

The Weil Conjecture for K3 Surfaces

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1. Statement of the Theorem

Let $\mathbb{F}_q$ be a field with q elements, let $\overline{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$, let $\varphi \in \text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ be the Frobenius substitution $x \mapsto x^q$, and let $F = \varphi^{-1}$ be the “geometric Frobenius.”

Let X be a scheme (separated and of finite type) over $\mathbb{F}_q$, and let $\overline{X}$ be the scheme over $\overline{\mathbb{F}}_q$ obtained from it by extension of scalars. For every closed point x of X , let $\deg(x) = [k(x) : \mathbb{F}_q]$ be the degree of the residue-field extension over $\mathbb{F}_q$. The zeta function $Z(X, t) \in \mathbb{Z}[[t]]$ is defined by

$$Z(X, t) = \prod_{x \in X} (1 - t^{\deg(x)})^{-1} \quad (x \text{ a closed point of } X),$$

$$\log Z(X, t) = \sum_{n>0} \#X(\mathbb{F}_{q^n}) \frac{t^n}{n}.$$

For every prime number ℓ prime to q , the ℓ -adic cohomology with proper supports $H_c^i(\overline{X}, \mathbb{Q}_\ell)$ of $\overline{X}$ is a finite-dimensional vector space over $\mathbb{Q}_\ell$, on which $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ acts by transport of structure. By Grothendieck, one has

$$Z(X, t) = \prod_i \det(1 - Ft, H_c^i(\overline{X}, \mathbb{Q}_\ell))^{(-1)^{i+1}}.$$

Take X to be a nonsingular projective surface. In this case it is known that the factors

$$P_i(t) = \det(1 - Ft, H^i(\overline{X}, \mathbb{Q}_\ell))$$

of $Z(X, t)$ are polynomials with integer coefficients independent of ℓ . The Weil conjecture asserts here that the roots of $P_i(t)$ have complex absolute value $q^{-i/2}$. This conjecture is invariant under finite extension of the finite base field. For X as above, and $i \neq 2$, it follows from Weil [9]. If, for simplicity, X is assumed geometrically connected, one indeed has

$$\begin{aligned} P_0(t) &= 1 - t, & P_4(t) &= 1 - q^2 t, \\ P_1(t) &= \det(1 - \varphi t; T_\ell(\text{Pic}^0(X)_{\text{red}})), & P_3(t) &= P_1(qt). \end{aligned}$$

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Let k be a field of characteristic 0, and let X be a nonsingular projective geometrically connected surface over k . We say that X is a K3 surface if X is regular and has trivial canonical divisor, that is, if $H^1(X, \mathcal{O}) = 0$ and Ω_X^2 is isomorphic to $\mathcal{O}_X$. For the remarkable properties of these surfaces, we refer to [8]. We shall mainly use the following facts.

1.1. Lemma. (i) One has $h^{2,0} = h^{0,2} = 1$.

(i) One has $H^2(X, T_X^1) = 0$ (infinitesimal deformations are unobstructed), and the map induced by interior product

$$H^1(X, T_X^1) \otimes H^0(X, \Omega_X^2) \xrightarrow{\sim} H^1(X, \Omega_X^1) \quad (1.1.1)$$

is bijective, while $H^0(X, T_X^1) = 0$ (X has no infinitesimal automorphisms).

One has

$$h^{2,0} =_{\text{def}} \dim H^0(X, \Omega_X^2) = \dim H^0(X, \mathcal{O}_X) = 1,$$

and (i) follows by Hodge symmetry. Since $H^1(X, \mathcal{O}_X) = 0$, Hodge symmetry and Serre duality give

$$h^{0,1} = h^{1,0} = h^{2,1} = h^{1,2} = 0.$$

Since $\Omega_X^2 = \mathcal{O}_X$, the “interior product” isomorphism

$$T_X^1 \otimes \Omega_X^2 \xrightarrow{\sim} \Omega_X^1$$

induces isomorphisms $T_X^1 \simeq \Omega_X^1$. The vanishing of $H^2(X, T_X^1)$ therefore follows from that of $h^{1,2}$, and it is clear that (1.1.1) is bijective. The same argument gives $H^0(X, T_X^1) = 0$.

1.2. Lemma. Let Y_0 be a nonsingular projective surface over a finite field $\mathbb{F}_q$. The following conditions are equivalent.

- (i) There exists a complete discrete valuation ring V of unequal characteristic with finite residue field k , a proper smooth morphism $f_V : Y \rightarrow \text{Spec}(V)$ whose generic fiber is a K3 surface, and $i : \mathbb{F}_q \rightarrow k$ such that $Y_0 \otimes_{\mathbb{F}_q} k \simeq Y \otimes_V k$.
- (ii) There exists an integral scheme S whose fraction field has characteristic 0, a point $s \in S$, a proper smooth morphism $f : Y \rightarrow S$ whose generic fiber is a K3 surface, and $i : \mathbb{F}_q \rightarrow k(s)$ such that $Y_0 \otimes_{\mathbb{F}_q} k(s) \simeq Y_s$.

The proof uses a standard passage to the limit. If the equivalent conditions of 1.2 are satisfied, we say that Y_0 lifts to a K3 surface.

1.3. Theorem. A nonsingular projective surface over a finite field $\mathbb{F}_q$ which lifts to a K3 surface satisfies the Weil conjecture.

This theorem was obtained independently by Pjateckii-Shapiro and Šafarevitch.

It applies, for example, to surfaces of degree 4 in $\mathbb{P}^3$. The Betti numbers of a K3 surface are $(1, 0, 22, 0, 1)$; the theorem therefore implies

that

$$|\#X(\mathbb{F}_q) - \#\mathbb{P}^2(\mathbb{F}_q)| \leq 21q.$$

Grothendieck's conjectural theory of motives (in particular his theory of the “motivic Galois group”) was very useful to me in constructing the proof. In this language (which will not be used below), the idea is, roughly speaking, to construct, by reduction from $\mathbb{C}$ to $\mathbb{F}_q$, an abelian variety A and an injective “morphism” of motives

$$H^2(X) \hookrightarrow H^1(A) \otimes H^1(A),$$

in order to deduce the Weil conjecture for X from Weil's theorem for A . However, we do not define this “morphism” by an algebraic cycle; it is therefore not a morphism in the sense of [5]. In Hodge theory, its mode of definition has the following analogue (which will not be used below).

1.4. Lemma. *Let H be a variation of Hodge structures of weight n on a scheme S . If the local system $H_{\mathbb{Z}}$ has no global sections other than the section s and its multiples, then at every point of S the section s is of type $(n/2, n/2)$.*

1.5. Notations. 1.5.1. Scalar extensions will be denoted by subscripts: if M_A is a module over a ring (commutative with identity) A (or a scheme over A , for example an algebraic group over A), then M_B denotes the module over the A -algebra B (or the scheme over B) obtained from it by extension of scalars.

1.5.2. If a group G acts on a set X , we write $g * x$ for the transform of x by g .

1.5.3. When an equality is the definition of one of its sides, we write it $=_{\text{def}}$.

2. Polarized Hodge Structures

To work with Hodge structures, we shall use the formalism set out in [4] and briefly recalled below.

2.1. Let $\underline{S}$ be the real algebraic group of invertible elements of the $\mathbb{R}$ -algebra $\mathbb{C}$, let $w : \mathbb{G}_{m_{\mathbb{R}}} \rightarrow \underline{S}$ be the inclusion of $\mathbb{R}^*$ into $\mathbb{C}^*$, and let $t : \underline{S} \rightarrow \mathbb{G}_{m_{\mathbb{R}}}$ be the inverse of the norm; then $tw(x) = x^{-2}$. Let V be a finite-dimensional real vector space. It amounts to the same thing to give:

- a) $\rho : \underline{S} \rightarrow \text{GL}(V)$ of weight n : $\rho w(x) * v = x^n \cdot v$;
- b) a Hodge bigrading of weight n of $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$:

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q} \quad \text{with} \quad \overline{V^{p,q}} = V^{q,p};$$

- c) a Hodge filtration of weight n of $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$: this is a decreasing filtration such that

$$F^p(V_{\mathbb{C}}) \oplus \overline{F^{n-p+1}(V_{\mathbb{C}})} \xrightarrow{\sim} V_{\mathbb{C}}.$$

One has

$$F^p(V_{\mathbb{C}}) = \bigoplus_{p' \geq p} V^{p', q'};$$

for $z \in \underline{S}(\mathbb{R}) = \mathbb{C}^*$ and $v^{p,q} \in V^{p,q}$, one has

$$z * v^{p,q} = z^p \bar{z}^q v^{p,q}.$$

Put $C = \rho(i)$; this is the *Weil operator* C . One has $Cv^{p,q} = i^{p-q}v^{p,q}$. For $\mathcal{E}$ a subset of $\mathbb{Z} \times \mathbb{Z}$, a Hodge bigrading is said to be of *type* $\mathcal{E}$ if the *Hodge numbers* $h^{p,q} = \dim V^{p,q}$ vanish for $(p,q) \notin \mathcal{E}$.

2.2. Here a *Hodge structure of weight* n H will mean a free finite-type $\mathbb{Z}$ -module $H_{\mathbb{Z}}$ together with a Hodge bigrading of weight n of $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C}$. The *Tate Hodge structure* $\mathbb{Z}(n)$ is the Hodge structure of type $\{(-n, -n)\}$ given by

$$\mathbb{Z}(n)_{\mathbb{C}} = \mathbb{C} \quad \text{and} \quad \mathbb{Z}(n)_{\mathbb{Z}} = (2\pi i)^n \mathbb{Z} \subset \mathbb{C}.$$

A polarization of a Hodge structure H of weight n is a morphism of Hodge structures

$$\psi : H \otimes H \longrightarrow \mathbb{Z}(-n)$$

such that, on $H_{\mathbb{R}} = H_{\mathbb{Z}} \otimes \mathbb{R}$, the real bilinear form

$$(2\pi i)^n \psi(x, Cy)$$

is symmetric and positive definite. The form ψ is then $(-1)^n$ -symmetric.

A *rational Hodge structure of weight* n consists of a $\mathbb{Q}$ -vector space $H_{\mathbb{Q}}$ and a Hodge bigrading of weight n of $H_{\mathbb{C}} = H_{\mathbb{Q}} \otimes \mathbb{C}$. We denote by $\mathbb{Q}(n)$ the rational Hodge structure underlying $\mathbb{Z}(n)$. The definition of polarizations is left to the reader.

The following statement reformulates a theorem of Riemann.

2.3. Scholium. *The functor $A \mapsto H^1(A, \mathbb{Z})$ identifies polarized abelian varieties with polarized Hodge structures of type $\{(0, 1), (1, 0)\}$.*

From the remainder of this subsection, we shall use only 2.11(iii') $\Rightarrow$ (ii). To prove this implication, one may restrict to connected groups, which makes 2.5 unnecessary or obvious.

2.4. Let G be a real linear algebraic group. We say that G is *compact* if $G(\mathbb{R})$ is compact and every connected component of $G(\mathbb{C})$ has a real point. The obvious functor is then an equivalence from the category of algebraic linear representations of G to the category of linear representations of the compact group $G(\mathbb{R})$.

2.5. Lemma. *Every real algebraic subgroup H of a compact real algebraic group G is compact.*

It is known that the map

$$(g, \ell) \longmapsto g \cdot \exp(i\ell) : G(\mathbb{R}) \times \text{Lie}(G) \longrightarrow G(\mathbb{C})$$

is bijective. It is enough to prove that, for every real subgroup H of G and every $h = g \cdot \exp(i\ell) \in H(\mathbb{C})$, one has $\ell \in \text{Lie}(H)$. Since $\bar{h} \in H(\mathbb{C})$, one has

$$\exp(2i\ell) = \bar{h}^{-1}h \in H(\mathbb{C}).$$

Regard $G(\mathbb{C})$ as the set of real points of a real algebraic group. Let $\bar{J} \subset H(\mathbb{C})$ be, within this group, the Zariski closure of $J = \{\exp(ni\ell) \mid n \in 2\mathbb{Z}\}$. The elements $g \in J$, and hence the elements $g \in \bar{J}$, satisfy $g^{-1} = \bar{g}$, whence $\text{Lie}(\bar{J}) \subset i\text{Lie}(G)$. The Lie algebra $\text{Lie}(\bar{J})$ is abelian, and $\exp(\text{Lie}(\bar{J}))$ is a group, necessarily of finite index in $\bar{J}$. There are therefore n and $\ell' \in \text{Lie}(\bar{J})$ such that

$$\exp(ni\ell) = \exp(\ell').$$

Thus $i\ell \in \text{Lie}(\bar{J}) \subset \text{Lie}(H)$ and $\ell \in \text{Lie}(H)$, completing the proof.

2.6. Lemma. *Let G be a real algebraic group. The following conditions are equivalent:*

- (i) G is compact;
- (ii) for every real (respectively complex) linear representation of G , there exists a positive-definite G -invariant symmetric bilinear (respectively hermitian sesquilinear) form;
- (iii) for a faithful representation of G , there exists a form as in (ii).

For G connected, these conditions are further equivalent to

- (iii') for a representation of G with finite kernel, there exists a form as in (ii).

For G compact, G -invariance is equivalent to $G(\mathbb{R})$ -invariance, and (ii) is classical. If (iii) holds, G is a subgroup of an orthogonal or unitary group, and 2.5 applies. Condition (iii') implies that $G(\mathbb{R})$ is compact (as a finite covering of a compact group); if G is connected, G is therefore compact.

2.7. Let G be a real linear algebraic group. For every involutive automorphism σ of G , let $G^{(\sigma)}$ be the real form of $G_{\mathbb{C}}$ defined by the antilinear involution $g \mapsto \sigma(\bar{g})$. We say that σ is a *Cartan involution* if $G^{(\sigma)}$ is compact.

Let C be an element of $G(\mathbb{R})$ whose square is central. A real (respectively complex) representation of G will be called C -polarizable if there exists on V a G -invariant bilinear (respectively sesquilinear) form ψ such that $\psi(x, Cy)$ is symmetric (respectively hermitian) and positive definite. We then say that ψ is a C -polarization of V . A special case of the following lemma is stated without proof in [4].

2.8. Lemma. *a) Whether ψ is a C -polarization of V depends only on the $G(\mathbb{R})$ -conjugacy class of C .
b) The following conditions are equivalent:*

- (i) $\text{ad } C$ is a Cartan involution of G ;
- (ii) every real (respectively complex) representation of G is C -polarizable;
- (iii) G has a faithful real (respectively complex) C -polarizable representation.

If G is connected, these conditions are also equivalent to

- (iii') G has a real (respectively complex) C -polarizable representation with finite kernel.

To prove a), note that $\psi(x, gCg^{-1}y) = \psi(g^{-1}x, Cg^{-1}y)$. A sesquilinear form ψ on a complex representation of G is G -invariant if and only if

$$\psi(gx, \bar{g}y) = \psi(x, y) \quad \text{for } g \in G(\mathbb{C}).$$

This condition is equivalent to

$$\psi(gx, CC^{-1}\bar{g}Cy) = \psi(x, Cy) \quad \text{for } g \in G(\mathbb{C})$$

(replace y by Cy), i.e. to the $G^{(\text{ad } C)}$ -invariance of $\psi(x, Cy)$. The “respectively” version of 2.8 therefore follows from 2.6. The non-“respectively” version follows from it:

- a) the complexification (respectively realification) of a real (respectively complex) C -polarizable representation is C -polarizable;
- b) a polarization of V induces a polarization of every subrepresentation of V ;
- c) for V real (respectively complex), one has $V \hookrightarrow V \otimes_{\mathbb{R}} \mathbb{C}$.

2.9. Let G be a reductive group over $\mathbb{Q}$, and let t and w be morphisms defined over $\mathbb{Q}$

$$\mathbb{G}_m \xrightarrow{w} G \xrightarrow{t} \mathbb{G}_m,$$

with $tw(x) = x^{-2}$ and w central. Suppose also given a commutative diagram

$$\begin{array}{ccccc} \mathbb{G}_{m,\mathbb{R}} & \xrightarrow{w} & \underline{S} & \xrightarrow{t} & \mathbb{G}_{m,\mathbb{R}} \\ \parallel & & \downarrow h & & \parallel \\ \mathbb{G}_{m,\mathbb{R}} & \xrightarrow{w} & G_{\mathbb{R}} & \xrightarrow{t} & \mathbb{G}_{m,\mathbb{R}}. \end{array}$$

A representation defined over $\mathbb{Q}$, $V_{\mathbb{Q}}$, of G is called *homogeneous of weight n* if $w(x) * v = x^n \cdot v$ ($v \in V$). Every representation is a sum of homogeneous representations. Let $V_{\mathbb{Q}}$ be a representation of weight n . Then h endows $V_{\mathbb{Q}}$ with a rational Hodge structure of weight n .

We shall regard $\mathbb{Q}(n)_{\mathbb{Q}}$ as a representation of G by the rule

$$g * x = t(g)^n \cdot x.$$

This convention is reasonable, since $\underline{S}$ acts on $\mathbb{Q}(n)_{\mathbb{R}}$ by the same formula. A *polarization* of a weight- n representation $V_{\mathbb{Q}}$ of G is a G -invariant form

$$\psi : V_{\mathbb{Q}} \otimes V_{\mathbb{Q}} \longrightarrow \mathbb{Q}(n)$$

such that $\psi(x, h(i)y)$ is a symmetric positive-definite form on $V_{\mathbb{R}}$. Being G -invariant, a polarization ψ is also $\underline{S}$ -invariant and hence a polarization of the rational Hodge structure $V_{\mathbb{Q}}$.

2.10. Lemma. *A representation $V_{\mathbb{Q}}$ is polarizable if and only if the representation $V_{\mathbb{R}}$ is $h(i)$ -polarizable.*

Let $P_{\mathbb{Q}}$ (resp. $P_{\mathbb{R}}$) be the space of G -invariant $(-1)^n$ -symmetric bilinear forms on $V_{\mathbb{Q}}$ (resp. $V_{\mathbb{R}}$). One has $P_{\mathbb{R}} = \mathbb{R} \otimes P_{\mathbb{Q}}$. The $h(i)$ -polarizations form an open subset of $P_{\mathbb{R}}$, and the polarizations of $V_{\mathbb{Q}}$ are the intersection of this open subset with $P_{\mathbb{Q}}$. The lemma follows.

2.11. Proposition. *a) Whether ψ is a polarization of V depends only on the $G(\mathbb{R})$ -conjugacy class of h .*

b) The following conditions are equivalent:

- (i) *$\text{ad } h(i)$ is a Cartan involution of $\text{Ker}(t)_{\mathbb{R}}$;*
- (ii) *every homogeneous representation of G is polarizable;*
- (iii) *G has a faithful family of polarizable homogeneous representations.*

If G is connected, these conditions are equivalent to

- (iii') *G has a polarizable representation ρ such that $\text{Ker}(\rho) \cap \text{Ker}(t)$ is finite.*

Assertion a) follows from 2.8.a). If G is connected, then $\text{Ker}(t)$ is connected: otherwise t would be of the form t_0^n , with $n > 1$. Since $twx = x^{-2}$, one would have $n = 2$ and $w(-1) \notin \text{Ker}(t)^0$. This is absurd, since

$$w(-1) \in h(\text{Ker}(t : \underline{S} \longrightarrow \mathbb{G}_{m, \mathbb{R}})),$$

which is connected. With this established, b) follows from 2.10 and 2.8.b).

3. Review of the Spinor Group

For the material recalled in this section, see [3].

3.1. Let V be a free module over a ring A , equipped with a quadratic form Q . The module V is identified with a submodule of the Clifford algebra $C(V)$; in $C(V)$ one has

$$v \cdot v = Q(v).$$

We denote by $C^+(V)$ the even part of $C(V)$.

If V is a free module over A , equipped with a symmetric bilinear form ψ , we denote by $C(V)$ (or by $C(V, \psi)$) the Clifford algebra of V equipped with the quadratic form $\psi(x, x)$.

3.2. Suppose that A is a field of characteristic 0, and that Q is nondegenerate. We denote by $\text{CSpin}(V)$ the *Clifford group*. It is the algebraic group of invertible elements g of $C^+(V)$ such that $gVg^{-1} = V$. We define a morphism from $\text{CSpin}(V)$ to $\text{SO}(V)$ by

$$g \mapsto (v \mapsto gvg^{-1}).$$

The kernel of this morphism consists only of the scalars: there is an exact sequence of algebraic groups

$$0 \longrightarrow \mathbb{G}_m \xrightarrow{w} \text{CSpin}(V) \longrightarrow \text{SO}(V) \longrightarrow 0.$$

The spin group $\text{Spin}(V)$ is the algebraic subgroup of $\text{CSpin}(V)$ that is the kernel of the spinor norm

$$N : \text{CSpin}(V) \longrightarrow \mathbb{G}_m.$$

Let t denote the inverse of N ; there is a commutative diagram

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 & & & \text{Spin}(V) & & & \\
 & & & \downarrow & \searrow & & \\
 0 & \longrightarrow & \mathbb{G}_m & \xrightarrow{w} & \text{CSpin}(V) & \longrightarrow & \text{SO}(V) \longrightarrow 0 \\
 & & \searrow & & \downarrow & & \\
 & & x \mapsto x^{-2} & & t & & \\
 & & & & \mathbb{G}_m & & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array} \tag{3.2.1}$$

By definition, one also has

$$\alpha : \text{CSpin}(V) \hookrightarrow C^+(V)^*. \tag{3.2.2}$$

3.3. We denote by $(C^+(V))_s$ the representation of $\text{CSpin}(V)$ on $C^+(V)$ given by

$$g * x = \alpha(g) \cdot x. \tag{3.3.1}$$

We denote by $(C^+(V))_{\text{ad}}$ the representation of $\text{CSpin}(V)$ on $C^+(V)$ obtained from the action of $\text{SO}(V)$ on $C^+(V)$ by transport of structure. One has

$$g *_{\text{ad}} x = \alpha(g) \cdot x \cdot \alpha(g)^{-1}. \quad (3.3.2)$$

The action of $\text{CSpin}(V)$ on $(C^+(V))_s$ is compatible with the right $C^+(V)$ -module structure of $(C^+(V))_s$. There is an isomorphism of representations

$$\text{End}_{C^+(V)}((C^+(V))_s) = (C^+(V))_{\text{ad}}. \quad (3.3.3)$$

There is an isomorphism of representations

$$(C^+(V))_{\text{ad}} = \bigoplus \bigwedge^{2i} V. \quad (3.3.4)$$

3.4. Suppose that A is algebraically closed. We distinguish two cases.

1) $\dim(V)$ *odd*: $C^+(V)$ is here a matrix algebra. Let W be a simple $C^+(V)$ -module; W is a *spin representation* of $\text{CSpin}(V)$:

$$g * W = \alpha(g) \cdot W.$$

There are isomorphisms of representations

$$(C^+(V))_s = \text{a sum of copies of } W, \quad (3.4.1)$$

$$(C^+(V))_{\text{ad}} = \text{End}_A(W). \quad (3.4.2)$$

2) $\dim(V)$ *even*: $C^+(V)$ is here a product of two matrix algebras. Let W_1 and W_2 be two nonisomorphic simple $C^+(V)$ -modules, and let $W = W_1 + W_2$; W_1 and W_2 are the *half-spin representations*.

There are isomorphisms of representations

$$(C^+(V))_s = \text{a sum of copies of } W, \quad (3.4.3)$$

$$(C^+(V))_{\text{ad}} = \text{End}_A(W_1) \times \text{End}_A(W_2). \quad (3.4.4)$$

In both cases, $C^+(V)$ is the bicommutant of the representation W .

3.5. Proposition. *Let Γ be a Zariski-dense subgroup of $\text{Spin}(V)$ and let a be an automorphism of the A -algebra $C^+(V)$ that commutes with the action of Γ given by $\gamma \mapsto \alpha(\gamma) \cdot x \cdot \alpha(\gamma)^{-1}$. Then a is the identity.*

It is enough to treat the case in which A is an algebraically closed field and Γ is the whole spin group (indeed, commuting with a is a Zariski-closed condition).

With the notation of 3.4, let us identify $C^+(V)$ with a subalgebra of $\text{End}_A(W)$.

There is an automorphism $\bar{a}$ of W such that

$$a(x) = \bar{a}x\bar{a}^{-1} \quad (x \in C^+(V)).$$

For $\gamma \in \text{Spin}(V)$, by hypothesis one has

$$\alpha(\gamma)\bar{a}\alpha(\gamma)^{-1}\bar{a}^{-1} \cdot x \cdot \bar{a}\alpha(\gamma)\bar{a}^{-1}\alpha(\gamma)^{-1} = x \quad (x \in C^+(V));$$

the commutator $(\alpha(\gamma), \bar{a}) = \alpha(\gamma)\bar{a}\alpha(\gamma)^{-1}\bar{a}^{-1}$ therefore lies in the center of $C^+(V)$.

1) $\dim(V)$ *odd*: here $C^+(V) = \text{End}_A(W)$, and $\det_W((\alpha(\gamma), \bar{a})) = 1$, so that $\alpha(\gamma)\bar{a}\alpha(\gamma)^{-1}\bar{a}^{-1}$ is a root of unity ν . Since $\text{Spin}(V)$ is connected, and since for $\gamma = e$ one has $\nu = 1$, one even has

$$(\alpha(\gamma), \bar{a}) = 1, \quad \text{i.e.} \quad \alpha(\gamma)\bar{a} = \bar{a}\alpha(\gamma).$$

2) $\dim(V)$ *even*: here $C^+(V) = \text{End}_A(W_1) \times \text{End}_A(W_2)$, and $\bar{a}$ either permutes the W_i or preserves them. In both cases, from

$$\det_{W_i}(\alpha(\gamma)) = 1 \quad (i = 1, 2)$$

one deduces that

$$\det_{W_i}((\alpha(\gamma), \bar{a})) = 1 \quad (i = 1, 2)$$

and one concludes as above that

$$\alpha(\gamma)\bar{a} = \bar{a}\alpha(\gamma).$$

In both cases, $\bar{a}$ lies in the commutant of the representation, and therefore commutes with the bicommutant $C^+(V)$: $\bar{a}$ is the identity.

4. Construction of abelian varieties

In this paragraph and part of the next, we present results of [7].

4.1. Let $V_{\mathbb{Z}}$ be a free $\mathbb{Z}$ -module of rank $n + 2$ endowed with a bilinear form of nonzero discriminant

$$B : V_{\mathbb{Z}} \otimes V_{\mathbb{Z}} \longrightarrow \mathbb{Z}.$$

Assume that B has index $(n+, 2-)$. We are interested in weight-0 morphisms $h : \underline{S} \rightarrow \text{SO}(V_{\mathbb{R}})$ such that

- 1) B is a polarization of the representation $V_{\mathbb{Q}}$ of $\text{SO}(V_{\mathbb{Q}})$;
- 2) the Hodge numbers of $V_{\mathbb{C}}$ are

$$h^{-1,1} = h^{1,-1} = 1 \quad \text{and} \quad h^{0,0} = n.$$

4.2. A morphism h as above lifts uniquely to $\tilde{h} : \underline{S} \rightarrow \text{CSpin}(V_{\mathbb{R}})$, making the diagram

$$\begin{array}{ccccc} \mathbb{G}_{m,\mathbb{R}} & \xrightarrow{w} & \underline{S} & \xrightarrow{t} & \mathbb{G}_{m,\mathbb{R}} \\ \parallel & & \downarrow \tilde{h} & & \parallel \\ \mathbb{G}_{m,\mathbb{R}} & \longrightarrow & \text{CSpin}(V_{\mathbb{R}}) & \xrightarrow{t} & \mathbb{G}_{m,\mathbb{R}}. \end{array}$$

commutative.

4.3. Lemma. *With respect to $\tilde{h}$, every representation of $\mathrm{CSpin}(V_{\mathbb{Q}})$ is polarizable (2.9).*

We apply criterion 2.11. b)(iii') to the representation $V_{\mathbb{Q}}$ of $\mathrm{CSpin}(V_{\mathbb{Q}})$.

4.4. Lemma. *The Hodge structure on $C^+(V_{\mathbb{Q}})_{\mathrm{ad}}$ defined by $\tilde{h}$ is of type $\{(-1, 1), (0, 0), (1, -1)\}$.*

We have

$$C^+(V_{\mathbb{Q}})_{\mathrm{ad}} \simeq \bigoplus \bigwedge^{2i} V_{\mathbb{Q}},$$

and the conclusion follows from $h^{-1,1}(V_{\mathbb{Q}}) = 1$.

4.5. Proposition. *The Hodge structure on $C^+(V_{\mathbb{Q}})_s$ defined by $\tilde{h}$ is of type $\{(0, 1), (1, 0)\}$. Thus the Hodge structure $C^+(V_{\mathbb{Q}})_s$, endowed with the lattice $C^+(V_{\mathbb{Z}})$, defines an abelian variety.*

First suppose that n is odd. After extension of scalars to $\mathbb{C}$, $C^+(V_{\mathbb{Q}})$ becomes a matrix algebra

$$C^+(V_{\mathbb{C}}) \simeq \mathrm{End}(W).$$

On W , $\mathrm{CSpin}(V_{\mathbb{Q}})_{\mathbb{C}}$ acts by $g \cdot w = \alpha(g) \cdot w$: this is the spin representation. The representation $C^+(V_{\mathbb{C}})_s$ is a sum of $\dim(W)$ copies of W , so it is enough to prove that the weight-one representation W is purely of type $\{(0, 1), (1, 0)\}$. Otherwise, the representation $\mathrm{End}(W) \simeq C^+(V_{\mathbb{C}})_{\mathrm{ad}}$ could not be purely of type $\{(-1, 1), (0, 0), (1, -1)\}$.

For n even, similarly,

$$C^+(V_{\mathbb{C}}) \simeq \mathrm{End}(W') \oplus \mathrm{End}(W''),$$

where W' and W'' are the half-spin representations. As above, one proves that W' and W'' are of type $\{(0, 1), (1, 0)\}$, and hence so is $C^+(V_{\mathbb{C}})_s$.

The second assertion follows from 2.3 and 3.3.

4.6. The construction given in this paragraph conceals the following fact (which we shall not use), visible from the Dynkin diagram and the real reason why everything works.

Let $V_{\mathbb{Z}}$ be a polarized Hodge structure of weight 0 and type $\{(1, -1), (0, 0), (-1, 1)\}$, with $h^{1,-1} = 0$. Let

$$r : \mathbb{G}_m \longrightarrow \mathrm{SO}(V_{\mathbb{C}})$$

be the homomorphism such that $r(x) \cdot v^{pq} = x^p v^{pq}$ for v^{pq} of type (p, q) . Let T be a maximal torus of $\mathrm{SO}(V_{\mathbb{C}})$ equipped with a system of simple roots Φ , let $\tilde{T}$ be the inverse image of T in $\mathrm{Spin}(V_{\mathbb{C}})$, and let $(w_{\alpha})_{\alpha \in \Phi}$ be the fundamental weights of $\tilde{T}$. The homomorphism r has a unique conjugate

$$r_0 : \mathbb{G}_m \longrightarrow T$$

such that the rational numbers $\langle w_\alpha, r_0 \rangle \in \frac{1}{2}\mathbb{Z}$ are all positive. One checks that r_0 is the homomorphism $H_{\alpha_1} : \mathbb{G}_m \rightarrow T$ corresponding to the root denoted below by α_1 . The numbers $\langle w_\alpha, r_0 \rangle$ are as follows:

$$\begin{array}{c}
\begin{array}{ccccccc}
1 & & 1 & & 1 & & \dots & & 1 & & 1 & & \frac{1}{2} \\
| & & | & & | & & & & | & & | & & | \\
\hline
\alpha_1 & & \alpha_2 & & \alpha_3 & & & & \alpha_{\ell-2} & & \alpha_{\ell-1} & & \alpha_\ell
\end{array} & \text{for } \mathrm{SO}(2, 2\ell - 1), \\
\\
\begin{array}{ccccccc}
1 & & 1 & & 1 & & \dots & & 1 & & 1 & & \frac{1}{2} \\
| & & | & & | & & & & | & & | & & | \\
\hline
\alpha_1 & & \alpha_2 & & \alpha_3 & & & & \alpha_{\ell-3} & & \alpha_{\ell-2} & & \alpha_{\ell-1} \\
& & & & & & & & & & & & \frac{1}{2} \\
& & & & & & & & & & & & \alpha_\ell
\end{array} & \text{for } \mathrm{SO}(2, 2\ell - 2).
\end{array}$$

Let w be a dominant weight of a spin or half-spin representation, and let σ be the opposition involution. The point is that

$$\langle w, r_0 \rangle + \langle \sigma w, r_0 \rangle = 1.$$

5. Construction of Families of Abelian Varieties

We shall use the following theorem of A. Borel (to appear).

5.1. Theorem (A. Borel). *Let X/Γ be the quotient of a Hermitian symmetric domain by a torsion-free arithmetic group. It is known [2] that X/Γ is naturally a quasi-projective algebraic variety. Let S be a reduced scheme over $\mathbb{C}$ and let $f : S^{\mathrm{an}} \rightarrow X/\Gamma$ be a morphism of analytic spaces. Then f is algebraic.*

Let S be smooth over $\mathbb{C}$. For the notion of a variation of Hodge structures $\underline{H}$ on S , and of a polarization of $\underline{H}$, we refer to Griffiths [6]. We write $\underline{H}_{\mathbb{Z}}$ for the local system of free $\mathbb{Z}$ -modules on S defined by $\underline{H}$, and $\underline{H}_s$ for the fiber Hodge structure of $\underline{H}$ at $s \in S$.

For X the Siegel upper half-space, Theorem 5.1 has the following corollary, which generalizes 2.3.

5.2. Scholium. *The functor $(a : A \rightarrow S) \mapsto R^1 a_* \mathbb{Z}$ identifies polarized abelian schemes A over S with polarized variations of Hodge structures of type $\{(0, 1), (1, 0)\}$ on S .*

5.3. In the category of variations of Hodge structures on S , all the usual tensor operations make sense. Thus

a) if $\underline{H}$ and $\underline{H}'$ are variations of Hodge structures, there are variations of Hodge structures

$$\underline{H} \otimes \underline{H}', \quad \underline{\mathrm{Hom}}(\underline{H}, \underline{H}'), \quad \bigwedge^n \underline{H}, \quad \underline{H}^\vee, \quad \underline{\mathrm{End}}(\underline{H}), \dots$$

- b) if $\underline{H}$ is a variation of Hodge structures of weight 0, and ψ is a polarization of $\underline{H}$ (or any symmetric morphism of Hodge structures $\underline{H} \otimes \underline{H} \rightarrow \mathbb{Z}(0)$), then $C(\underline{H}, \psi)$ and $C^+(\underline{H}, \psi)$ are defined. At every point $t \in S$, with the notation of 3.4, one has

$$C^+(\underline{H}, \psi)_t = C^+(\underline{H}_t, \psi)_{\text{ad}}.$$

5.4. Let $(V_{\mathbb{Z}}, B)$ be as in 4.1, and let X be the set of $h : \underline{S} \rightarrow \text{SO}(V_{\mathbb{R}})$ of the type considered in 4.1. Giving such an h is equivalent to giving a two-dimensional subspace $V_{\mathbb{R}}^-$ of $V_{\mathbb{R}}$, on which B is negative definite, together with an orientation of $V_{\mathbb{R}}^-$: for $z = \tau \cdot e^{i\theta} \in \underline{S}(\mathbb{R}) = \mathbb{C}^*$, $h(z)$ induces the identity on the orthogonal complement of $V_{\mathbb{R}}^-$ and the rotation through angle 2θ on $V_{\mathbb{R}}^-$. The group $\text{O}(V_{\mathbb{R}})$ acts on X by transport of structure: $(g * h)(z) = g \cdot h(z) \cdot g^{-1}$, and the description above makes it geometrically evident that X is a homogeneous space under $\text{SO}(V_{\mathbb{R}})$. The stabilizer of $h \in X$ is a subgroup $\text{SO}(2) \times \text{SO}(n)$, the identity component of a *maximal* compact subgroup.

5.5. For $h \in X$, let F_h be the corresponding Hodge filtration of $V_{\mathbb{C}}$. The function $h \mapsto F_h^1$ identifies X with an open subset of the space of isotropic lines in $V_{\mathbb{C}}$. This construction endows X with a complex structure for which F_h is a holomorphic function of h . It is known that, for this structure, the (two) connected components of X are Hermitian symmetric domains.

5.6. Let $W_{\mathbb{Q}}$ be a weight- n representation of the algebraic group $\text{CSpin}(V_{\mathbb{Q}})$: $w(x) * y = x^n \cdot y$. Let $\underline{W}_{\mathbb{Q}}$ be the constant local system on X with fiber $W_{\mathbb{Q}}$. The group $\text{CSpin}(V_{\mathbb{Q}})$ acts on $(X, \underline{W}_{\mathbb{Q}})$ by

$$\gamma * (w \text{ at } h \in X) = (\gamma w \text{ at } \gamma h \gamma^{-1} \in X).$$

Each $h \in X$ defines $\tilde{h} : \underline{S} \rightarrow \text{CSpin}(V_{\mathbb{R}})$, hence a $\mathbb{Q}$ -Hodge structure on the fiber of $\underline{W}_{\mathbb{Q}}$ at $h \in X$. Thus one obtains a variation of $\mathbb{Q}$ -Hodge structures of weight n on X , again denoted $\underline{W}_{\mathbb{Q}}$:

- a) the Hodge filtration is a holomorphic function of h ;
- b) the transversality axiom is satisfied (we shall use it only in a trivial case).

This construction is compatible with tensor products. In particular, a polarization $\psi : W_{\mathbb{Q}} \otimes W_{\mathbb{Q}} \rightarrow \mathbb{Q}(-n)$ of the representation $W_{\mathbb{Q}}$ defines a polarization of $\underline{W}_{\mathbb{Q}}$.

Let $W_{\mathbb{Z}}$ be an integral lattice in $W_{\mathbb{Q}}$ and let Γ be a subgroup of $\text{CSpin}(V_{\mathbb{Q}})$, discrete in $\text{CSpin}(V_{\mathbb{R}})$, acting freely on X and satisfying $\Gamma W_{\mathbb{Z}} = W_{\mathbb{Z}}$. Then the constant local system $\underline{W}_{\mathbb{Z}}$ on X , with fiber $W_{\mathbb{Z}}$, underlies a Γ -equivariant variation of Hodge structure $\underline{W}$ on X as above. Passing to the quotient, it defines a variation of Hodge structures $\underline{W}$ on X/Γ .

If $W_{\mathbb{Q}}$ is a representation of $\mathrm{SO}(V_{\mathbb{Q}})$, the same construction applies to discrete subgroups of $\mathrm{SO}(V_{\mathbb{Q}})$.

5.7. Proposition. *Let $(\underline{H}, \psi)$ be a polarized variation of Hodge structures of type $\{(-1, 1), (0, 0), (1, -1)\}$, with $h^{1, -1} = 1$, on a smooth connected scheme S . There exist*

- a) *a finite surjective étale cover $u : S_1 \rightarrow S$;*
- b) *an abelian scheme A over S_1 ;*
- c) *a $\mathbb{Z}$ -algebra C , and $\mu : C \hookrightarrow \mathrm{End}_{S_1}(A)$;*
- d) *an isomorphism of variations of Hodge structures*

$$u : C^+(\underline{H}, \psi) \xrightarrow{\sim} \underline{\mathrm{End}}_C(R^1 a_* \mathbb{Z})$$

which induces an isomorphism of local systems of algebras

$$u_{\mathbb{Z}} : C^+(\underline{H}_{\mathbb{Z}}, \psi) \xrightarrow{\sim} \underline{\mathrm{End}}_C(R^1 a_* \mathbb{Z}).$$

Let $(V_{\mathbb{Z}}, B)$ be a $\mathbb{Z}$ -module equipped with a symmetric bilinear form, isomorphic to $(H_{s\mathbb{Z}}, \psi)$, and retain the notation of 5.4. Let n be an integer,

$$\Gamma = \{\gamma \in \mathrm{SO}(V_{\mathbb{Z}}) \mid \gamma \equiv 1 \pmod{n}\}$$

and

$$\tilde{\Gamma} = \{\gamma \in \mathrm{CSpin}(V_{\mathbb{Q}}) \mid \alpha(\gamma) \equiv 1 \pmod{n} \text{ in } C^+(V_{\mathbb{Z}})\}.$$

Assume n is large enough that Γ and $\tilde{\Gamma}$ are torsion-free.

There is a finite étale cover $v : S'_1 \rightarrow S$ such that the local system $v^* \underline{H}_{\mathbb{Z}/n\mathbb{Z}}$ is trivial. Choose an isomorphism σ_n between this local system and the constant local system $\underline{V}_{\mathbb{Z}/n\mathbb{Z}}$. Let $s \in S'_1$ and let $\sigma : V_{\mathbb{Z}} \xrightarrow{\sim} H_{s\mathbb{Z}}$ be an isometry such that $\sigma \equiv \sigma_n \pmod{n}$ (we choose σ_n so that such σ exist). The Hodge structure h_s on H_s defines $\sigma^{-1}(h_s) \in X$. The image of $\sigma^{-1}(h_s)$ in X/Γ depends only on s . We thus define

$$m : S'_1 \longrightarrow X/\Gamma$$

and an isomorphism compatible with the polarizations

$$\underline{H} \simeq m^* \underline{V}.$$

Let

$$S_1 = S'_1 \times_{X/\Gamma} X/\tilde{\Gamma} :$$

$$\begin{array}{ccccc} & & S_1 & & \\ & p_1 \swarrow & & \searrow p_2 & \\ S & \xleftarrow{v} & S'_1 & & X/\tilde{\Gamma} \\ & & \searrow m & \swarrow & \\ & & X/\Gamma & & \end{array}$$

On S_1 , one has

- a) an isomorphism $(vp_1)^*H \simeq (mp_1)^*\underline{V}$;
- b) a variation of Hodge structure, the pullback by p_2 of the one on $X/\tilde{\Gamma}$ defined by the representation $(C^+(V_{\mathbb{Q}}))_s$ and the lattice $C^+(V_{\mathbb{Z}})$.

By 5.2, 5.6 and 4.5, this variation of Hodge structures corresponds to an abelian scheme $a : A \rightarrow S_1$.

c) Let C be the algebra $C^+(V_{\mathbb{Z}})$. The abelian scheme constructed in b) has complex multiplication by C , and the variation of Hodge structures $\underline{\text{End}}_C(R^1a_*\mathbb{Z})$ is

$$p_2^*((C^+(V_{\mathbb{Z}}))_{\text{ad}}).$$

Combining this with a), one obtains an isomorphism of local systems of algebras and of variations of Hodge structures

$$\underline{C}^+(vp_1^*H) \simeq \underline{\text{End}}_C(R^1a_*\mathbb{Z}).$$

This proves 5.7.

6. K3 Surfaces

6.1. A polarized K3 surface over a field K of characteristic 0 is a K3 surface over K , say Y , endowed with an element $\eta \in \text{NS}(Y) = \text{Pic}(Y)$ which is the class of an ample invertible sheaf. For $K = \mathbb{C}$, we shall identify $\text{NS}(Y)$ with a subset of $H^2(Y, \mathbb{Z})$.

6.2. A family of K3 surfaces parametrized by a scheme S of characteristic 0 is a proper flat morphism $f : Y \rightarrow S$ whose fibers are K3 surfaces. A polarization of $f : Y \rightarrow S$ is a section η of $\text{Pic}_S(Y)$ which, for every $s \in S$, is a polarization of $Y_s = f^{-1}(s)$.

For every prime number ℓ , we identify η with a section of $R^2f_*\mathbb{Z}_{\ell}$; we denote by $P^2f_*\mathbb{Z}_{\ell}$ the orthogonal complement of η (for the cup product); it is a subsheaf of $R^2f_*\mathbb{Z}_{\ell}$ in $\mathbb{Z}_{\ell}$ -modules. The cup product

$$R^2f_*\mathbb{Z}_{\ell} \otimes R^2f_*\mathbb{Z}_{\ell} \longrightarrow R^4f_*\mathbb{Z}_{\ell}$$

and the trace morphism

$$R^4f_*\mathbb{Z}_{\ell} \xrightarrow{\sim} \mathbb{Z}_{\ell}(-2)$$

induce

$$\psi_{\ell} : P^2f_*\mathbb{Z}_{\ell}(1) \otimes P^2f_*\mathbb{Z}_{\ell}(1) \longrightarrow \mathbb{Z}_{\ell}. \quad (6.2.1)$$

6.3. Take S to be a scheme of finite type over $\mathbb{C}$. Then $R^2f_*\mathbb{Z}$ is a variation of Hodge structures on S ; η is everywhere of type $(1, 1)$ and its orthogonal complement $P^2f_*\mathbb{Z}$ is again a variation of Hodge structures. As above, the cup product defines

$$\psi : P^2f_*\mathbb{Z}(1) \otimes P^2f_*\mathbb{Z}(1) \longrightarrow \mathbb{Z}. \quad (6.3.1)$$

This time, the negative of ψ is a polarization of the variation of Hodge structures $P^2 f_* \mathbb{Z}(1)$. Moreover, ψ_ℓ is obtained from ψ via the isomorphism

$$(P^2 f_* \mathbb{Z}(1)_{\mathbb{Z}}) \otimes \mathbb{Z}_\ell \simeq P^2 f_* \mathbb{Z}_\ell(1).$$

For every $s \in S$, the fiber at s , $(P^2 f_* \mathbb{Z}(1))_s$, is the primitive part of the cohomology $H^2(Y_s, \mathbb{Z})$. The Hodge structure on $(P^2 f_* \mathbb{Z}(1))_s$ is of type $\{(-1, 1), (0, 0), (1, -1)\}$, with $h^{-1,1} = 1$ ((1.1)(i)).

The action of $\pi_1(S, s)$ on $H^2(Y_s, \mathbb{Z})$ induces

$$\pi : \pi_1(S, s) \longrightarrow \mathrm{O}(P^2(Y_s, \mathbb{Z})(1)_{\mathbb{Z}}, \psi). \quad (6.3.2)$$

6.4. Proposition. *For every complex polarized K3 surface Y_0 , there exists a family of polarized K3 surfaces $f : Y \rightarrow S$ as above, with $Y_s \simeq Y_0$ for some $s \in S$, such that the image of π in (6.3.2) has finite index.*

Let $f_T^\wedge : Y_T^\wedge \rightarrow T^\wedge$ be the versal formal deformation of Y_0 . Since $H^0(Y_0, T^1) = 0$, $T^\wedge$ is even a formal moduli scheme for Y_0 , but this is immaterial. Since $H^2(Y_0, T^1) = 0$ ((1.1)(i)), $T^\wedge$ is the spectrum of a formal power-series ring over $\mathbb{C}$. From $f_T^\wedge$ one obtains a (non-polarized) variation of Hodge structures on $T^\wedge$. By (1.1.1), this is the universal deformation of {the Hodge structure $H^2(Y_0, \mathbb{Z})$, endowed with the cup product}.

Let $\bar{\eta}$ be the image of η under the composite map

$$H^2(Y_0, \mathbb{Z}) \longrightarrow H_{\mathrm{DR}}^2(Y_T^\wedge / T^\wedge) \longrightarrow R^2 f_{T*}^\wedge \mathcal{O}.$$

Let $S^\wedge$ be the formal subscheme of $T^\wedge$ with equation $\bar{\eta} = 0$, and let $f^\wedge : Y^\wedge \rightarrow S^\wedge$ be induced by $f_T^\wedge$. Since $h^{0,2} = 1$, $S^\wedge \subset T^\wedge$ is defined by one equation; this equation is part of a regular system of parameters, so $S^\wedge$ is a power-series ring. On $S^\wedge$, η comes from an ample invertible sheaf $\mathcal{O}(1)$ on $Y^\wedge$; hence $f^\wedge$ is algebraizable. Moreover, on $S^\wedge$, the polarized variation of Hodge structure $(P^2(Y^\wedge / S^\wedge, \mathbb{Z})(1), \psi)$ is a universal deformation of $(P^2(Y_0, \mathbb{Z})(1), \psi)$.

Put $S^\wedge = \mathrm{Spec}(A^\wedge)$, and express $A^\wedge$ as an inductive limit of rings of finite type over $\mathbb{C}$. Since $Y^\wedge$ is algebraizable, there is a Cartesian diagram

$$\begin{array}{ccc} Y^\wedge & \longrightarrow & Y \\ f^\wedge \downarrow & & \downarrow f \\ S^\wedge & \longrightarrow & S \end{array}$$

with S of finite type over $\mathbb{C}$ and Y a polarized family of K3 surfaces over S (we do not assume that $S^\wedge$ is a completion of S). We may take S normal and connected; let $s \in S$ be the image of the closed point of $S^\wedge$.