA^{-2}BA^8.S_{i^2}T \\
&= ABA^{-2}BA^6.ABA^3BA.A^8S_{i^2}T \\
&= ABA^{-2}B.BA^{-3}BA^{-7}.A^9S_{i^2}T \\
&= ABA^{-5}BABA^8BA^9BA.T \\
&= ABA^{-6}BA^7BA^9BAT = ATAT \\
&= BS_1BS_1 = I.
\end{aligned}$$

Relation (i, i^2+i) may be written

$$S_iT.S_{i^2+i}TS_{i^2}.S_{i^2}.TS_1.S_{i^2}.S_{i^2}TS_{i^2+1}.S_{i^2}.S_iT = I.$$

Substituting, and transforming by A^{-2}, it becomes

$$A^{12}BA^4B(A^8B)^4A^9BA^9BA^{-4}B(A^9B)^3 = I.$$

Replacing $A^{12}BA^4B$ by its inverse, transforming by BA^8, replacing the new front product $A^5BA^{13}B$ by its inverse, and transforming by BA^8, we get

$$A^{13}BA^5A^{-2}BA^8BA^8BA^9BA^9BA^{-4}BA^9B = I.$$

Replacing $A^{-2}BA^8BA^8B$ by $BA^9BA^9BA^2$, in view of

$$(BS_{i^2+i})^2 = I,$$

and replacing $A^{13}BA^5B$ by its inverse, and transforming by BA^8B, we get

$$BA^{-1}A^5BA^{13}BA^9BA^{11}BA^9BA^{-4} = I.$$

Replacing $A^5BA^{13}B$ by its inverse, and $BA^{-1}B$ by ABA,

$$ABA^5BA^4BA^{11}BA^9BA^{-4} = I.$$

Transforming by A^{10}, and replacing BA^9BA^6 by its inverse,

$$A^8BA^{-7}A^{12}BA^4BA^5BA^8B = I.$$

Replacing $A^{12}BA^4B$ by its inverse, and transforming by A^{11},

$$A^{-3}BA^{-7}B.A^{-4}BA^{10}.BA^8BA^{11} = I.$$

Replacing the first and third products by their inverses,

$$BA^7BA^{-1}BA^{-1}BA^9B = BA^7.A.A^9B = I.$$

We may write $(i+1,\ i^2+1)$ thus:

$$S_{i+1}.TS_1.S_{i^2}TS_{i^2+i}.TS_i.(S_{i^2}.S_{i^2}T)^2 = I,$$

$$BA^{-1}BA^8BA^{-4}BA^8A^{-1}(A^{-1}BA^8BA^9BA^{10}BA^2BA^{10})^2 = I,$$

$$ABA^9BA^{-4}BA^7(A^{-1}BA^8BA^9BA^9BA^{-2}BA^9)^2 = I.$$

In the expanded square occurs the factor

$$A^{-2}BA^8BA^8B = BA^9BA^9BA^2,$$

as remarked in the preceding paragraph. The relation becomes

$$ABA^9BA^{-4}BA^6BA^8BA^9BABA^9BA^{11}BA^9BA^{-2}BA^9 = I.$$

Replacing BAB by $A^{-1}BA^{-1}$, and then BA^8BA^{11} by its inverse, and transforming by A, we get

$$BA^9BA^{-4}BA^6BA^8BA^{14}BABA^{-2}BA^{10} = I.$$

Replacing BAB by $A^{-1}BA^{-1}$, and $BA^{-3}BA^{10}$ by its inverse,

$$BA^9BA^{-4}BA^6BA^8BA^3BA^5B = I.$$

Transforming by BA^4, replacing $A^5BA^{-4}B$ and BA^8BA^7 by their inverses, the relation becomes

$$BA^4BABABA^{-3}B = BA^4A^{-1}A^{-3}B = I.$$

Consider $(i^2+i,\ i^2+i)$, viz.,

$$(TS_{i^2+i})^5 = I.$$

Now
$$(TS_{i^2+i})^2 = TS_i.S_{i^2}TS_{i^2+i} = A^7BA^{-4}BA^8,$$
$$(TS_{i^2+i})^4 = A^7BA^{-4}.BA^{-2}B.A^{-4}BA^8$$
$$= A^7BA^{-4}.ABA^2BA.A^{-4}BA^8.$$

The original condition thus becomes

$$BA^{11}BA^7BA^9B.A^9BA^2BA^9.A^7BA^{-3}BA^2BA^{-3}BA^8 = I.$$

Replacing B^1A^6B by ABA, and transforming by BA^{-6},

$$BA^7BA^9BA^9BA^3BA^{-2}BA^2BA^{-3}BA^8BA^{-6} = I.$$

Replacing BA^8BA^{-6} by its inverse, and transforming by BA^8B,

$$BA^{16}BA^9BA^9BA^3BA^{-3}BA^2BA^3 = I.$$

Replacing $BA^{16}B$ by ABA, and transforming by ABA^7,

$$A^3BA^9BA^3BA^{-3}BA^2BA^4BA^7 = I.$$

Replacing BA^2B by $A^{-1}BA^{-2}BA^{-1}$, we get

$$A^3BA^9BA^3BA^{-3}BA^{-2}BA^3BA^7 = I.$$

Replacing BA^3BA^7 by its inverse, we get

$$(A^3BA^9BA^3)B(A^3BA^9BA^3)^{-1}B = I \quad \text{[by (16)]}.$$

Consider $(i^2+i+1,\ i^2+i+1)$, viz.,

$$W^5 = I,$$

where
$$W = S_{i^2+i+1}T,$$
$$W^2 = S_{i+1}.S_{i^2}TS_{i^2+i}.S_1T = BA^8BA^{-4}BA^8BA,$$
$$W^4 = BA^8BA^{-4}BA^8.A^{-1}BA^{-1}.A^8BA^{-4}BA^8BA,$$
$$W^5 = W^4S_{i^2+i}S_iT = W^4BA^9BA^2BA^{10}$$
$$= BA^8BA^{-4}BA^7BA^7BA^{-4}BA^7BA^8BA^2BA^{10}.$$

Transforming both members of $W^5 = I$ by A^6, replacing $A^{-6}BA^8B$ by its inverse, and then transforming by AB,

$$ABA^{10}BA^{-1}A^8BA^7BA^7BA^{-4}BA^7BA^8BA^2 = I.$$

Replacing A^8BA^7B by its inverse, and $BA^{-1}B$ by ABA,

$$ABA^{11}BA^{-1}A^{12}BA^4BA^{-4}BA^7BA^8BA^2 = I.$$

Replacing $A^{12}BA^4B$ by its inverse, then $BA^{-1}B$ by ABA, then BAB by $A^{-1}BA^{-1}$, then BA^6BA^9 by its inverse, we get

$$ABA^{12}BA^4BA^{-6}BA^{-1}BA^2 = I.$$

Replacing $A^{12}BA^4B$ by its inverse, and $BA^{+1}B$ by ABA, we obtain an identity. Hence the theorem is proved.

On Perpetuants. *By* J. H. GRACE. Communicated December 11th, 1902. Received December 16th, 1902. Revised March 4th, 1903.

1. In a recent paper I proved that perpetuants of unit degree in each quantic involved are of the form

$$(ab)^\lambda (ac)^\mu \dots (ak)^\rho (ah)^\sigma (al)^\tau,$$

where $\tau \geqslant 1$, $\sigma \geqslant 2$, $\rho \geqslant 4$, ...

It is easily seen by the methods there explained that the form

$$(ab)^\lambda (bc)^\mu \dots (kh)^\sigma (hl)^\tau,$$

with the same conditions imposed on the exponents, is equally suited to the expression of perpetuants.

In the present paper I shall find the general form of a perpetuant when all the letters do not refer to different quantics. The results lead incidentally to the generating functions discovered by MacMahon and Stroh.

2. *Statement of Results for one Quantic.*

The symbols $a_1, a_2, a_3, \dots$ all referring to the same quantic, the general form of a perpetuant is

$$(a_1a_2)^{2\alpha_1}(a_2a_3)^{\alpha_2}(a_3a_4)^{\alpha_3} \dots (a_ra_{r+1})^{\alpha_r},$$