

HODGE THEORY, II

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Introduction

According to Hodge, the cohomology space $H^n(X, \mathbb{C})$ of a compact Kähler variety X is endowed with a *Hodge structure* of weight n , that is, with a natural bigrading

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}$$

satisfying $\overline{H^{p,q}} = H^{q,p}$. We show here that the complex cohomology of a nonsingular algebraic variety, not necessarily compact, is endowed with a somewhat more general kind of structure, which presents $H^n(X, \mathbb{C})$ as a successive extension of Hodge structures of decreasing weights, lying between $2n$ and n , whose Hodge numbers $h^{p,q} = \dim H^{p,q}$ are zero for $p > n$ or $q > n$.

The reader will find in [14] an exposition of the yoga underlying this construction.

The proof, essentially algebraic, rests on the one hand on Hodge theory and on the other hand on Hironaka's resolution of singularities, which permits one, via a spectral sequence, to *express* the cohomology of a nonsingular quasi-projective algebraic variety in terms of the cohomology of nonsingular projective varieties.

Section 1 contains, besides a few elementary facts about filtrations collected for the reader's convenience, two key results:

- a) Theorem (1.2.10), which will be used only through its corollary (2.3.5), giving the fundamental properties of mixed Hodge structures;
- b) the *lemma on two filtrations* (1.3.16).

Section 2 recalls Hodge theory and introduces mixed Hodge structures. The heart of the paper is no. 3.2, which defines the mixed Hodge structure on $H^n(X, \mathbb{C})$ and establishes certain degeneracies of spectral sequences.

Section 4 gives various applications, all deduced from (4.1.1) and from the theory of the (K/k) -trace for Hodge structures which follows from it (4.1.2). The principal ones are (4.2.6) and (4.4.15).

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1. Filtrations

1.1. Filtered objects

(1.1.1) Let $\mathcal{A}$ be an abelian category. We shall have to consider filtrations of type $\mathbb{Z}$, usually finite, on the objects of $\mathcal{A}$.

Definition (1.1.2). A decreasing, respectively increasing, filtration F of an object A of $\mathcal{A}$ is a family $(F^n(A))_{n \in \mathbb{Z}}$, respectively $(F_n(A))_{n \in \mathbb{Z}}$, of subobjects of A , satisfying

$$\forall n, m \quad n \leq m \Rightarrow F^n(A) \supset F^m(A)$$

(resp. $\forall n, m \quad n \leq m \Rightarrow F_n(A) \subset F_m(A)$).

A filtered object is an object endowed with a filtration. When no confusion is possible, the same letter will often denote filtrations on different objects of $\mathcal{A}$.

If F is a decreasing, respectively increasing, filtration on A , put $F^\infty(A) = 0$ and $F^{-\infty}(A) = A$ (resp. $F_{-\infty}(A) = 0$ and $F_\infty(A) = A$).

The shifted filtrations of a decreasing filtration W are defined by

$$W[n]^p(A) = W^{n+p}(A).$$

(1.1.3) If F is a decreasing, respectively increasing, filtration of A , then the $F_n(A) = F^{-n}(A)$, respectively the $F^n(A) = F_{-n}(A)$, form an increasing, respectively decreasing, filtration of A . Thus in principle it is enough to consider decreasing filtrations; unless explicitly stated otherwise, the word *filtration* will henceforth mean *decreasing filtration*.

(1.1.4) A filtration F of A is called finite if there exist n and m such that $F^n(A) = A$ and $F^m(A) = 0$.

(1.1.5) A morphism from a filtered object (A, F) to a filtered object (B, F) is a morphism f from A to B satisfying

$$f(F^n(A)) \subset F^n(B) \quad \text{for } n \in \mathbb{Z}.$$

The filtered objects (respectively, the objects with finite filtrations) of $\mathcal{A}$ form an additive category in which finite inductive and projective limits exist (and hence so do the kernels, cokernels, images, and coimages of a morphism).

A morphism $f : (A, F) \rightarrow (B, F)$ is called strict, or strictly compatible with the filtrations, if the canonical arrow from $\text{Coim}(f)$ to $\text{Im}(f)$ is an isomorphism of filtered objects (cf. (1.1.11)).

(1.1.6) Let $^\circ$ be the contravariant identity functor from $\mathcal{A}$ to the dual category $\mathcal{A}^\circ$. If (A, F) is a filtered object of $\mathcal{A}$, the objects $(A/F^{1-n}(A))^\circ$ are identified with subobjects of A° . The filtration on A° dual to F is defined by

$$F^n(A^\circ) = (A/F^{1-n}(A))^\circ.$$

The bidual of (A, F) is identified with (A, F) . This construction identifies the dual of the category of filtered objects of $\mathcal{A}$ with the category of filtered objects of $\mathcal{A}^\circ$.

(1.1.7) If (A, F) is a filtered object of $\mathcal{A}$, its associated graded object is the object of $\mathcal{A}^\mathbb{Z}$ defined by

$$\text{Gr}^n(A) = F^n(A)/F^{n+1}(A).$$

The convention (1.1.6) is justified by the simple formula

$$\text{Gr}^n(A^\circ) = \text{Gr}^{-n}(A)^\circ,$$

which is verified on the autodual diagram

$$\begin{array}{ccccc}
 A/F^n(A) & \longleftarrow & A/F^{n+1}(A) & & 0 \\
 & \swarrow & \uparrow & \nwarrow & \uparrow \\
 & & A & & \text{Gr}^n(A) \\
 & \nearrow & \uparrow & \nearrow & \nwarrow \\
 F^{n+1}(A) & \longrightarrow & F^n(A) & & 0
 \end{array} \tag{1.1.7.1}$$

(1.1.8) Let (A, F) be a filtered object and let $j : X \hookrightarrow A$ be a subobject of A . The filtration induced on X by F is the unique filtration on X such that j is strictly compatible with the filtrations; one has

$$F^n(X) = j^{-1}(F^n(A)) = X \cap F^n(A).$$

Dually, the quotient filtration on A/X , the unique filtration such that $p : A \rightarrow A/X$ is strictly compatible with the filtrations, is given by

$$F^n(A/X) = p(F^n(A)) \simeq (X + F^n(A))/X \simeq F^n(A)/(X \cap F^n(A)).$$

Lemma (1.1.9). If X and Y are two subobjects of A , with $X \subset Y$, then on $Y/X \simeq \text{Ker}(A/X \rightarrow A/Y)$ the quotient filtration from Y coincides with the filtration induced from A/X . In the diagram

$$\begin{array}{ccccc}
 A/Y & \longleftarrow & A/X & & \\
 & \swarrow & \nearrow & \nwarrow & \swarrow \\
 & & A & & Y/X \\
 & \nearrow & \nwarrow & \nearrow & \\
 X & \longrightarrow & Y & &
 \end{array}$$

the arrows are strict.

(1.1.10) The filtration of (1.1.9) on Y/X will be called the filtration induced by that of A . By (1.1.9), its definition is autodual.

In particular, if $\Sigma : A \xrightarrow{f} B \xrightarrow{g} C$ is a 0-sequence and if B is filtered, then

$$H(\Sigma) = \text{Ker}(g)/\text{Im}(f) = \text{Ker}(\text{Coker}(f) \rightarrow \text{Coim}(g))$$

is endowed with a canonical induced filtration.

The reader will verify the following.

Proposition (1.1.11).

- (i) Let $f : (A, F) \rightarrow (B, F)$ be a morphism of finitely filtered objects. For f to be strict, it is necessary and sufficient that the sequence

$$0 \rightarrow \text{Gr}(\text{Ker}(f)) \rightarrow \text{Gr}(A) \rightarrow \text{Gr}(B) \rightarrow \text{Gr}(\text{Coker}(f)) \rightarrow 0$$

be exact.

- (ii) Let $\Sigma : (A, F) \rightarrow (B, F) \rightarrow (C, F)$ be a 0-sequence of strict morphisms. Then one has canonically

$$H(\text{Gr}(\Sigma)) \simeq \text{Gr}(H(\Sigma)).$$

In particular, if Σ is exact in $\mathcal{A}$, then $\text{Gr}(\Sigma)$ is exact in $\mathcal{A}^{\mathbb{Z}}$.

In a category of modules, to say that a morphism $f : (A, F) \rightarrow (B, F)$ is strict means that every $b \in B$ of filtration $\geq n$ ($b \in F^n(B)$) which is in the image of A is already in the image of $F^n(A)$:

$$f(F^n(A)) = f(A) \cap F^n(B).$$

(1.1.12) If $\otimes : \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \rightarrow \mathcal{B}$ is a multiadditive right exact functor, and if A_i is a finitely filtered object of $\mathcal{A}_i$ ($1 \leq i \leq n$), one defines a filtration on $\bigotimes_{i=1}^n A_i$ by

$$F^k \left(\bigotimes_{i=1}^n A_i \right) = \sum_{\sum k_i = k} \text{Im} \left(\bigotimes_{i=1}^n F^{k_i}(A_i) \rightarrow \bigotimes_{i=1}^n A_i \right)$$

(sum of subobjects).

Dually, if H is left exact, put

$$F^k(H(A_i)) = \bigcap_{\sum k_i = k} \text{Ker} (H(A_i) \rightarrow H(A_i/F^{k_i}(A_i))).$$

For exact H , the two definitions are equivalent. These definitions are extended to functors contravariant in certain variables by (1.1.6). In particular, for the left exact functor Hom , one puts

$$F^k(\text{Hom}(A, B)) = \{f : A \rightarrow B \mid \forall n, f(F^n(A)) \subset F^{n+k}(B)\}.$$

Thus

$$\text{Hom}((A, F), (B, F)) = F^0(\text{Hom}(A, B)).$$

Under the preceding hypotheses, one has evident morphisms

$$\bigotimes_{i=1}^n \text{Gr}(A_i) \rightarrow \text{Gr} \left(\bigotimes_{i=1}^n A_i \right)$$

and

$$\text{Gr } H(A_i) \rightarrow H(\text{Gr}(A_i)).$$

For exact H , these are mutually inverse isomorphisms.

These constructions are compatible with the composition of functors, in a sense which we leave to the reader not to make explicit.

1.2. Opposite filtrations

(1.2.1) Let A be an object of $\mathcal{A}$ endowed with two filtrations F and G . By definition, $\text{Gr}_F^m(A)$ is a quotient of a subobject of A , and as such is endowed with a filtration induced by G (1.1.10). Passing to the associated graded object, one defines a bigraded object $(\text{Gr}_G^n \text{Gr}_F^m(A))_{n,m \in \mathbb{Z}}$. By a lemma of Zassenhaus, $\text{Gr}_G^n \text{Gr}_F^m(A)$ and $\text{Gr}_F^m \text{Gr}_G^n(A)$ are canonically isomorphic: if one defines the induced filtrations (1.1.10) as quotient filtrations of filtrations induced on a subobject, one has

$$\begin{aligned} \text{Gr}_G^n \text{Gr}_F^m(A) &\simeq \frac{F^m(A) \cap G^n(A)}{(F^{m+1}(A) \cap G^n(A)) + (F^m(A) \cap G^{n+1}(A))} \\ &\quad \parallel \\ \text{Gr}_F^m \text{Gr}_G^n(A) &\simeq \frac{G^n(A) \cap F^m(A)}{(G^{n+1}(A) \cap F^m(A)) + (G^n(A) \cap F^{m+1}(A))}. \end{aligned}$$

(1.2.2) Let H be a third filtration of A . It induces a filtration on $\text{Gr}_F(A)$, and hence on $\text{Gr}_G \text{Gr}_F(A)$. It also induces a filtration on $\text{Gr}_F \text{Gr}_G(A)$. One should note that these filtrations do not in general correspond under the isomorphism (1.2.1). In the expression $\text{Gr}_H \text{Gr}_G \text{Gr}_F(A)$, therefore, G and H play symmetric roles, but F and G do not.

Definition (1.2.3). Two finite filtrations F and $\bar{F}$ on A are called n -opposite if

$$\text{Gr}_F^p \text{Gr}_{\bar{F}}^q(A) = 0 \quad \text{for } p + q \neq n.$$

(1.2.4) If $A^{p,q}$ is a bigraded object of $\mathcal{A}$ such that:

- a) $A^{p,q} = 0$ except for finitely many pairs (p, q) ;
- b) $A^{p,q} = 0$ for $p + q \neq n$,

then one defines two finite n -opposite filtrations of $A = \sum_{p,q} A^{p,q}$ by putting

$$F^p(A) = \sum_{p' \geq p} A^{p',q'}, \quad (1.2.4.1)$$

$$\overline{F}^q(A) = \sum_{q' \geq q} A^{p',q'}. \quad (1.2.4.2)$$

One has

$$\mathrm{Gr}_F^p \mathrm{Gr}_{\overline{F}}^q(A) = A^{p,q}. \quad (1.2.4.3)$$

Conversely:

Proposition (1.2.5).

- (i) Let F and $\overline{F}$ be two finite filtrations on A . For F and $\overline{F}$ to be n -opposite, it is necessary and sufficient that

$$\forall p, q \quad p + q = n + 1 \Rightarrow F^p(A) \oplus \overline{F}^q(A) \xrightarrow{\sim} A.$$

- (ii) If F and $\overline{F}$ are n -opposite, and if one puts

$$\begin{cases} A^{p,q} = 0 & \text{for } p + q \neq n, \\ A^{p,q} = F^p(A) \cap \overline{F}^q(A) & \text{for } p + q = n, \end{cases}$$

then A is the direct sum of the $A^{p,q}$, and F and $\overline{F}$ are deduced from the bigrading $A^{p,q}$ of A by the procedure (1.2.4).

Proof. (i) The condition $\mathrm{Gr}_F^p \mathrm{Gr}_{\overline{F}}^q(A) = 0$ for $p + q > n$ means that

$$F^p \cap \overline{F}^q = (F^{p+1} \cap \overline{F}^q) + (F^p \cap \overline{F}^{q+1}) \quad (p + q > n).$$

By hypothesis, $F^p \cap \overline{F}^q$ is zero for $p + q$ sufficiently large; by descending induction one deduces that the condition $\mathrm{Gr}_F^p \mathrm{Gr}_{\overline{F}}^q(A) = 0$ for $p + q > n$ is equivalent to the condition $F^p(A) \cap \overline{F}^q(A) = 0$ for $p + q > n$. Dually ((1.1.6), (1.1.7), (1.1.10)) the condition $\mathrm{Gr}_F^p \mathrm{Gr}_{\overline{F}}^q(A) = 0$ for $p + q < n$ is equivalent to $A = F^p(A) + \overline{F}^q(A)$ for $(1 - p) + (1 - q) > -n$, that is, for $p + q \leq n + 1$, and (i) follows.

- (ii) If F and $\overline{F}$ are n -opposite, we prove by descending induction on p that

$$\bigoplus_{p' \geq p} A^{p',q'} \xrightarrow{\sim} F^p(A). \quad (1.2.5.1)$$

- a) For $F^p(A) = 0$, the assertion is evident.

- b) The decomposition $A = F^{p+1}(A) \oplus \overline{F}^{n-p}(A)$ induces on $F^p(A) \supset F^{p+1}(A)$ a decomposition

$$F^p(A) = F^{p+1}(A) \oplus (F^p(A) \cap \overline{F}^{n-p}(A)),$$

and the induction concludes the proof. For p sufficiently small, $F^p(A) = A$. By (1.2.5.1), the $A^{p,q}$ therefore form a bigrading of A , and F satisfies (1.2.4.1). That $\overline{F}$ satisfies (1.2.4.2) follows by symmetry.

(1.2.6) The constructions (1.2.4) and (1.2.5) establish quasi-inverse equivalences of categories between objects of $\mathcal{A}$ endowed with two finite n -opposite filtrations and bigraded objects of $\mathcal{A}$ of the type considered in (1.2.4).

Definition (1.2.7). Three finite filtrations W , F , and $\bar{F}$ on an object A of $\mathcal{A}$ are called opposite if

$$\mathrm{Gr}_F^p \mathrm{Gr}_{\bar{F}}^q \mathrm{Gr}_W^n(A) = 0 \quad \text{for } p + q + n \neq 0.$$

This condition is symmetric in F and $\bar{F}$. It means that F and $\bar{F}$ induce on $W^n(A)/W^{n+1}(A)$ two $(-n)$ -opposite filtrations. Put

$$A^{p,q} = \mathrm{Gr}_F^p \mathrm{Gr}_{\bar{F}}^q \mathrm{Gr}_W^{-p-q}(A),$$

whence decompositions (1.2.4), (1.2.5)

$$W^n(A)/W^{n+1}(A) = \bigoplus_{p+q=-n} A^{p,q} \quad (1.2.7.1)$$

which make $\mathrm{Gr}_W(A)$ into a bigraded object.

Lemma (1.2.8). Let W , F , and $\bar{F}$ be three finite opposite filtrations on A , and let σ be a sequence $(p_i, q_i)_{i \geq 0}$ of pairs of integers satisfying:

- a) $p_i \leq p_j$ and $q_i \leq q_j$ for $i \geq j$;
- b) For $i > 0$, $p_i + q_i = p_0 + q_0 - i + 1$.

Put $p = p_0$, $q = q_0$, $n = -p - q$, and

$$A_\sigma = \left(\sum_{0 \leq i} (W^{n+i}(A) \cap F^{p_i}(A)) \right) \cap \left(\sum_{0 \leq i} (W^{n+i}(A) \cap \bar{F}^{q_i}(A)) \right).$$

Then the projection from $W^n(A)$ to $\mathrm{Gr}_W^n(A)$ induces an isomorphism

$$A_\sigma \xrightarrow{\sim} A^{p,q} \subset \mathrm{Gr}_W^n(A).$$

We prove by induction on k the following assertion:

$(*_k)$ The projection from $W^n(A)/W^{n+k}(A)$ to $\mathrm{Gr}_W^n(A)$ induces an isomorphism from

$$\frac{((\sum_{i < k} (W^{n+i}(A) \cap F^{p_i}(A))) + W^{n+k}(A)) \cap ((\sum_{i < k} (W^{n+i}(A) \cap \bar{F}^{q_i}(A))) + W^{n+k}(A))}{W^{n+k}(A)}$$

to $A^{p,q} \subset \mathrm{Gr}_W^n(A)$.

For $k = 1$, this is exactly the definition of $A^{p,q}$. By (1.2.5)(i), one has

$$F^{p_k}(\mathrm{Gr}_W^{n+k}(A)) \oplus \bar{F}^{q_k}(\mathrm{Gr}_W^{n+k}(A)) \xrightarrow{\sim} \mathrm{Gr}_W^{n+k}(A). \quad (1.2.8.1)$$

Put

$$\begin{aligned} B &= \sum_{i < k} (W^{n+i}(A) \cap F^{p_i}(A)), \\ C &= \sum_{i < k} (W^{n+i}(A) \cap \bar{F}^{q_i}(A)), \\ B' &= (W^{n+k}(A) \cap F^{p_k}(A)) + W^{n+k+1}(A), \\ C' &= (W^{n+k}(A) \cap \bar{F}^{q_k}(A)) + W^{n+k+1}(A), \\ D &= W^{n+k}(A), \\ E &= W^{n+k+1}(A). \end{aligned}$$

Formula (1.2.8.1) becomes

$$B' + C' = D,$$

and

$$B' \cap C' = E.$$

Moreover, since $p_k \leq p_i$ ($i \leq k$),

$$B \cap D \subset F^{p_k}(A) \cap W^{n+k}(A) \subset B'$$

and since $q_k \leq q_i$ ($i \leq k$),

$$C \cap D \subset \overline{F}^{q_k}(A) \cap W^{n+k}(A) \subset C'.$$

The assertion $(*_k)$ therefore follows from $(*_k)$ and from the following lemma.

Lemma (1.2.9). Let B, C, B', C', D , and E be subobjects of A . Assume that

$$\begin{aligned} B' + C' &= D, & B' \cap C' &= E, \\ B \cap D &\subset B', & C \cap D &\subset C'. \end{aligned}$$

Then,

$$((B + B') \cap (C + C'))/E \xrightarrow{\sim} ((B + D) \cap (C + D))/D.$$

To prove surjectivity, one writes

$$\begin{aligned} ((B + B') \cap (C + C')) + D &= (((B + B') \cap (C + C')) + B') + C' \\ &= ((B + B') \cap (C + C' + B')) + C' \\ &= (B + B' + C') \cap (C + C' + B') \\ &= (B + D) \cap (C + D). \end{aligned}$$

To prove injectivity, one writes

$$(B + B') \cap (C + C') \cap D = ((B + B') \cap D) \cap ((C + C') \cap D).$$

Since $B' \subset D$, one has

$$(B + B') \cap D = (B \cap D) + B' = B';$$

similarly,

$$(C + C') \cap D = C',$$

and

$$(B + B') \cap (C + C') \cap D = B' \cap C' = E.$$

Finally, the proof of (1.2.8) is completed by observing that (1.2.8) is equivalent to $(*_k)$ for large k .

Theorem (1.2.10). Let $\mathcal{A}$ be an abelian category, and provisionally denote by $\mathcal{A}'$ the category of objects of $\mathcal{A}$ endowed with three opposite filtrations W, F , and $\overline{F}$. The morphisms in $\mathcal{A}'$ are the morphisms in $\mathcal{A}$ compatible with the three filtrations.

- (i) $\mathcal{A}'$ is an abelian category.
- (ii) The kernel (respectively, the cokernel) of an arrow $f : A \rightarrow B$ in $\mathcal{A}'$ is the kernel (respectively, the cokernel) of f in $\mathcal{A}$, endowed with the filtrations induced from those of A (respectively, the quotient filtrations induced from those of B).
- (iii) Every morphism $f : A \rightarrow B$ in $\mathcal{A}'$ is strictly compatible with the filtrations W, F , and $\overline{F}$; the morphism $\text{Gr}_W(f)$ is compatible with the bigradings of $\text{Gr}_W(A)$ and $\text{Gr}_W(B)$; the morphisms $\text{Gr}_F(f)$ and $\text{Gr}_{\overline{F}}(f)$ are strictly compatible with the filtration induced by W .
- (iv) The functors *forget the filtrations*, Gr_W , Gr_F , $\text{Gr}_{\overline{F}}$, and

$$\begin{aligned} \text{Gr}_W \text{Gr}_F &\simeq \text{Gr}_F \text{Gr}_W \simeq \text{Gr}_{\overline{F}} \text{Gr}_F \text{Gr}_W \\ &\simeq \text{Gr}_{\overline{F}} \text{Gr}_W \simeq \text{Gr}_W \text{Gr}_{\overline{F}} \end{aligned}$$

from $\mathcal{A}'$ to $\mathcal{A}$ are exact.

Let $\sigma_0(p, q)$ and $\sigma_1(p, q)$ denote the sequences

$$\sigma_0(p, q) = (p, q), (p, q), (p, q-1), (p, q-2), (p, q-3), \dots$$

$$\sigma_1(p, q) = (p, q), (p, q), (p-1, q), (p-2, q), (p-3, q), \dots$$

and, with the notation of (1.2.8), put

$$A_i^{p,q} = A_{\sigma_i(p,q)} \quad (i = 0, 1).$$

If $f : A \rightarrow B$ is compatible with W , F , and $\overline{F}$, then

$$f(A_i^{p,q}) \subset B_i^{p,q} \quad (i = 0, 1). \quad (1.2.10.1)$$

Assertion (iii) therefore follows from the following lemma.

Lemma (1.2.11). The $A_i^{p,q}$ form a bigrading of A . One has

$$W^n(A) = \sum_{n+p+q \leq 0} A_i^{p,q} \quad (i = 0, 1), \quad (1.2.11.1)$$

$$F^p(A) = \sum_{p' \geq p} A_0^{p',q'}, \quad (1.2.11.2)$$

$$\overline{F}^q(A) = \sum_{q' \geq q} A_1^{p',q'}. \quad (1.2.11.3)$$

By symmetry, it is enough to prove the assertions for $i = 0$. Put $A_0 = \bigoplus A_0^{p,q}$, and define on A_0 filtrations W and F by the formulae of (1.2.11). The canonical map i from A_0 to A is compatible with the filtrations W and F . Moreover, by (1.2.8), $\text{Gr}_W(i)$ is an isomorphism, and it induces isomorphisms of graded objects

$$\sum_{p+q=n} A_0^{p,q} \xrightarrow{\sim} \text{Gr}_W^{-n}(A) = \sum_{p+q=n} A^{p,q}. \quad (1.2.11.4)$$

The morphism i is therefore an isomorphism, and the $A_0^{p,q}$ form a bigrading of A . Formula (1.2.11.1) then expresses that $\text{Gr}_W(i)$ is an isomorphism. By (1.2.11.4), $\text{Gr}_F \text{Gr}_W(i)$ is an isomorphism, hence so are $\text{Gr}_W \text{Gr}_F(i)$ and $\text{Gr}_F(i)$. Formula (1.2.11.2) follows.

(1.2.12) We prove (1.2.10). Let $f : A \rightarrow B$ be in $\mathcal{A}'$ and endow $K = \text{Ker}(f)$ with the filtrations induced from those of A . By (1.2.11), $\text{Gr}_W(K) \hookrightarrow \text{Gr}_W(A)$; moreover, the filtration F (respectively, $\overline{F}$) of K induces on $\text{Gr}_W(K)$ the inverse-image filtration of the filtration F on $\text{Gr}_W(A)$. Finally, the subobject $\text{Gr}_W(K)$ of $\text{Gr}_W(A)$ is compatible with the bigrading of $\text{Gr}_W(A)$:

$$\text{Gr}_W(K) = \bigoplus_{p,q} (\text{Gr}_W(K) \cap A^{p,q}).$$

It follows that

$$\text{Gr}_F^p \text{Gr}_F^q \text{Gr}_W^n(K) \hookrightarrow \text{Gr}_F^p \text{Gr}_F^q \text{Gr}_W^n(A);$$

the filtrations W , F , and $\overline{F}$ of K are therefore opposite, and K is a kernel of f in $\mathcal{A}'$. Together with the dual result, this proves (ii).

If f is an arrow of $\mathcal{A}'$, the canonical morphism from $\text{Coim}(f)$ to $\text{Im}(f)$ is an isomorphism in $\mathcal{A}$; by (iii), it is also an isomorphism in $\mathcal{A}'$, which is therefore abelian.

The functor *forget the filtrations* is exact by (ii). The exactness of the other functors in (iv) follows at once from (ii), (iii), and (1.1.11)(i) or (ii).

(1.2.13) Let A be an object of $\mathcal{A}$ endowed with a finite increasing filtration W , and with two finite decreasing filtrations F and $\overline{F}$. The construction (1.1.3) associates to $W_\bullet$ a decreasing filtration $W^\bullet$. We shall say that the filtrations $W_\bullet$, F , and $\overline{F}$ are opposite if the filtrations $W^\bullet$, F , and $\overline{F}$ are opposite, that is, if for every n the filtrations induced by F and $\overline{F}$ on

$$\mathrm{Gr}_n^W(A) = W_n(A)/W_{n-1}(A)$$

are n -opposite.

Theorem (1.2.10) carries over trivially to this variant.

1.3. The lemma on two filtrations

(1.3.1) Let K be a differential complex of objects of $\mathcal{A}$, endowed with a filtration F . This filtration is called biregular if it induces on each component of K a finite filtration. Recall the definition of the terms $E_r^{pq}(K, F)$, or simply E_r^{pq} , of the spectral sequence defined by F . Put

$$Z_r^{pq} = \mathrm{Ker} \left(d : F^p(K^{p+q}) \rightarrow K^{p+q+1}/F^{p+r}(K^{p+q+1}) \right),$$

and define B_r^{pq} dually by the formula

$$K^{p+q}/B_r^{pq} = \mathrm{coker} \left(d : F^{p-r+1}(K^{p+q-1}) \rightarrow K^{p+q}/F^{p+1}(K^{p+q}) \right).$$

These formulae still make sense for $r = \infty$.

One should note that the use made here of the notation B_r^{pq} is different from that of Godement [TF].

By definition one has:

$$E_r^{pq} = \mathrm{Im} \left(Z_r^{pq} \rightarrow K^{p+q}/B_r^{pq} \right) \quad (1.3.1.1)$$

$$= Z_r^{pq} / (B_r^{pq} \cap Z_r^{pq}) \quad (1.3.1.2)$$

$$= \mathrm{Ker} \left(K^{p+q}/B_r^{pq} \rightarrow K^{p+q}/(Z_r^{pq} + B_r^{pq}) \right). \quad (1.3.1.3)$$

One can also write

$$\begin{aligned} B_r^{p*} \cap Z_r^{p*} &\stackrel{\mathrm{def}}{=} (dF^{p-r+1} + F^{p+1}) \cap (d^{-1}F^{p+r} \cap F^p) \\ &= (dF^{p-r+1} \cap F^p) + F^{p+1} \cap d^{-1}F^{p+r}, \end{aligned} \quad (1.3.1.4)$$

since $dF^{p-r+1} \subset d^{-1}F^{p+r}$ and $F^{p+1} \subset F^p$.

For $r < \infty$, the E_r form a complex graded by the degree $p - r(p+q)$, and E_{r+1} is expressed as the cohomology of this complex:

$$E_{r+1}^{p,q} = H \left(E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1} \right). \quad (1.3.1.5)$$

For $r = 0$ one has

$$E_0^{**} = \mathrm{Gr}_F^*(K^*). \quad (1.3.1.6)$$

Proposition (1.3.2). Let K be a complex endowed with a biregular filtration F . The following conditions are equivalent:

- (i) the spectral sequence defined by F degenerates ($E_1 = E_\infty$).
- (ii) the morphisms $d : K^i \rightarrow K^{i+1}$ are strictly compatible with the filtrations.

We check this when $\mathcal{A}$ is a category of modules. For fixed p and q , the hypothesis that the arrows d_r with source $E_r^{p,q}$ are zero for $r \geq 1$ means that, if $x \in F^p(K^{p+q})$ satisfies $dx \in F^{p+1}(K^{p+q+1})$, then there exists y in K^{p+q} such that $dy = 0$ and such that x and y have the same image in $E_1^{p,q}$. Modifying y by a boundary and putting $z = x - y$, one obtains

$$\forall x \in F^p(K^{p+q}) \left(dx \in F^{p+1}(K^{p+q+1}) \Rightarrow \exists z \in F^{p+1}(K^{p+q}), dz = dx \right)$$

or, in other words,

$$F^{p+1}(K^{p+q+1}) \cap dF^p(K^{p+q}) = dF^{p+1}(K^{p+q}). \quad (1)$$

If this condition holds for all p and q , then by induction on r one has

$$F^{p+r} \cap dF^p = dF^{p+r},$$

which, for $p+r$ large, becomes

$$F^p \cap dK = dF^p. \quad (2)$$

Assertion (2) trivially implies (i), and is equivalent to (ii). This proves (1.3.2).

(1.3.3) If (K, F) is a filtered complex, we shall denote by $\text{Dec}(K)$ the complex K endowed with the shifted filtration

$$\text{Dec}(F)^p K^n = Z_1^{p+n, -p}.$$

This filtration is compatible with the differentials:

$$dZ_1^{p+n, -p} \subset F^{p+n+1}(K^{n+1}) \cap \text{Ker}(d) \subset Z_\infty^{p+n+1, -p} \subset Z_1^{p+n+1, -p}.$$

Since

$$Z_1^{p+1+n, -p-1} \subset F^{p+1+n}(K^n) \subset B_1^{p+n, -p} \subset Z_1^{p+n, -p}, \quad (1.3.3.1)$$

the evident map from $Z_1^{p+n, -p}/Z_1^{p+1+n, -p-1}$ to $Z_1^{p+n, -p}/B_1^{p+n, -p}$ is a morphism

$$u : E_0^{p, n-p}(\text{Dec } K) \longrightarrow E_1^{p+n, -p}(K). \quad (1.3.3.2)$$

Proposition (1.3.4).

- (i) The morphisms (1.3.3.2) form a morphism of graded complexes from $E_0(\text{Dec } K)$ to $E_1(K)$.
- (ii) This morphism induces an isomorphism on cohomology.
- (iii) Step by step, via (1.3.1.5), it induces isomorphisms of graded complexes

$$E_r(\text{Dec}(K)) \xrightarrow{\sim} E_{r+1}(K) \quad (r \geq 1).$$

Proof. Let F' be the filtration of K defined by

$$F'^p(K^n) = \text{Dec}(F)^{p-n}(K^n) = Z_1^{p, n-p}.$$

There are trivial isomorphisms, compatible with the d_r and with (1.3.1.5),

$$E_r^{p, n-p}(\text{Dec } K) = E_{r+1}^{p+n, -p}(K, F'). \quad (1.3.4.1)$$

The map u is deduced from (1.3.4.1) and from the identity morphism

$$(K, F') \longrightarrow (K, F).$$

This proves (i), and it remains to verify that for $r \geq 2$,

$$E_r^{p, q}(K, F') \simeq E_r^{p, q}(K, F).$$

Indeed one has

$$Z_r^{p, q}(K, F') = Z_r^{p, q}(K, F) \quad (r \geq 1),$$

and

$$Z_r^{p, q}(K, F') \cap B_r^{p, q}(K, F') = Z_r^{p, q}(K, F) \cap B_r^{p, q}(K, F) \quad (r \geq 2),$$

and then applies (1.3.1.2).

(1.3.5) The construction (1.3.3) is not self-dual. The dual construction consists in defining

$$\text{Dec}^*(F)^p K^n = B_1^{p+n-1, -p+1}.$$

One then has morphisms

$$E_0^{p, n-p}(\text{Dec } K) \longrightarrow E_1^{p+n, p}(K) \longrightarrow E_0^{p, n-p}(\text{Dec}^* K)$$

and, for $r \geq 1$, isomorphisms

$$E_r^{p, n-p}(\text{Dec } K) \xrightarrow{\sim} E_{r+1}^{p+n, p}(K) \xrightarrow{\sim} E_r^{p, n-p}(\text{Dec}^* K).$$

Recall that a morphism of complexes is called a quasi-isomorphism if it induces an isomorphism on cohomology.

Definition (1.3.6).

- (i) A morphism $f : (K, F) \rightarrow (K', F')$ of filtered complexes with biregular filtration is a filtered quasi-isomorphism if $\text{Gr}_F(f)$ is a quasi-isomorphism, i.e. if the $E_1^{p,q}(f)$ are isomorphisms.
- (ii) A morphism $f : (K, F, W) \rightarrow (K', F', W')$ between biregular bifiltered complexes is a bifiltered quasi-isomorphism if $\text{Gr}_F \text{Gr}_{W'}(f)$ is a quasi-isomorphism.

(1.3.7) Let K be a differential complex of objects of $\mathcal{A}$, endowed with two filtrations F and W . Let $E_r^{p,q}$ be the spectral sequence defined by W . The filtration F induces several filtrations on the $E_r^{p,q}$, which we shall compare.

(1.3.8) Formula (1.3.1.2) identifies $E_r^{p,q}$ with a quotient of a subobject of K^{p+q} . Thus the term $E_r^{p,q}$ receives a filtration F_d induced by F , called the first direct filtration.

(1.3.9) Dually, formula (1.3.1.3) identifies $E_r^{p,q}$ with a subobject of a quotient of K^{p+q} , and hence gives a new filtration F_{d*} induced by F , the second direct filtration.

Lemma (1.3.10). On E_0 and E_1 one has $F_d = F_{d*}$. For $r = 0$ or 1 , one has $B_r^{p,q} \subset Z_r^{p,q}$, and one applies (1.1.9).

(1.3.11) Formula (1.3.1.5) identifies $E_{r+1}^{p,q}$ with a quotient of a subobject of $E_r^{p,q}$. The recurrent filtration F_r of the $E_r^{p,q}$ is defined by the following conditions:

- (i) on $E_0^{p,q}$, $F_r = F_d = F_{d*}$.
- (ii) on $E_{r+1}^{p,q}$, the recurrent filtration is the one induced by the recurrent filtration of $E_r^{p,q}$.

(1.3.12) The definitions (1.3.8) and (1.3.9) still make sense for $r = \infty$. If the filtration of K is biregular, the direct filtrations of $E_\infty^{p,q}$ coincide with those of $E_r^{p,q} = E_\infty^{p,q}$ for r large enough, and the recurrent filtration of $E_\infty^{p,q}$ is defined by requiring it to coincide with that of $E_r^{p,q}$ for r large enough.

The filtrations F and W each induce a filtration of $H^*(K)$, and $E_\infty^{**} = \text{Gr}_W^*(H^*(K))$. The filtration F of $H^*(K)$ therefore induces a new filtration on $E_\infty^{p,q}$.

Proposition (1.3.13).

- (i) For the first direct filtration, the morphisms d_r are compatible with filtrations. If $E_{r+1}^{p,q}$ is regarded as a quotient of a subobject of $E_r^{p,q}$, then the first direct filtration on $E_{r+1}^{p,q}$ is finer than the filtration F' induced by the first direct filtration on $E_r^{p,q}$; one has $F_d(E_{r+1}^{p,q}) \subset F'(E_{r+1}^{p,q})$.
 - (ii) Dually, the morphisms d_r are compatible with the second direct filtration, and the second direct filtration on $E_{r+1}^{p,q}$ is less fine than the filtration induced by that of $E_r^{p,q}$.
 - (iii) $F_d(E_r^{p,q}) \subset F_r(E_r^{p,q}) \subset F_{d*}(E_r^{p,q})$.
 - (iv) On $E_\infty^{p,q}$, the filtration induced by the filtration F of $H^*(K)$ (1.3.12) is finer than the first direct filtration and less fine than the second.
- (i) is evident, (ii) is its dual, and (iii) follows by induction. The first assertion of (iv) is easy to check, and the second is dual to it.

(1.3.14) Denote by $\text{Dec}(K)$, respectively by $\text{Dec}^*(K)$, the complex K endowed with the filtrations $\text{Dec}(W)$ and F , respectively $\text{Dec}^*(W)$ and F .

It is clear from (1.3.4.1) that the isomorphism (1.3.4) transforms the first direct filtration of $E_r(\text{Dec } K)$ into the first direct filtration of $E_{r+1}(K)$, for $r \geq 1$. The dual isomorphism (1.3.5) transforms the second direct filtration of $E_r(\text{Dec}^* K)$ into the second direct filtration of $E_{r+1}(K)$.

Lemma (1.3.15). If the filtration F is biregular, and if, on the $\text{Gr}_W^p(K)$, the morphisms d are strictly compatible with the filtration induced by F , then:

- (i) the morphism (1.3.3.2) of graded complexes filtered by F

$$u : \text{Gr}_{\text{Dec}(W)}(K) \longrightarrow E_1(K, W)$$

is a filtered quasi-isomorphism;

- (ii) dually, the morphism (1.3.5)

$$u : E_1(K, W) \longrightarrow \text{Gr}_{\text{Dec}^*(W)}(K)$$

is a filtered quasi-isomorphism.

By duality it is enough to prove (i). By (1.3.3) and (1.3.4), the complex $E_1(K, W)$, filtered by F , is a quotient of the filtered complex $\text{Gr}_{\text{Dec}(W)}(K)$. Let U be the filtered kernel complex; it is acyclic by (1.3.4) (ii). The long exact cohomology sequence associated with the exact sequence of complexes

$$0 \rightarrow \text{Gr}_F(U) \rightarrow \text{Gr}_F(\text{Gr}_{\text{Dec}(W)}(K)) \rightarrow \text{Gr}_F(E_1(K, W)) \rightarrow 0$$

shows that u is a filtered quasi-isomorphism if and only if $\text{Gr}_F(U)$ is an acyclic complex. By (1.3.2), and because U is acyclic, this amounts to requiring that the differentials of U be strictly compatible with the filtration F . From (1.3.3.1) one obtains that U is the sum, over p , of the complexes

$$(U^p)^n = B_1^{p+n, -p} / Z_1^{p+1+n, -p-1},$$

endowed with the filtration induced by F .

Each differential d of each of the complexes U^p is included in a commutative diagram of filtered objects of the following type, where, for simplicity, the total or complementary degree has been omitted:

$$\begin{array}{ccccc}
B^p/Z^{p+1} & \xrightarrow{\quad d \quad} & B^{p+1}/Z^{p+1} & & \\
& \searrow & \nearrow & & \\
& \text{Coim}(d) & \xrightarrow{\quad \textcircled{5} \quad} & \text{Im}(d) & \\
& \nearrow & \searrow & & \\
W^{p+1}/Z^{p+1} & \xrightarrow{\quad \textcircled{3} \quad} & B^{p+1}/W^{p+2} & & \\
& \nearrow & \searrow & & \\
& & \textcircled{2} & & \\
W^{p+1}/W^{p+2} & \xrightarrow{\quad \textcircled{1} \quad} & W^{p+1}/W^{p+2} & &
\end{array}$$

By hypothesis, the morphism (1) is strict. Since the square (2) is just the canonical decomposition of (1), the arrow (3) is a filtered isomorphism. The arrows of the trapezium (4) are isomorphisms; hence they are filtered isomorphisms, because (3) is one. That (5) is a filtered isomorphism means that d is strict. This proves (1.3.15).

Theorem (1.3.16). Let K be a complex endowed with two filtrations W and F , with F biregular. Let $r_0 \geq 0$ be an integer, and suppose that, for $0 \leq r < r_0$, the differentials of the graded complex $E_r(K, W)$ are strictly compatible with the filtration F_r . Then, for $r \leq r_0 + 1$, one has

$$F_d = F_r = F_{d*} \quad \text{on } E_r^{p,q}.$$

The theorem is proved by induction on r_0 . For $r_0 = 0$ the hypothesis is empty, and one applies (1.3.10) and (1.3.13), (iii). For $r_0 \geq 1$, by the induction hypothesis, one has $F_d = F_r = F_{d*}$ on $E_r^{p,q}$ for $r \leq r_0$.

By (1.3.15), the morphism $u : E_0(\text{Dec } K) \rightarrow E_1(K)$ is a filtered quasi-isomorphism. It therefore induces a filtered isomorphism from $H^*(\text{Dec } K)$ to $H^*(E_1(K))$:

$$u : (E_1(\text{Dec } K), F_r) \xrightarrow{\sim} (E_2(K), F_r).$$

Step by step, it follows that the canonical isomorphism from $E_s(\text{Dec } K)$ to $E_{s+1}(K)$ ($s \geq 1$) is a filtered isomorphism for the recurrent filtration. On $E_1(\text{Dec}(K))$, $F_r = F_d$ (1.3.10), and we already know (1.3.14) that u' is a filtered isomorphism

$$u' : (E_1(\text{Dec } K), F_d) \xrightarrow{\sim} (E_2(K), F_d).$$

Thus on $E_2(K)$ one has $F_d = F_r$. Together with the dual result, this proves (1.3.16) for $r_0 = 1$.

Assume now that $r_0 \geq 2$. Then the arrows d_1 of $E_1(K)$ are strictly compatible with the filtrations; hence so are the arrows d_0 of $E_0(\text{Dec } K)$ (indeed, u induces an isomorphism of spectral sequences, and one applies the criterion (1.3.2)).

For $0 < s < r_0 - 1$, the isomorphism $(E_s(\text{Dec } K), F_r) \simeq (E_{s+1}(K), F_r)$ shows that the d_s are strictly compatible with the recurrent filtrations. By the induction hypothesis one therefore has $F_d = F_r$ on $E_s(\text{Dec } K)$ for $s \leq r_0$. The isomorphism $(E_s(\text{Dec } K), F_d) \simeq (E_{s+1}(K), F_d)$ (1.3.13) then shows that $F_d = F_r$ on $E_r(K)$ for $r \leq r_0 + 1$. Together with the dual result, this proves (1.3.16).

Corollary (1.3.17). Under the general hypotheses of (1.3.16), suppose that, for every r , the differentials d_r are strictly compatible with the recurrent filtrations of the E_r . Then, on E_∞ , the filtrations F_d , F_r , and F_{d*} coincide, and coincide with the filtration induced by the filtration F of $H^*(K)$.

This follows at once from (1.3.16) and (1.3.13) (iv).

1.4. Hypercohomology of filtered complexes.

In this section we recall some standard constructions in hypercohomology. We do not use the language of derived categories, which would be the most natural here.

Throughout this section, by a “complex” we shall mean a “complex bounded below”.

(1.4.1) Let T be a left exact functor from an abelian category $\mathcal{A}$ to an abelian category $\mathcal{B}$. Assume that every object of $\mathcal{A}$ injects into an injective object; the derived functors $R^iT : \mathcal{A} \rightarrow \mathcal{B}$ are therefore defined. An object A of $\mathcal{A}$ will be called *acyclic* for T if $R^iT(A) = 0$ for $i > 0$.

(1.4.2) Let (A, F) be a filtered object with a finite filtration, and let TF be the filtration of TA by its subobjects $TF^p A$ (these are subobjects because T is left exact). If $\text{Gr}_F(A)$ is T -acyclic, then the $F^p(A)$ are T -acyclic as successive extensions of T -acyclic objects. The image under T of the sequence

$$0 \rightarrow F^{p+1}(A) \rightarrow F^p(A) \rightarrow \text{Gr}^p(A) \rightarrow 0$$

is therefore exact, and

$$\text{Gr}_{TF} TA \xrightarrow{\sim} T \text{Gr}_F A. \quad (1.4.2.1)$$

(1.4.3) Let A be an object with two finite filtrations F and W such that $\text{Gr}_F \text{Gr}_W A$ is T -acyclic. The objects $\text{Gr}_F A$ and $\text{Gr}_W A$ are then T -acyclic, as are the $F^q(A) \cap W^p(A)$. The sequences

$$0 \rightarrow T(F^q \cap W^{p+1}) \rightarrow T(F^q \cap W^p) \rightarrow T((F^q \cap W^p)/(F^q \cap W^{p+1})) \rightarrow 0$$

are therefore exact, and $T(F^q(\text{Gr}_W^p(A)))$ is the image in $T(\text{Gr}_W^p(A))$ of $T(F^p \cap W^q)$. The diagram

$$\begin{array}{ccccc} T(F^q \cap W^p) & \longrightarrow & T(F^q \text{Gr}_W^p A) & \longrightarrow & T \text{Gr}_W^p A \\ \downarrow & & & & \downarrow \\ TF^q \cap TW^p & \longrightarrow & & \longrightarrow & \text{Gr}_{TW}^p TA \end{array}$$

then shows that the isomorphism (1.4.2.1) relative to W carries the filtration $\text{Gr}_{TW}(TF)$ to the filtration $T(\text{Gr}_W(F))$.

(1.4.4) Let K be a complex of objects of $\mathcal{A}$. The *hypercohomology objects* $R^iT(K)$ are computed as follows:

- a) Choose a quasi-isomorphism $i : K \rightarrow K'$ such that the components of K' are acyclic for T . For example, one may take for K' the simple complex associated with a Cartan–Eilenberg injective resolution of K .
- b) Put

$$R^iT(K) = H^i(T(K')).$$

One checks that $R^iT(K)$ does not depend on the choice of K' , depends functorially on K , and that a quasi-isomorphism $f : K_1 \rightarrow K_2$ induces *isomorphisms*

$$R^iT(f) : R^iT(K_1) \rightarrow R^iT(K_2).$$

(1.4.5) Let F be a biregular filtration of K . A *filtered T -acyclic resolution* of K is a filtered quasi-isomorphism $i : K \rightarrow K'$ from K to a biregular filtered complex such that the $\mathrm{Gr}^p(K'^n)$ are acyclic for T . If K' is such a resolution, the K'^n are acyclic for T and the filtered complex (cf. (1.4.2)) $T(K')$ defines a spectral sequence

$$E_1^{p,q} = R^{p+q}T(\mathrm{Gr}^p K) \Rightarrow R^{p+q}T(K).$$

This is independent of the choice of K' . It is called the *hypercohomology spectral sequence of the filtered complex* K . It depends functorially on K , and a filtered quasi-isomorphism induces an isomorphism of spectral sequences.

The differentials d_1 of this spectral sequence are the connecting morphisms defined by the short exact sequences

$$0 \rightarrow \mathrm{Gr}^{p+1} K \rightarrow F^p K / F^{p+2} K \rightarrow \mathrm{Gr}^p K \rightarrow 0.$$

(1.4.6) Let K be a complex. Denote by $\tau_{\leq p}(K)$ the following subcomplex:

$$\tau_{\leq p}(K)^n = \begin{cases} K^n, & n < p, \\ \mathrm{Ker}(d), & n = p, \\ 0, & n > p. \end{cases}$$

The *filtration*, called *canonical*, of K by the $\tau_{\leq p}(K)$ is obtained by shifting the trivial filtration G for which $G^0(K) = K$ and $G^1(K) = 0$. For the canonical filtration one has

$$\begin{aligned} E_1^{p,q} &= 0 & \text{if } p+q \neq -p, \\ H^{-p} & & \text{if } p+q = -p. \end{aligned}$$

A quasi-isomorphism $f : K \rightarrow K'$ is automatically a filtered quasi-isomorphism for the canonical filtrations.

(1.4.7) The subcomplexes $\sigma_{\geq p}(K)$ of K :

$$\sigma_{\geq p}(K)^n = \begin{cases} 0, & n < p, \\ K^n, & n \geq p, \end{cases}$$

define a biregular filtration, the *stupid filtration* of K .

The hypercohomology spectral sequences attached to the stupid or canonical filtrations of K are the two *hypercohomology spectral sequences* of K .

Example (1.4.8). — Let $f : X \rightarrow Y$ be a continuous map between topological spaces, and let $\mathcal{F}$ be an abelian sheaf on X . Let $\mathcal{F}^*$ be a resolution of $\mathcal{F}$ by f_* -acyclic sheaves. Then $R^i f_* \mathcal{F} \simeq \mathcal{H}^i(f_* \mathcal{F}^*)$. Take as T the functor $\Gamma(Y, \cdot)$. The hypercohomology spectral sequence of the complex $f_* \mathcal{F}^*$, endowed with its canonical filtration (1.4.6),

$$E_1^{p,q} = H^{2p+q}(Y, R^{-p} f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

is, up to the renumbering $E_r^{p,q} \mapsto E_{r+1}^{2p+q, -p}$, the *Leray spectral sequence* for f and $\mathcal{F}$.

(1.4.9) Let (K, W, F) be a biregular bifiltered complex. To this complex one associates:

a) A spectral sequence

$${}_W E_1^{p,n-p} = H^n(\mathrm{Gr}_W^p(K)) \Rightarrow H^n(K),$$

whose differentials ${}_W d_1$ are the connecting morphisms deduced from the short exact sequences

$$0 \rightarrow \mathrm{Gr}_W^{p+1}(K) \rightarrow W^p(K)/W^{p+2}(K) \rightarrow \mathrm{Gr}_W^p(K) \rightarrow 0;$$

b) An analogous spectral sequence for the filtration F ;

c) Exact squares

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Gr}_F^{p+1} \mathrm{Gr}_W^{q+1} K & \longrightarrow & F^p/F^{p+2}(\mathrm{Gr}_W^{q+1} K) & \longrightarrow & \mathrm{Gr}_F^p \mathrm{Gr}_W^{q+1} K \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Gr}_F^{p+1}(W^q/W^{q+2}(K)) & \longrightarrow & F^p/F^{p+2}(W^q/W^{q+2}(K)) & \longrightarrow & \mathrm{Gr}_F^p(W^q/W^{q+2}(K)) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Gr}_F^{p+1} \mathrm{Gr}_W^q K & \longrightarrow & F^p/F^{p+2} \mathrm{Gr}_W^q K & \longrightarrow & \mathrm{Gr}_F^p \mathrm{Gr}_W^q K \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

The exterior rows and columns of this square define connecting morphisms

$${}_{F,W} d_1 : H^n \mathrm{Gr}_F^p \mathrm{Gr}_W^q K \longrightarrow H^{n+1} \mathrm{Gr}_F^{p+1} \mathrm{Gr}_W^q K,$$

$${}_{W,F} d_1 : H^n \mathrm{Gr}_F^p \mathrm{Gr}_W^q K \longrightarrow H^{n+1} \mathrm{Gr}_F^p \mathrm{Gr}_W^{q+1} K.$$

These morphisms satisfy

$${}_{FW} d_1 \circ {}_{WF} d_1 + {}_{WF} d_1 \circ {}_{FW} d_1 = 0,$$

$${}_{FW} d_1^2 = 0, \quad {}_{WF} d_1^2 = 0.$$

The morphisms ${}_{FW} d_1$ are the d_1 morphisms of the spectral sequences $E^{(q)}$, with term $E_1^{(q)p,n-p}$ equal to

$$E_1^{p,q,n-p-q} \stackrel{\text{dfn}}{=} H^n(\mathrm{Gr}_F^p \mathrm{Gr}_W^q K) \Rightarrow H^n(\mathrm{Gr}_W^q K) = {}_W E_1^{q,n-q}, \quad (1.4.9.1)$$

defined by the filtered complex $\mathrm{Gr}_W^q(K)$. This spectral sequence abuts to the filtration induced by F on $H^* \mathrm{Gr}_W^q K$. Similarly, the ${}_{WF} d_1$ are the d_1 of spectral sequences with the same initial terms

$$E_1^{p,q,n-p-q} = H^n(\mathrm{Gr}_F^p \mathrm{Gr}_W^q K) \Rightarrow H^n(\mathrm{Gr}_F^p K). \quad (1.4.9.2)$$

$$\begin{array}{ccccc} & & H^n \mathrm{Gr}_W^q K & & \\ & \nearrow (F, {}_W d_1) & & \searrow ({}_W d_1) & \\ H^n \mathrm{Gr}_F^p \mathrm{Gr}_W^q K & & & & H^n K \\ & \searrow ({}_W, {}_F d_1) & & \nearrow ({}_F d_1) & \\ & & H^n \mathrm{Gr}_F^p K & & \end{array}$$

These constructions are symmetric in F and W , *via* the isomorphism

$$\mathrm{Gr}_F^p \mathrm{Gr}_W^q \simeq \mathrm{Gr}_W^q \mathrm{Gr}_F^p.$$

(1.4.10) One may also interpret the ${}_{W,F}d_1$ as the initial morphisms of a morphism of spectral sequences (1.4.9.2), abutting to ${}_Wd_1$. Indeed, let C^q be the cone of the morphism

$$W^q(K)/W^{q+2}(K) \longrightarrow \mathrm{Gr}_W^q(K).$$

In the diagram

$$\Sigma : \mathrm{Gr}_W^{q+1}(K)[1] \xrightarrow{u} C^q \xleftarrow{i} \mathrm{Gr}_W^q(K),$$

u is a quasi-isomorphism, and one has

$${}_Wd_1 = H(u)^{-1} \circ H(i).$$

In fact, u is even a filtered quasi-isomorphism (for F), and the preceding construction defines a morphism from the spectral sequence defined by $(\mathrm{Gr}_W^q(K), F)$ to the one defined by $(\mathrm{Gr}_W^{q+1}(K)[1], F)$, a morphism which abuts to ${}_Wd_1$. The initial term of this morphism, deduced from $\mathrm{Gr}_F(\Sigma)$, is precisely ${}_{W,F}d_1$.

(1.4.11) These constructions pass unchanged to hypercohomology. Let K be a complex endowed with two biregular filtrations F and W .

A *bifiltered T -acyclic resolution* of K is a bifiltered quasi-isomorphism $i : K \rightarrow K'$ such that the $\mathrm{Gr}_F^p \mathrm{Gr}_W^n(K'^m)$ are T -acyclic. Such resolutions always exist. In the special case where $\mathcal{A}$ is the category of sheaves of A -modules on a topological space X , and where T is the functor Γ from $\mathcal{A}$ to A -modules, one example of a bifiltered T -acyclic resolution of K is the simple complex associated with the Godement double-resolution complex $\mathcal{C}^*(K)$ of K , filtered by the $\mathcal{C}^*(F^p(K))$ and the $\mathcal{C}^*(W^n(K))$. Since $\mathcal{C}^*$ is exact, one indeed has

$$\mathrm{Gr}_F \mathrm{Gr}_W(\mathcal{C}^*(K)) \simeq \mathcal{C}^*(\mathrm{Gr}_F \mathrm{Gr}_W(K)).$$

No other case will be needed here.

If K' is a bifiltered T -acyclic resolution of K , the complex TK' is filtered by the TF^pK' and the TW^qK' (1.4.3). Moreover, $\mathrm{Gr}_W^n(K')$ is a filtered T -acyclic resolution (for F) of $\mathrm{Gr}_W^n(K)$, $\mathrm{Gr}_F^n(K')$ is a filtered T -acyclic resolution (for W) of $\mathrm{Gr}_F^n(K)$, $\mathrm{Gr}_F^n \mathrm{Gr}_W^m(K')$ is a T -acyclic resolution of $\mathrm{Gr}_F^n \mathrm{Gr}_W^m(K)$, and

$$\begin{aligned} T \mathrm{Gr}_F K' &\simeq \mathrm{Gr}_F TK' && \text{(as a } W\text{-filtered complex)} \\ T \mathrm{Gr}_W K' &\simeq \mathrm{Gr}_W TK' && \text{(as an } F\text{-filtered complex)} \\ T \mathrm{Gr}_F \mathrm{Gr}_W K' &\simeq \mathrm{Gr}_F \mathrm{Gr}_W TK'. \end{aligned}$$

Lemma (1.4.12). — Under the hypotheses of (1.4.11):

(i) The initial terms of the hypercohomology spectral sequences

$${}_WE_1^{q,n-q} = R^n T(\mathrm{Gr}_W^q K) \Rightarrow R^n T(K) \quad (1)$$

$${}_FE_1^{p,n-p} = R^n T(\mathrm{Gr}_F^p K) \Rightarrow R^n T(K) \quad (2)$$

are the abutments of the hypercohomology spectral sequences of the filtered complexes $\mathrm{Gr}_W^q K$ and $\mathrm{Gr}_F^p K$, with E_1 term given by

$$E_1^{p,q,n-p-q} \stackrel{\mathrm{def}}{=} R^n T(\mathrm{Gr}_F^p \mathrm{Gr}_W^q K) \Rightarrow {}_WE_1^{q,n-q} \quad (q \text{ fixed}) \quad (3)$$

$$E_1^{p,q,n-p-q} \stackrel{\mathrm{def}}{=} R^n T(\mathrm{Gr}_F^p \mathrm{Gr}_W^q K) \Rightarrow {}_FE_1^{p,n-p} \quad (p \text{ fixed}). \quad (4)$$

- (ii) The filtration of ${}_WE_1^{p,n-p}$, the abutment of the spectral sequence (3), is the filtration of ${}_WE_1^{p,n-p}(TK')$ induced by the filtration F of TK' .
- (iii) The differentials of the complexes $\mathrm{Gr}_W^n(T(K))$ are strictly compatible with the filtration F if and only if the hypercohomology spectral sequences (3) degenerate at E_1 .
- (iv) The morphisms d_1 of the spectral sequence (4) are the initial terms of degree-one morphisms of spectral sequences (3) abutting to the morphisms d_1 of the spectral sequence (1).

Assertions (i) and (iv) follow from (1.4.9) and (1.4.10) applied to TK' , *via* the isomorphisms (1.4.11). Assertion (ii) is then immediate from the definition of the recurrent filtration F (identical with the direct filtrations by (1.3.10) and (1.3.13) (iii)), and assertion (iii) follows from (1.3.2).

2. Hodge structures

2.1. Pure structures.

(2.1.1) In what follows, $\mathbb{C}$ will denote an algebraic closure of $\mathbb{R}$: we do not assume that a root i of the equation $x^2 + 1 = 0$ has been chosen. The theory will be invariant under complex conjugation (cf. (2.1.14)).

(2.1.2) Let S denote the real algebraic group $\mathbb{C}^*$, obtained by Weil restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$ of the group $\mathbb{G}_m$:

$$S = \prod_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m.$$

$$S(\mathbb{R}) = \mathbb{C}^*.$$

The group S is a torus, i.e. it is connected and of multiplicative type. It is therefore described by the finitely generated free abelian group

$$X(S) = \text{Hom}(S_{\mathbb{C}}, \mathbb{G}_m) = \text{Hom}(S, \mathbb{G}_m)(\mathbb{C})$$

of its complex characters, endowed with the action of $\text{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/(2)$.

The group $X(S)$ has generators z and $\bar{z}$, inducing respectively the identity and complex conjugation:

$$\mathbb{C}^* = S(\mathbb{R}) \longrightarrow S(\mathbb{C}) \longrightarrow \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*.$$

Complex conjugation exchanges z and $\bar{z}$.

(2.1.3) There is a canonical map

$$w : \mathbb{G}_m \longrightarrow S \tag{2.1.3.1}$$

which, on real points, induces the inclusion of $\mathbb{R}^*$ into $\mathbb{C}^*$. One has

$$zw = \bar{z}w = \text{Id}. \tag{2.1.3.2}$$

There is also a map

$$N : S \longrightarrow \mathbb{G}_m \tag{2.1.3.3}$$

which, on real points, is identified with the norm $N_{\mathbb{C}/\mathbb{R}} : \mathbb{C}^* \rightarrow \mathbb{R}^*$. One has

$$N = z\bar{z} \tag{2.1.3.4}$$

and

$$N \circ w = (x \mapsto x^2). \tag{2.1.3.5}$$

Definition (2.1.4). — *A real Hodge structure is a finite-dimensional real vector space V endowed with an action of the real algebraic group S .*

(2.1.5) By the general theory of groups of multiplicative type, giving a real Hodge structure on V is equivalent to giving a bigrading V^{pq} of $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ such that $\overline{V^{pq}} = V^{qp}$. The action of S and the bigrading determine one another *via* the condition:

$$\text{On } V^{pq}, \quad S \text{ acts by multiplication by } z^p \bar{z}^q. \tag{2.1.5.1}$$

(2.1.6) Let $(\mathbb{C}, \times)$ be the multiplicative monoid, and put

$$\bar{S} = \prod_{\mathbb{C}/\mathbb{R}} (\mathbb{C}, \times).$$

If V is a real vector space, giving an action of $\bar{S}$ on V is equivalent to giving on V a bigrading such that $\overline{V^{pq}} = V^{qp}$ and $V^{pq} = 0$ for $p < 0$ or $q < 0$.

(2.1.7) Let V be a real Hodge structure, defined by a representation σ of S and by a bigrading V^{pq} . The grading of $V_{\mathbb{C}}$ by the $V_{\mathbb{C}}^n = \sum_{p+q=n} V^{pq}$ is then defined over $\mathbb{R}$. It is called the *weight grading*. On $V^n = V \cap V_{\mathbb{C}}^n$, the representation σw of $\mathbb{G}_m$ is multiplication by x^n .

We shall say that V is of *weight* n if $V^{pq} = 0$ for $p + q \neq n$, i.e. if σw is multiplication by x^n .

(2.1.8) Let V be a real Hodge structure. The *Hodge filtration* on $V_{\mathbb{C}}$ is defined by

$$F^p(V_{\mathbb{C}}) = \sum_{p' \geq p} V^{p'q'}.$$

By (1.2.6), one has

Proposition (2.1.9). — *Let n be an integer. The construction (2.1.8) establishes an equivalence of categories between:*

- a) *the category of real Hodge structures of weight n ;*
- b) *the category of pairs consisting of a finite-dimensional real vector space V and a filtration F on $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ which is n -opposed to its complex conjugate $\bar{F}$.*

Definition (2.1.10). — *A Hodge structure H of weight n consists of*

- a) *a finitely generated $\mathbb{Z}$ -module $H_{\mathbb{Z}}$ (the “integral lattice”);*
- b) *a real Hodge structure of weight n on $H_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$.*

(2.1.11) A morphism $f : H \rightarrow H'$ is a homomorphism $f : H_{\mathbb{Z}} \rightarrow H'_{\mathbb{Z}}$ such that $f_{\mathbb{R}} : H_{\mathbb{R}} \rightarrow H'_{\mathbb{R}}$ is compatible with the action of S (i.e. such that $f_{\mathbb{C}}$ is compatible with the bigrading, or with the Hodge filtration).

Hodge structures of weight n form an abelian category. If H has weight n and H' has weight n' , one defines a Hodge structure $H \otimes H'$ of weight $n + n'$ by the formulas:

- a) $(H \otimes H')_{\mathbb{Z}} = H_{\mathbb{Z}} \otimes H'_{\mathbb{Z}}$;
- b) the action of S on $(H \otimes H')_{\mathbb{R}} = H_{\mathbb{R}} \otimes H'_{\mathbb{R}}$ is the tensor product of the actions of S on $H_{\mathbb{R}}$ and $H'_{\mathbb{R}}$.

The bigrading (respectively, the Hodge filtration) of $(H \otimes H')_{\mathbb{C}} = H_{\mathbb{C}} \otimes H'_{\mathbb{C}}$ is the tensor product of the bigradings (respectively, of the Hodge filtrations (cf. (1.1.12))) of $H_{\mathbb{C}}$ and $H'_{\mathbb{C}}$.

Similarly one defines the Hodge structure $\text{Hom}(H, H')$ (of weight $n' - n$), the Hodge structures $\bigwedge^p H$ (of weight pn), and the dual Hodge structure H^* .

The preceding internal *Hom*, and the group of homomorphisms, are related by the *Remark (2.1.11.1)*. — $\text{Hom}(H, H')$ is the subgroup of

$$\text{Hom}(H, H')_{\mathbb{Z}} = \text{Hom}_{\mathbb{Z}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}})$$

formed by the elements of type $(0, 0)$.

The actions of S on $H_{\mathbb{R}}$, $H'_{\mathbb{R}}$, and $\text{Hom}(H, H')_{\mathbb{R}} = \text{Hom}(H_{\mathbb{R}}, H'_{\mathbb{R}})$ are indeed related by

$$s(f(x)) = s(f)(s(x)).$$

For f to be of type $(0, 0)$, i.e. invariant under S , therefore means that it commutes with the action of S .

(2.1.12) For A a noetherian subring of $\mathbb{R}$, an *A-Hodge structure of weight n* is defined as consisting of a finite-type A -module H_A and of a real Hodge structure of weight n on $H_{\mathbb{R}} = \mathbb{R} \otimes_A H_A$. These definitions are used mainly for $A = \mathbb{Q}$. An *A-Hodge structure* consists of a finite-type A -module H_A and of a real Hodge structure on $H_{\mathbb{R}} = H_A \otimes_A \mathbb{R}$, such that the weight grading is defined over the field of fractions of A .

Definition (2.1.13). — *The Tate Hodge structure $\mathbb{Z}(1)$ is the Hodge structure of weight -2 , of rank 1, purely of bidegree $(-1, -1)$, with integral lattice $2\pi i\mathbb{Z} \subset \mathbb{C}$.*

The action of S is therefore multiplication by the inverse of the norm (2.1.3.3). For $n \in \mathbb{Z}$, define $\mathbb{Z}(n)$ to be the n -th tensor power of $\mathbb{Z}(1)$: $\mathbb{Z}(n)$ is the Hodge structure of weight $-2n$, rank 1, purely of bidegree $(-n, -n)$, with integral lattice $(2\pi i)^n \mathbb{Z} \subset \mathbb{C}$. The action of S is multiplication by $N(x)^{-n}$.

(2.1.14) The choice in $\mathbb{C}$ of a solution i of the equation $x^2 + 1 = 0$ determines on every complex variety X purely of dimension n an orientation $\text{or}_i(X)$. When i is changed to $-i$, one has

$$\text{or}_{-i}(X) = (-1)^n \text{or}_i(X).$$

The choice of i also defines an element C of order 4 in $S(\mathbb{R})$: the image of i under the isomorphism $S(\mathbb{R}) \simeq \mathbb{C}^*$.

Finally it defines an isomorphism between $\mathbb{Z}$ and the integral lattice of $\mathbb{Z}(n)$: multiplication by $(2\pi i)^n$.

Whenever i , an orientation of X , C , or an identification $\mathbb{Z} \simeq \mathbb{Z}(n)_{\mathbb{Z}}$ occurs in a formula, it will in principle be understood that they are subordinated to the same choice of i , and that changing i to $-i$ would give an equivalent definition or formula.

Definition (2.1.15). — *A polarization of a Hodge structure H of weight n is a homomorphism*

$$(x, y) : H \otimes H \longrightarrow \mathbb{Z}(-n)$$

such that the real bilinear form $(2\pi i)^n(x, Cy)$ on $H_{\mathbb{R}}$ is symmetric and positive definite.

(2.1.16) The *real Tate Hodge structure* is the real Hodge structure $\mathbb{R}(1)$ underlying $\mathbb{Z}(1)$. Similarly one defines $\mathbb{R}(n)$ underlying $\mathbb{Z}(n)$. A polarization of a *real* Hodge structure H of weight n is a homomorphism

$$(x, y) : H \otimes H \longrightarrow \mathbb{R}(-n)$$

such that the real bilinear form $(2\pi i)^n(x, Cy)$ on $H_{\mathbb{R}}$ is symmetric and positive definite. A polarization is entirely determined by the positive-definite quadratic form $(2\pi i)^n(x, Cy)$ on $H_{\mathbb{R}}$, subject only to the condition that it be invariant under the compact subtorus of S , the kernel of N .

One has

$$(x, y) = (Cx, Cy) = (y, C^2x) = (-1)^n(y, x).$$

The form (x, y) is therefore symmetric or alternating according to the parity of n .

(2.1.17) The reader will generalize these definitions to A -Hodge structures of weight n (2.1.12).

2.2. Hodge theory.

(2.2.1) Let X be a compact Kähler variety, for example smooth projective. By the holomorphic Poincaré lemma, the De Rham complex Ω_X^* is a resolution of the constant sheaf $\mathbb{C}$. Thus one has an isomorphism

$$H^*(X, \mathbb{C}) \sim \mathbb{H}^*(X, \Omega_X^*),$$

and the bête filtration of Ω_X^* (1.4.5) defines the hypercohomology spectral sequence

$$(2.2.1.1) \quad E_1^{pq} = H^q(X, \Omega_X^p) \Rightarrow H^{p+q}(X, \mathbb{C}),$$

whose abutment is the *Hodge filtration* of $H^*(X, \mathbb{C})$.

By Hodge theory [9], [12], one has:

(A) The spectral sequence (2.2.1.1) degenerates: $E_1 = E_{\infty}$.

(B) The Hodge filtration of $H^n(X, \mathbb{C})$ is n -opposed to the complex conjugate filtration.

(2.2.2) Let V be a local system of real vector spaces on X , i.e. a sheaf of $\mathbb{R}$ -vector spaces locally isomorphic to a constant sheaf $\mathbb{R}^n$. Suppose that on V there exists a bilinear form

$$Q : V \otimes V \longrightarrow \mathbb{R}$$

which is locally constant and *definite*. For X connected, this is the case if V is defined by a representation of a finite quotient of the fundamental group of X .

The assertions (A) and (B) above remain valid exactly as stated, without any change in the proofs, for cohomology with coefficients in $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$: the spectral sequence

$$E_1^{pq} = H^q(X, \Omega_X^p(V)) \Rightarrow H^{p+q}(X, V_{\mathbb{C}})$$

deduced from the De Rham resolution of $V_{\mathbb{C}}$ by $\Omega_X^*(V_{\mathbb{C}})$ degenerates and abuts to a filtration on $H^n(X, V_{\mathbb{C}})$ which is n -opposed to the complex conjugate filtration.

The vector space $H^n(X, V)$ is therefore endowed with a canonical real Hodge structure of weight n .

(2.2.3) It is shown in [2] that the statements (2.2.1) remain valid for X a complete nonsingular algebraic variety, not necessarily Kähler. The proof in *loc. cit.*, based on a reduction to the projective case *via* Chow's lemma and resolution of singularities, extends to the setting (2.2.2).

(2.2.4) Let $\mathcal{L}$ be an invertible sheaf. Here are two ways to define the class $c_1(\mathcal{L})$.

(2.2.4.1) The sheaf $\mathcal{L}$ defines an element c in $H^1(X, \mathcal{O}^*)$. Its image under $df/f : \mathcal{O}^* \rightarrow \Omega_X^1$ lies in $H^1(X, \Omega_X^1)$. More precisely, df/f defines a morphism of complexes

$$d \log : \mathcal{O}^*[-1] \longrightarrow [0 \rightarrow \Omega_X^1 \rightarrow \Omega_X^2 \rightarrow \cdots] = \sigma_{\geq 1}(\Omega_X^*).$$

This complex maps to Ω_X^* , whence

$$d \log : \mathcal{O}^*[-1] \longrightarrow \Omega_X^*,$$

and the image of c under $d \log$ lies in $H^2(\Omega_X^*)$. This construction would still make sense for an algebraic variety over an arbitrary field k . For $k = \mathbb{C}$, one also has

$$H^2(X, \mathbb{C}) \sim H^2(X, \Omega_X^*)$$

and hence a class

$$c'_1(\mathcal{L}) \in H^2(X, \mathbb{C}).$$

(2.2.4.2) The exponential exact sequence

$$0 \rightarrow \mathbb{Z}(1) \rightarrow \mathcal{O} \rightarrow \mathcal{O}^* \rightarrow 0$$

defines a homomorphism

$$\partial : H^1(X, \mathcal{O}^*) \longrightarrow H^2(X, \mathbb{Z}(1)),$$

hence a class

$$\partial c = c''_1(\mathcal{L}) \in H^2(X, \mathbb{Z}(1)).$$

If i is chosen, this class is identified with

$$c'''_1(\mathcal{L}) \in H^2(X, \mathbb{Z}),$$

and for $\mathcal{L} = \mathcal{O}(D)$, $c'''_1(\mathcal{L})$ is just the integral cohomology class defined by D (the orientations being defined by i).

(2.2.5) We prove that:

(2.2.5.1) For α the natural injection of $\mathbb{Z}(1)$ into $\mathbb{C}$, one has

$$-\alpha c''_1(\mathcal{L}) = c'_1(\mathcal{L}).$$

(2.2.5.2) For β the natural injection of $\mathbb{Z}$ into $\mathbb{C}$, one has

$$-\beta c'''_1(\mathcal{L}) = \frac{1}{2\pi i} c'_1(\mathcal{L}).$$

This is verified by contemplating the following diagram of complexes, in which $\simeq$ denotes a quasi-isomorphism, and whose upper rectangle is anticommutative up to homotopy

$$\begin{array}{ccccc}
\mathbb{C} & \xrightarrow{\simeq} & \Omega_X^* & \xleftarrow{\quad} & \sigma_{\geq 1} \Omega^* \\
\parallel & & & & \parallel \\
\mathbb{C} & \xleftarrow{\quad} & [\mathbb{C} \rightarrow \mathcal{O}] & \xrightarrow[\simeq]{d} & \sigma_{\geq 1} \Omega^* \\
\uparrow & & \uparrow & & \uparrow d \log \\
\mathbb{Z}(1) & \xleftarrow{\quad} & [\mathbb{Z}(1) \rightarrow \mathcal{O}] & \xrightarrow[\simeq]{\exp} & \mathcal{O}^*[-1]
\end{array}$$

(2.2.6) Let X be a nonsingular projective variety purely of dimension n . A choice of i defines an orientation of X and an isomorphism $\mathbb{Z}(1) \simeq \mathbb{Z}$. The corresponding trace morphism:

$$H^{2n}(X, \mathbb{Z}(n)) \longrightarrow \mathbb{Z}$$

which follows from it does not depend on the choice of i .

According to Hodge, for $i \leq n$ the morphism

$$L^{n-i} = \wedge c_1''(\mathcal{O}(1))^{n-i} : H^i(X, \mathbb{Z}) \longrightarrow H^{2n-i}(X, \mathbb{Z}(n-i)),$$

is an isomorphism and, combined with Poincaré duality

$$H^i(X, \mathbb{Z}) \otimes H^{2n-i}(X, \mathbb{Z}(n-i)) \longrightarrow H^{2n}(X, \mathbb{Z}(n-i)) \longrightarrow \mathbb{Z}(-i),$$

it furnishes a polarization on the *primitive part* $\text{Ker}(L^{n-i+1})$ of $H^i(X, \mathbb{Z})$. It follows that the rational Hodge structures $H^i(X, \mathbb{Q})$ are polarizable.

2.3. Mixed structures.

Definition (2.3.1). — A mixed Hodge structure H consists of:

- a) a finitely generated $\mathbb{Z}$ -module $H_{\mathbb{Z}}$ (the “integral lattice”).
- b) a finite increasing filtration W_n of $H_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$, called the weight filtration.
- c) a finite filtration F^p of $H_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$, called the Hodge filtration.

It is required that on $H_{\mathbb{C}}$, the filtration $W_{\mathbb{C}}$ deduced from W by extension of scalars, the filtration F , and its complex conjugate $\bar{F}$ form a system $(W_{\mathbb{C}}, F, \bar{F})$ of three opposed filtrations ((1.2.7) and (1.2.13)).

(2.3.2) Let W again denote the filtration of $H_{\mathbb{Z}}$ which is the inverse image of the filtration W of $H_{\mathbb{Q}}$. The axiom of mixed Hodge structures means that, for every n , the filtration F induces on $\mathbb{C} \otimes_{\mathbb{Z}} \text{Gr}_n^W(H_{\mathbb{Z}})$ a filtration n -opposed to its complex conjugate. By (2.1.9), $\text{Gr}_n^W(H_{\mathbb{Z}})$ is endowed with a Hodge structure of weight n , with Hodge filtration induced by F .

Example (2.3.3). — If H is a Hodge structure of weight n , one defines a mixed Hodge structure with the same integral lattice and the same Hodge filtration by putting

$$W_i(H_{\mathbb{Q}}) = 0 \quad \text{for } i < n \quad \text{and} \quad W_i(H_{\mathbb{Q}}) = H_{\mathbb{Q}} \quad \text{for } i \geq n.$$

(2.3.4) A morphism $f : H \rightarrow H'$ of Hodge structures is a homomorphism $f : H_{\mathbb{Z}} \rightarrow H'_{\mathbb{Z}}$ compatible with the filtrations W and F (and therefore compatible with $\bar{F}$). One immediately deduces from (1.2.10) the following theorem.

Theorem (2.3.5). —

- (i) The category of mixed Hodge structures is abelian.
- (ii) The kernel (respectively, cokernel) of a morphism $f : H \rightarrow H'$ has as integral lattice the kernel (respectively, cokernel) K of $f : H_{\mathbb{Z}} \rightarrow H'_{\mathbb{Z}}$, with $K \otimes \mathbb{Q}$ and $K \otimes \mathbb{C}$ endowed with the induced (respectively, quotient) filtrations from the filtrations W and F of $H_{\mathbb{Q}}$ and $H_{\mathbb{C}}$ (respectively, of $H'_{\mathbb{Q}}$ and $H'_{\mathbb{C}}$).
- (iii) Every morphism $f : H \rightarrow H'$ is strictly compatible with the filtration W of $H_{\mathbb{Q}}$ and $H'_{\mathbb{Q}}$, and with the filtration F of $H_{\mathbb{C}}$ and $H'_{\mathbb{C}}$. It induces morphisms of $\mathbb{Q}$ -Hodge structures:

$$\mathrm{Gr}_n^W(f) : \mathrm{Gr}_n^W(H_{\mathbb{Q}}) \longrightarrow \mathrm{Gr}_n^W(H'_{\mathbb{Q}})$$

and morphisms strictly compatible with the filtration induced by $W_{\mathbb{C}}$:

$$\mathrm{Gr}_F^p(f) : \mathrm{Gr}_F^p(H_{\mathbb{C}}) \longrightarrow \mathrm{Gr}_F^p(H'_{\mathbb{C}}).$$

- (iv) The functor Gr_n^W is an exact functor from the category of mixed Hodge structures to the category of $\mathbb{Q}$ -Hodge structures of weight n .
- (v) The functor Gr_F^p is exact.

(2.3.6) Let H be a mixed Hodge structure. The $W_n(H_{\mathbb{Z}})$, endowed with the filtrations induced by W and F , then form mixed Hodge substructures $W_n(H)$ of H . The quotient $W_n(H)/W_{n-1}(H)$ is identified with $\mathrm{Gr}_n^W(H_{\mathbb{Z}})$ endowed with its Hodge structure (2.3.2), (2.3.3). This Hodge structure will be denoted by $\mathrm{Gr}_n^W(H)$.

(2.3.7) Put $H^{pq} = \mathrm{Gr}_F^p \mathrm{Gr}_F^q \mathrm{Gr}_{p+q}^W(H_{\mathbb{C}}) = (\mathrm{Gr}_{p+q}^W(H))^{p,q}$. The Hodge numbers of H are the integers

$$h^{pq} = \dim_{\mathbb{C}} H^{pq}.$$

Thus the Hodge number h^{pq} of H is the Hodge number h^{pq} of the Hodge structure $\mathrm{Gr}_{p+q}^W(H)$.

(2.3.8) A $\mathbb{Q}$ -mixed Hodge structure H is defined as consisting of a finite-dimensional vector space $H_{\mathbb{Q}}$ over $\mathbb{Q}$, a finite increasing filtration W of $H_{\mathbb{Q}}$, and a finite decreasing filtration F of $H_{\mathbb{C}}$, the filtrations $W_{\mathbb{C}}$, F , and $\bar{F}$ being opposed. Theorem (2.3.5) generalizes trivially to this variant.

3. Hodge theory of nonsingular algebraic varieties

3.1. Logarithmic poles and residues.

(3.1.1) We recall some classical properties of “logarithmic poles” of holomorphic differential forms. The reader will find proofs in [3], II, (3.1) to (3.7), for example.

(3.1.2) A divisor Y in a smooth complex analytic variety X will be said to have normal crossings if the inclusion of Y into X is locally isomorphic to the inclusion of a union of coordinate hyperplanes in $\mathbb{C}^n$; this does not imply that Y is a union of smooth divisors. Let Y be a normal-crossings divisor in X , and let j be the inclusion of $X^* = X - Y$ in X . Let $\Omega_X^1\langle Y \rangle$ denote the locally free $\mathcal{O}$ -submodule of $j_*\Omega_{X^*}^1$ generated by Ω_X^1 and by the $\frac{dz_i}{z_i}$, where z_i is a local equation of a local irreducible component of Y . The sheaf $\Omega_X^p\langle Y \rangle$ of differential p -forms on X with logarithmic pole along Y is, by definition, the locally free subsheaf $\bigwedge^p \Omega_X^1\langle Y \rangle$ of $j_*\mathcal{O}_{X^*}^p$.

Proposition (3.1.3). —

- (i) A section α of $j_*\Omega_{X^*}^p$ belongs to $\Omega_X^p\langle Y \rangle$ if and only if α and $d\alpha$ have at worst simple poles along the divisor Y .
- (ii) The $\Omega_X^p\langle Y \rangle$ form the smallest subcomplex of $j_*\Omega_{X^*}^*$, stable under exterior product, containing Ω_X^* , and containing the logarithmic differential df/f of every local section, meromorphic along Y , of $j_*\mathcal{O}_{X^*}^*$.

The complex $\Omega_X^*\langle Y \rangle$ is called the logarithmic De Rham complex of X along Y . By (3.1.3), (ii), this complex is contravariant in the pair (X, X^*) .

(3.1.4) Locally on X , Y is a union of smooth divisors Y_i , and Y^n (resp. $\tilde{Y}^n$) denotes the union (resp. the disjoint sum) of the intersections n at a time of the Y_i . The Y^n glue to a subspace Y^n of X , and the $\tilde{Y}^n$ glue to the normalization of Y^n . One has $\tilde{Y}^0 = Y^0 = X$ and puts $\tilde{Y} = \tilde{Y}^1$.

The two-element set of *orientations* of a finite set E with n elements is defined to be the set of generators of $\bigwedge^n \mathbb{Z}^E$. For $n \geq 2$, this set is the set of conjugacy classes, under the alternating group, of total orderings of E .

If to each point y of $\tilde{Y}^n$ one associates the set of the n local components of Y which contain the image in X of a neighborhood of y in $\tilde{Y}^n$, then one defines on $\tilde{Y}^n$ a local system E_n of n -element sets. The local system of orientations of these sets is a torsor under $\mathbb{Z}/(2)$. This torsor defines, *via* the inclusion of $\mathbb{Z}/(2)$ into $\mathbb{C}^*$, a rank-one complex local system ε^n on $\tilde{Y}^n$, endowed with an isomorphism

$$(\varepsilon^n)^{\otimes 2} \simeq \mathbb{C}.$$

One has

$$\varepsilon^n \simeq \bigwedge^n \mathbb{C}^{E_n}.$$

Locally on $\tilde{Y}^n$, ε^n is endowed with two opposite isomorphisms $\pm\alpha : \varepsilon^n \xrightarrow{\sim} \mathbb{C}$. Put

$$\varepsilon_{\mathbb{Z}}^n = \alpha^{-1}((2\pi i)^{-n} \mathbb{Z}).$$

The local system ε^n , endowed with $\varepsilon_{\mathbb{Z}}^n$, should be regarded as a twisted form of $\mathbb{Z}(-n)$ on $\tilde{Y}^n$.

Let ε_X^n (resp. $(\varepsilon_X^n)_{\mathbb{Z}}$) denote the direct image of ε^n (resp. $\varepsilon_{\mathbb{Z}}^n$) under the map from $\tilde{Y}^n$ to X . One has

$$(3.1.4.1) \quad \varepsilon_X^n \simeq \bigwedge^n \varepsilon_X^1 \quad (n \geq 0).$$

If Y is a union of distinct smooth divisors $(Y_i)_{i \in I}$, the choice of a total order on I trivializes the ε^n .

(3.1.5) Let $W_n(\Omega_X^p \langle Y \rangle)$ denote the submodule of $\Omega_X^p \langle Y \rangle$ formed by linear combinations of products

$$\alpha \wedge \frac{dt_{i(1)}}{t_{i(1)}} \wedge \cdots \wedge \frac{dt_{i(m)}}{t_{i(m)}} \quad (m \leq n),$$

with α holomorphic and with the $t_{i(j)}$ local equations of distinct local components Y_j of Y . The *weight filtration* of $\Omega_X^*(Y)$ is the *increasing* filtration by the subcomplexes $W_n(\Omega_X^*(Y))$. One has

$$(3.1.5.1) \quad W_n(\Omega_X^p \langle Y \rangle) \wedge W_m(\Omega_X^q \langle Y \rangle) \subset W_{n+m}(\Omega_X^{p+q} \langle Y \rangle).$$

Letting i_n denote the map from $\tilde{Y}^n$ to X , one checks that the correspondence

$$\alpha \wedge \frac{dt_{i(1)}}{t_{i(1)}} \wedge \cdots \wedge \frac{dt_{i(n)}}{t_{i(n)}} \longmapsto (\alpha \mid Y_{i(1)} \cap \cdots \cap Y_{i(n)}) \otimes (\text{orientation } i(1) \dots i(n))$$

defines isomorphisms of complexes

$$(3.1.5.2) \quad \text{Res} : \text{Gr}_n^W(\Omega_X^*(Y)) \simeq i_{n*} \Omega_{\tilde{Y}^n}^*(\varepsilon^n)[-n]$$

(the “*Poincaré residue*”).

(3.1.6) The following interpretation, in terms of (3.1.5.2), of the Leray spectral sequence for the inclusion of X^* in X was pointed out to me by N. Katz. It will make it possible to verify a point which I had initially considered evident ((3.2.5), (ii), first part).

(3.1.7) Every point of X admits a fundamental system of Stein open neighborhoods whose trace on X^* is Stein. For $\mathcal{F}$ a coherent analytic sheaf on X^* , one therefore has $R^i j_* \mathcal{F} = 0$ for $i > 0$. The De Rham complex $\Omega_{X^*}^*$ is therefore a resolution of the constant sheaf $\mathbb{C}$ by sheaves acyclic for the functor j_* . Hence

$$(3.1.7.1) \quad H^*(X^*, \mathbb{C}) \xrightarrow{\sim} \mathbb{H}^*(X^*, \Omega_{X^*}^*) \xleftarrow{\sim} \mathbb{H}^*(X, j_* \Omega_{X^*}^*)$$

and the Leray spectral sequence for the morphism j is identified with the hypercohomology spectral sequence for $j_* \Omega_{X^*}^*$, corresponding to the filtration τ by the subcomplexes $\tau_{\leq -n}(j_* \Omega_{X^*}^*)$ (1.4.6).

Proposition (3.1.8). — *The morphisms of filtered complexes*

$$(\Omega_X^*(Y), W) \xleftarrow{\alpha} (\Omega_X^*(Y), \tau) \xrightarrow{\beta} (j_*\Omega_{X^*}^*, \tau)$$

are filtered quasi-isomorphisms. They define an isomorphism between the Leray spectral sequence for j in complex cohomology and the hypercohomology spectral sequence of the filtered complex $(\Omega_X^*(Y), W)$ on X .

By (1.4.5) and (3.1.7), it suffices to prove the first assertion. In [3], II, (6.9), or in [1], one finds the proof that β is a quasi-isomorphism, hence a filtered quasi-isomorphism. One can also compute directly the cohomology sheaves of the two sides: those of $\Omega_X^*(Y)$ are determined by (3.1.5.2), whereas those of $j_*\Omega_{X^*}^*$ are the $R^i j_* \mathbb{C}$, which can be computed topologically.

For $n \geq p$ one has

$$W_n(\Omega_X^p(Y)) = \Omega_X^p(Y),$$

so that α is a morphism from $\Omega_X^*(Y)$, endowed with τ , to $\Omega_X^*(Y)$, endowed with the decreasing filtration associated with W (1.1.3). By (3.1.5.2), one has

$$(3.1.8.1) \quad \mathcal{H}^i(\mathrm{Gr}_n^W(\Omega_X^*(Y))) = \begin{cases} 0, & \text{for } i \neq n, \\ \varepsilon_X^n, & \text{for } i = n; \end{cases}$$

from the first line of this formula one deduces that α is a filtered quasi-isomorphism. This proves (3.1.8).

By (3.1.7), the isomorphism (3.1.8.1) defines an isomorphism

$$(3.1.8.2) \quad R^n j_* \mathbb{C} \simeq \mathcal{H}^n(j_*\Omega_{X^*}^*) \simeq \mathcal{H}^n(\Omega_X^*(Y)) \simeq \varepsilon_X^n.$$

The isomorphisms (3.1.4.1) correspond, via (3.1.8.2), to the cup product.

Proposition (3.1.9). — *The canonical morphism from $R^n j_* \mathbb{Z}$ to $R^n j_* \mathbb{C}$ identifies, via (3.1.8.2), the sheaf $R^n j_* \mathbb{Z}$ with $(\varepsilon_X^n)_{\mathbb{Z}}$ (3.1.4).*

The question is local on X . One can therefore suppose that X is an open polydisc D^m , with $D = \{z \in \mathbb{C} \mid |z| < 1\}$, and that $Y = \bigcup_{k=1}^{\ell} Y_k$, $Y_k = \mathrm{pr}_k^{-1}(0)$. The fiber at 0 of $R^n j_* \mathbb{Z}$ is then the integral cohomology of $X^* = D^{*\ell} \times D^{(m-\ell)}$, with $D^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$. The space X^* has the homotopy type of a torus; its cohomology is therefore torsion-free, and the cup product defines isomorphisms

$$\bigwedge^n (R^1 j_* \mathbb{Z})_0 \xrightarrow{\sim} (R^n j_* \mathbb{Z})_0.$$

It is therefore enough to prove (3.1.9) for $n = 1$.

The integral homology $H_1(X^*)$ is generated by the loops γ_k winding around the various Y_k . One has

$$\oint_{\gamma_k} \frac{dz_k}{z_k} = \pm 2\pi i;$$

the integral cohomology is therefore generated by the $\frac{1}{2\pi i} \frac{dz_k}{z_k}$, and this proves (3.1.9).

(3.1.10) Let $\mathcal{F}$ be a coherent analytic sheaf on X^* , given as the restriction to X^* of a coherent analytic sheaf $\mathcal{F}'$ on X . The meromorphic direct image $j_*^m \mathcal{F}$ of $\mathcal{F}$ is the inductive limit

$$j_*^m \mathcal{F} = \varinjlim_n \mathcal{F}'(nY).$$

Locally on X , Y is the sum of a finite family $(Y_i)_{i \in I}$ of smooth divisors, and the filtration by order of pole P on $j_*^m \mathcal{O}_{X^*}$ is defined by the formula

$$(3.1.10.1) \quad P^p(j_*^m \mathcal{O}_{X^*}) = \sum_{\mathbf{n} \in A_p} \mathcal{O}_X \left(\sum_i (n_i + 1) Y_i \right)$$

where

$$A_p = \{(n_i)_{i \in I} \mid \sum_i n_i \leq -p \text{ and } \forall i, n_i \geq 0\}.$$

This construction globalizes and provides on $j_*^m \mathcal{O}_{X^*}$ an exhaustive filtration such that $P^p = 0$ for $p > 0$.

The *filtration by order of pole* of the complex $j_*^m \Omega_{X^*}^* = j_*^m \mathcal{O}_{X^*} \otimes \Omega_X^*$ is the filtration

$$(3.1.10.2) \quad P^p(j_*^m \Omega_{X^*}^k) = P^{p-k}(j_*^m \mathcal{O}_{X^*}) \otimes \Omega_X^k.$$

The filtration P induces on the subcomplex $\Omega_X^*(Y)$ of $j_*^m \Omega_{X^*}^*$ the bête filtration by the $\sigma_{\geq p}(\Omega_X^*(Y))$, a filtration also called the *Hodge filtration* F .

Proposition (3.1.11). — *The inclusion morphism*

$$(\Omega_X^*(Y), F) \longrightarrow (j_*^m \Omega_{X^*}^*, P)$$

is a filtered quasi-isomorphism.

This statement was suggested to me by [4]. A proof appears in [3], II, (3.13).

3.2. Mixed Hodge theory.

Recall that from now on, by a scheme we mean a scheme of finite type over $\mathbb{C}$, and by a sheaf on S a sheaf on S^{an} .

(3.2.1) Let X be a smooth separated scheme. By Nagata [11], X is a Zariski open subset of a complete scheme $\overline{X}$. By Hironaka [8], one can take $\overline{X}$ smooth, and such that $Y = \overline{X} - X$ is a normal-crossings divisor.

The reader who would like to avoid the reference to Nagata may suppose X quasi-projective. The smooth completion $\overline{X}$ can then be chosen projective and such that Y is a union of smooth divisors. When one restricts oneself to such compactifications, Hodge theory is needed only in its standard form (2.2.1).

(3.2.2) By (3.1.7) and (3.1.8), one has

$$H^*(X, \mathbb{C}) \simeq \mathbb{H}^*(\overline{X}, \Omega_{\overline{X}}^*(Y)).$$

The *Hodge filtration* F on the complex $\Omega_{\overline{X}}^*(Y)$ is defined to be the filtration $F^p = \sigma_{\geq p}$ by bête truncations (1.4.5). Thus $\Omega_{\overline{X}}^*(Y)$ carries two filtrations: F and W (3.1.5).

(3.2.3) We shall use the existence of bifiltered resolutions $i : \Omega_{\overline{X}}^*(Y) \longrightarrow K^*$ such that the $\mathrm{Gr}_F^p \mathrm{Gr}_n^W(K^j)$ are sheaves acyclic for the functor Γ :

$$H^i(\overline{X}, \mathrm{Gr}_F^p \mathrm{Gr}_n^W(K^j)) = 0 \quad \text{for } i > 0.$$

Here are two ways to construct them:

- a) One may take for K^* the canonical Godement resolution $\mathcal{C}^*(\Omega_{\overline{X}}^*(Y))$, filtered by the $\mathcal{C}^*(W_n(\Omega_{\overline{X}}^*(Y)))$ and the $\mathcal{C}^*(F^p(\Omega_{\overline{X}}^*(Y)))$. This is a bifiltered resolution because $\mathcal{C}^*$ is an exact functor.
- b) One may take for K^* the d'' -resolution of $\Omega_{\overline{X}}^*(Y)$. Let $\Omega_{\overline{X}}^{pq}$ be the sheaf of C^∞ forms of type (p, q) ; then K^* is the simple complex associated with the double complex of the $\Omega_{\overline{X}}^p(Y) \otimes_{\mathcal{O}} \Omega_{\overline{X}}^{0,q}$ (a subcomplex of the $j_* \Omega_X^{**}$). This complex is filtered by the $F^p(\Omega_{\overline{X}}^*(Y)) \otimes \Omega_{\overline{X}}^{0,*}$ and by the $W_n(\Omega_{\overline{X}}^*(Y)) \otimes \Omega_{\overline{X}}^{0,*}$; to prove that this is a bifiltered resolution, one uses the fact that the sheaf $\mathcal{O}_\infty$ of complex-valued C^∞ functions on $\overline{X}$ is flat over $\mathcal{O}$ (a corollary of Malgrange's C^∞ preparation theorem). The sheaves $\mathrm{Gr}_F^p \mathrm{Gr}_n^W(K^*)$ are fine, because they are sheaves of modules over the soft sheaf $\mathcal{O}_\infty$.

(3.2.4) With the notation of (3.2.3), the complex cohomology of X appears as the cohomology of the bifiltered complex $\Gamma(\overline{X}, K^*)$. Thus there are two spectral sequences abutting to $H^*(X, \mathbb{C})$. With the notation of (3.1.4), they are

$$(3.2.4.1) \quad {}_w E_1^{pq} = H^{p+q}(\overline{X}, \varepsilon_{\overline{X}}^{-p}[p]) = H^{2p+q}(\widetilde{Y}^p, \varepsilon^{-p}) \Rightarrow H^n(X, \mathbb{C})$$

$$(3.2.4.2) \quad {}_F E_1^{pq} = H^q(\overline{X}, \Omega_X^p \langle Y \rangle) \Rightarrow H^n(X, \mathbb{C}).$$

The first of these, up to the renumbering ${}_W E_1^{pq} \mapsto E_2^{2p+q, -p}$, is just the Leray spectral sequence of the inclusion j .

Theorem (3.2.5). —

- (i) On the terms ${}_W E_r^{pq}$ of the spectral sequence (3.2.4.1), the first direct filtration, the second direct filtration, and the recurrent filtration defined by F coincide.
- (ii) The filtration on $H^n(X, \mathbb{C})$ which is the abutment of the spectral sequence ${}_W E$ is deduced from a filtration W of $H^n(X, \mathbb{Q})$. Neither it, nor the filtration F which is the abutment of the spectral sequence ${}_F E$, depends on the chosen compactification $\overline{X}$ of X or on the choice of K^* .
- (iii) The filtrations $W[n]$ (1.1.2) and F define on $H^n(X, \mathbb{Z})$ a mixed Hodge structure, functorial in X .

By (3.1.7), the spectral sequence ${}_W E$ is the Leray spectral sequence for j_* (up to renumbering). It is therefore obtained by tensoring with $\mathbb{C}$ a spectral sequence of $\mathbb{Q}$ -vector spaces, and the first assertion of (ii) is true. Complex conjugation acts on the ${}_W E_r$; it can be computed via (3.1.10).

Lemma (3.2.6). — *The hypercohomology spectral sequences of the filtered complexes $\mathrm{Gr}_n^W(\Omega_{\overline{X}}^*(Y))$, endowed with the filtration induced by the Hodge filtration, degenerate at E_1 .*

Define Y^n and $\tilde{Y}^n$ as in (3.1.4), and let $i_n : \tilde{Y}^n \rightarrow \overline{X}$. By (3.1.5.2), one has

$$\mathrm{Gr}_n^W(\Omega_{\overline{X}}^*(Y)) \simeq i_{n*} \Omega_{\tilde{Y}^n}^*(\varepsilon^n)[-n].$$

Moreover, the Hodge filtration induces the bête filtration (1.4.5), so that the spectral sequence (3.2.6) is obtained, by translations in the degrees, from the classical spectral sequence

$$E_1^{pq} = H^q(\tilde{Y}^n, \Omega_{\tilde{Y}^n}^p(\varepsilon^n)) \Rightarrow H^{p+q}(\tilde{Y}^n, \varepsilon^n).$$

If $\overline{X}$ is projective and Y is a union of smooth divisors, then $\tilde{Y}^n$ is projective, ε^n is a trivial local system, and classical Hodge theory (2.2.1) gives the degeneration (3.2.6). For the general case, one must refer to [2] (see (2.2.2), (2.2.3)).

Hodge theory further gives that the Hodge filtration on $H^k(\tilde{Y}^n, \varepsilon^n)$ is k -opposed to its complex conjugate (complex conjugation being defined in terms of $\varepsilon_{\mathbb{Z}}^n$ (3.1.4)). Taking account here of the translations in the degrees, one has:

Lemma (3.2.7). — *The filtration on*

$${}_W E_1^{-n, k+n} = H^k(\overline{X}, \mathrm{Gr}_n^W(\Omega_{\overline{X}}^*(Y))) \simeq H^{k-n}(\tilde{Y}^n, \varepsilon^n),$$

which is the abutment of the spectral sequence (3.2.6), is $(k+n)$ -opposed to its complex conjugate.

Lemma (3.2.8). — *The differentials d_1 of the spectral sequence ${}_W E$ are strictly compatible with the filtration F .*

On the terms E_1 , there is only one filtration induced by F to consider ((1.3.10) and (1.3.13), (iii)), and d_1 is compatible with this filtration ((1.3.13), (i)). This filtration is the abutment of the spectral sequence (3.2.6) ((1.4.8), (iii)). By (3.2.7), the arrow d_1

$$d_1 : H^k(\overline{X}, \mathrm{Gr}_n^W(\Omega_{\overline{X}}^*(Y))) \longrightarrow H^{k+1}(\overline{X}, \mathrm{Gr}_{n-1}^W(\Omega_{\overline{X}}^*(Y)))$$

that is,

$$(3.2.8.1) \quad d_1 : H^{k-n}(\tilde{Y}^n, \varepsilon^n) \longrightarrow H^{k-n+2}(\tilde{Y}^{n-1}, \varepsilon^{n-1})$$

is compatible with filtrations that are $(k+n)$ -opposed to their complex conjugates. Since d_1 commutes with complex conjugation, it respects the bigrading, and hence the filtration defined by F and $\overline{F}$; this proves (3.2.8). Moreover, the cohomology of the complex E_1 will still be bigraded.

Lemma (3.2.9). — *On ${}_WE_2^{pq}$, the recurrent filtration F is q -opposed to its complex conjugate.*

We prove by induction on r that:

Lemma (3.2.10). — *For $r \geq 0$, the differentials d_r of the spectral sequence ${}_WE$ are strictly compatible with the recurrent filtration F . For $r \geq 2$, they are zero.*

For $r = 0$ (resp. $r = 1$), one applies (3.2.6) and (1.4.8), (iii) (resp. (3.2.8)). For $r \geq 2$, it suffices to prove that $d_r = 0$. By induction and by (1.3.16), one may suppose that on the terms ${}_WE_s$ ($s \geq r+1$) one has $F_d = F_r = F_{d*}$, and that ${}_WE_r = {}_WE_2$. By (1.3.13), (i), d_r is therefore compatible with the filtration F_r .

On ${}_WE_r^{pq} = {}_WE_2^{pq}$, the filtration F_r is q -opposed to its complex conjugate. The morphism

$$d_r : {}_WE_r^{pq} \longrightarrow {}_WE_r^{p+r, q-r+1}$$

therefore satisfies, for $r-1 > 0$,

$$\begin{aligned} d_r({}_WE_r^{pq}) &= d_r \left(\sum_{a+b=q} (F^a({}_WE_r^{pq}) \cap \overline{F}^b({}_WE_r^{pq})) \right) \\ &\subset \sum_{a+b=q} (F^a({}_WE_r^{p+r, q-r+1}) \cap \overline{F}^b({}_WE_r^{p+r, q-r+1})) = 0. \end{aligned}$$

This proves (3.2.10), which by (1.3.16) implies (3.2.5) (i).

By (1.3.17), the filtration of ${}_WE_\infty^{pq}$ induced by the filtration F of $H^{p+q}(X, \mathbb{C})$ is q -opposed to its complex conjugate. Since $q = -p + (p+q)$, this proves the first part of (3.2.5) (iii).

(3.2.11) We prove (ii) and (iii), which will complete the proof.

A. Independence of the choice of K^* . The filtrations F and W of $H^*(X, \mathbb{C})$ are the abutments of the hypercohomology spectral sequences of $\Omega_X^* \langle Y \rangle$ for the filtrations F and W . These entire spectral sequences do not depend on the choice of K^* .

B. Functoriality. Let $f : X_1 \rightarrow X_2$ be a morphism of schemes. Suppose given a morphism of smooth compactifications

$$(3.2.11.1) \quad \begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ j_1 \downarrow & & \downarrow j_2 \\ \overline{X}_1 & \xrightarrow{\overline{f}} & \overline{X}_2, \end{array}$$

where $Y_i = \overline{X}_i - X_i$ are normal-crossings divisors. The canonical morphism (see (3.1.3)) from $\overline{f}^* \Omega_{\overline{X}_2}^* \langle Y_2 \rangle$ to $\Omega_{\overline{X}_1}^* \langle Y_1 \rangle$ is then a morphism of bifiltered complexes; on hypercohomology it induces a morphism compatible with F and W , and therefore $f^* : H^n(X_2, \mathbb{Z}) \rightarrow H^n(X_1, \mathbb{Z})$ is a morphism of mixed Hodge structures, for the structures defined by the compactifications $\overline{X}_i$.

C. Independence of the compactification. With the notation of B, if f is an isomorphism, then f^* is a bijective morphism of mixed Hodge structures, hence an isomorphism (2.3.5).

If $\overline{X}_1$ and $\overline{X}_2$ are two smooth compactifications of X , with $Y_i = \overline{X}_i - X$ a normal-crossings divisor, there exists a third smooth compactification $\overline{X}$, with $Y = \overline{X} - X$ a normal-crossings divisor, fitting into a commutative diagram

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & \downarrow & \searrow & \\ \overline{X}_1 & \longleftarrow & \overline{X} & \longrightarrow & \overline{X}_2 \end{array}$$

Namely, one takes for $\overline{X}$ a resolution of the singularities of the closure of the diagonal image of X in $\overline{X}_1 \times \overline{X}_2$. The identity map from $H^n(X, \mathbb{Z})$, endowed with the mixed Hodge structure defined by $\overline{X}_1$, to $H^n(X, \mathbb{Z})$, endowed with that defined by $\overline{X}_2$, is therefore a composition of two isomorphisms.

To finish the proof of (ii) and (iii), one observes that every morphism f fits into a diagram (3.2.11.1): choose compactifications $\overline{X}'_1$ and $\overline{X}_2$ of X_1 and X_2 , and then take for $\overline{X}_1$ a resolution of the singularities of the closure of the image of X_1 in $\overline{X}'_1 \times \overline{X}_2$.

Definition (3.2.12). — *The mixed Hodge structure on the cohomology of a smooth separated algebraic variety is the mixed Hodge structure (3.2.5), (iii).*

Corollary (3.2.13). — *With the preceding notation:*

- (i) *The spectral sequence (3.2.4.1) degenerates at E_2 , i.e. the Leray spectral sequence for the inclusion $j : X^* \hookrightarrow \overline{X}$ degenerates at E_3 ($E_3 = E_\infty$).*
- (ii) *The spectral sequence (3.2.4.2)*

$${}_F E_1^{pq} = H^q(\overline{X}, \Omega_{\overline{X}}^p(Y)) \implies H^{p+q}(X, \mathbb{C})$$

degenerates at E_1 .

- (iii) *The spectral sequence defined by the sheaf $\Omega_{\overline{X}}^p(Y)$, endowed with the filtration W :*

$$\begin{aligned} E_1^{-n, k+n} &= H^k(\overline{X}, \mathrm{Gr}_n^W(\Omega_{\overline{X}}^p(Y))) \\ &\simeq H^k(\tilde{Y}^n, \Omega_{\tilde{Y}^n}^{p-n}(\varepsilon^n)) \implies H^k(\overline{X}, \Omega_{\overline{X}}^p(Y)) \end{aligned}$$

degenerates at E_2 .

Assertion (i) was proved in (3.2.10).

Consider the four spectral sequences of (1.4.12), relative to the bifiltered complex $\Omega_{\overline{X}}^*(Y)$. By (i), one has

$$\sum_n \dim H^n(X, \mathbb{C}) = \sum_{p,q} \dim {}_W E_2^{p,q}.$$

The terms ${}_W E_1^{n, k-n}$ are, for fixed n , the abutment of a spectral sequence *degenerate at E_1* (3.2.6), with initial terms $E_1^{p, n, k-p-n}$ (notation of (1.4.12)), the initial terms of spectral sequence (iii). Since ${}_W d_1$ is strictly compatible with the filtration F that is the abutment of this spectral sequence (3.2.7), and is (1.4.12) the abutment of a morphism between these spectral sequences, beginning with the differentials of (iii), one has

$$\mathrm{Gr}_F^p({}_W E_2^{n, k-n}) \simeq H^*(E_1^{p, n-1, k-n-p} \rightarrow E_1^{p, n, k-n-p} \rightarrow E_1^{p, n+1, k-n-p})$$

and $\mathrm{Gr}_F^*({}_W E_2^{**})$ is the sum of the terms E_2^{***} of the spectral sequences (iii). Therefore

$$(3.2.13.1) \quad \sum_n \dim H^n(X, \mathbb{C}) = \sum \dim E_2^{***}.$$

On the other hand,

$$\sum_n \dim H^n(X, \mathbb{C}) \leq \sum \dim {}_F E_1^{**},$$

with equality if and only if the spectral sequence (ii) degenerates at E_1 , and

$$\sum \dim {}_F E_1^{**} \leq \sum \dim E_2^{***},$$

with equality if and only if the spectral sequences (iii) degenerate at E_2 . Comparing with (3.2.13.1), one obtains (3.2.13).

Corollary (3.2.14). — *Let ω be a meromorphic differential p -form on $\overline{X}$, holomorphic on X and having at worst logarithmic poles along Y . Then the restriction $\omega|_X$ of ω to X is closed, and if the cohomology class in $H^p(X, \mathbb{C})$ defined by ω is zero, then $\omega = 0$.*

This is the special case ${}_F E_1^{p0} = {}_F E_\infty^{p0}$ of (3.2.13), (ii).

Corollary (3.2.15). —

- (i) If X is a complete smooth algebraic variety, the mixed Hodge structure on $H^n(X, \mathbb{Z})$ is the classical Hodge structure of weight n .
- (ii) The Hodge numbers h^{pq} of the mixed Hodge structure of $H^n(X, \mathbb{Z})$ (X smooth algebraic) can be nonzero only for $p \leq n$, $q \leq n$, and $p + q \geq n$.

Assertion (i) is clear. To prove (ii), one observes that, for Y a union of smooth divisors, the rational Hodge structure $\mathrm{Gr}_k^W(H^n(X, \mathbb{Q}))$ is a quotient of a subobject of a Hodge structure of weight $n + k$, namely

$$H^{n-k}(\tilde{Y}^n, \mathbb{Q}) \otimes \mathbb{Q}(-k).$$

(3.2.16) Let X be a smooth separated scheme. It is known that X admits smooth compactifications $\overline{X}$, and that the subgroup of $H^n(X, \mathbb{Z})$ which is the image of $H^n(\overline{X}, \mathbb{Z})$ is independent of the choice of $\overline{X}$ (cf. [6], (9.1) to (9.4)).

Corollary (3.2.17). — Under the hypotheses (3.2.16), the image of $H^n(\overline{X}, \mathbb{Q})$ in $H^n(X, \mathbb{Q})$ is $W_n(H^n(X, \mathbb{Q}))$ (W denoting the weight filtration (3.2.12)).

One may suppose that $\overline{X} - X$ is a normal-crossings divisor. The assertion then follows from the fact that $W[-n]$ is the abutment of the Leray spectral sequence for the inclusion $j : X \hookrightarrow \overline{X}$.

Corollary (3.2.18). — Let f be a morphism from a proper smooth scheme Y to a smooth scheme X admitting a smooth compactification $\overline{X}$:

$$Y \xrightarrow{f} X \xrightarrow{j} \overline{X}.$$

Then the groups $H^n(X, \mathbb{Q})$ and $H^n(\overline{X}, \mathbb{Q})$ have the same image in $H^n(Y, \mathbb{Q})$.

Since f^* and $(jf)^*$ are strictly compatible with the weight filtration ((3.2.5), (iii)), it is enough to prove that $\mathrm{Gr}^W(f^*)$ and $\mathrm{Gr}^W(f^*j^*)$ have the same image in $\mathrm{Gr}^W(H^n(Y, \mathbb{Q}))$. By (3.2.17) and (3.2.15), $\mathrm{Gr}_n^W(j^*)$ is an isomorphism, while $\mathrm{Gr}_m^W(f^*) = 0$ for $m \neq n$ because $\mathrm{Gr}_m^W(H^n(Y, \mathbb{Q})) = 0$ for $m \neq n$.

Remark (3.2.19). — By (3.1.11) and (1.4.5), under the hypotheses of (3.2.5), the Hodge filtration on $H^n(X, \mathbb{C})$ is the abutment of the hypercohomology spectral sequence of the complex $j_*^m \Omega_{X^*}^*$, endowed with the filtration by order of pole (3.1.10): this spectral sequence coincides with (3.2.4.2).

4. Applications and complements

4.1. The fixed-part theorem.

Theorem (4.1.1). — Let S be a smooth separated scheme, and let $f : X \rightarrow S$ be a proper smooth morphism.

- (i) In rational cohomology, the Leray spectral sequence

$$E_2^{pq} = H^p(S, R^q f_* \mathbb{Q}) \implies H^{p+q}(X, \mathbb{Q})$$

degenerates ($E_2 = E_\infty$).

- (ii) If $\overline{X}$ is a nonsingular compactification of X , the canonical morphism

$$H^n(\overline{X}, \mathbb{Q}) \longrightarrow H^0(S, R^n f_* \mathbb{Q})$$

is surjective.

For f projective and smooth, assertion (i) is proved in [2]. The proof of [2] is rewritten more readably in [5]; it does not use the smoothness of S .

Fix f , and prove that (i) $\implies$ (ii). We reduce to the case where S is connected and nonempty. If $s \in S$ is a point of S , the local system $R^n f_* \mathbb{Q}$ is entirely described by its fiber $(R^n f_* \mathbb{Q})_s$ at s , and by the action of the fundamental group $\pi = \pi_1(S, s)$ on this fiber. One has

$$H^0(S, R^n f_* \mathbb{Q}) \xrightarrow{\sim} [(R^n f_* \mathbb{Q})_s]^\pi.$$

If $X_s = f^{-1}(s)$, the composite arrow

$$H^0(S, R^n f_* \mathbb{Q}) \longrightarrow (R^n f_* \mathbb{Q})_s \simeq H^n(X_s, \mathbb{Q})$$

is therefore injective. Consider the arrows

$$H^n(\overline{X}, \mathbb{Q}) \xrightarrow{a} H^n(X, \mathbb{Q}) \xrightarrow{b} H^0(S, R^n f_* \mathbb{Q}) \xrightarrow{c} H^n(X_s, \mathbb{Q}).$$

The arrows cb and cba have the same image by (3.2.18). The arrow b is an edge homomorphism of the Leray spectral sequence, hence is surjective by hypothesis. Since c is injective, b and ba have the same image and ba is surjective. This proves (ii) for f projective. We now deduce the general case. Again reduce to the case where S is connected and nonempty.

By Chow's lemma and resolution of singularities, there exists a scheme X' which is quasi-projective and smooth, and a projective birational morphism

$$p : X' \longrightarrow X.$$

There then exists (by Bertini, or Sard) a nonempty Zariski open subset S_1 of S such that $X'_1 = (fp)^{-1}(S_1)$ is smooth over S_1 . Finally, let $X_1 = f^{-1}(S_1)$, let $\overline{X}$ be a smooth compactification of X , and let $\overline{X}'_1$ be a smooth compactification of X'_1 which dominates $\overline{X}$.

$$\begin{array}{ccccc} \overline{X} & \xleftarrow{\quad \overline{p} \quad} & & & \overline{X}'_1 \\ \uparrow & & & & \uparrow \\ X & \xleftarrow{\quad} & X_1 & \xleftarrow{\quad p \quad} & X'_1 \\ f \downarrow & & f_1 \downarrow & & f'_1 \downarrow \\ S & \xleftarrow{\quad i \quad} & S_1 & \xlongequal{\quad} & S_1 \end{array}$$

If $s \in S_1$, the morphism $i_* : \pi_1(S_1, s) \rightarrow \pi_1(S, s)$ is surjective because the topological codimension of $S - S_1$ is ≥ 2 . Therefore

$$H^0(S, R^n f_* \mathbb{Q}) \xrightarrow{\sim} H^0(S_1, R^n f_{1*} \mathbb{Q}),$$

and it suffices to prove that

$$H^n(\overline{X}, \mathbb{Q}) \longrightarrow H^0(S_1, R^n f_{1*} \mathbb{Q})$$

is surjective.

The vertical arrows of the diagram

$$\begin{array}{ccccc} H^n(\overline{X}, \mathbb{Q}) & \longrightarrow & H^n(X_1, \mathbb{Q}) & \longrightarrow & H^0(S_1, R^n f_{1*} \mathbb{Q}) \\ \overline{p}^* \downarrow & & p^* \downarrow & & \downarrow p^* \\ H^n(\overline{X}'_1, \mathbb{Q}) & \longrightarrow & H^n(X'_1, \mathbb{Q}) & \longrightarrow & H^0(S_1, R^n f'_{1*} \mathbb{Q}) \end{array}$$

admit as left inverses the *Gysin morphisms* $\overline{p}_!$, $p_!$, and $p_!$, defined by Poincaré duality as transposes of the arrows analogous to the preceding ones in cohomology with proper support. Moreover, the diagram

$$\begin{array}{ccc} H^n(\overline{X}, \mathbb{Q}) & \xrightarrow{\quad u \quad} & H^0(S_1, R^n f_{1*} \mathbb{Q}) \\ \overline{p}_! \uparrow & & \uparrow p_! \\ H^n(\overline{X}'_1, \mathbb{Q}) & \xrightarrow{\quad v \quad} & H^0(S_1, R^n f'_{1*} \mathbb{Q}) \end{array}$$

is commutative. Thus u is a direct factor of v . The latter being surjective (since f'_1 is projective), the same is true of u .

We now prove (i) in the general case. We reduce to the case where f is purely of relative dimension n . Let $f \times f$ be the projection of $X \times_S X$ onto S .

Let δ be the image of the cohomology class of the diagonal of $X \times_S X$ in $H^0(S, R^n(f \times f)_*\mathbb{Q})$. By Künneth,

$$R^n(f \times f)_*\mathbb{Q} = \sum_{p+q=n} (R^p f_*\mathbb{Q} \otimes R^q f_*\mathbb{Q}).$$

Let δ'_{pq} ($p + q = n$) denote the components of δ in this decomposition, and let δ_{pq} be classes in $H^n(X \times_S X)$ whose images are the δ'_{pq} .

The δ_{pq} define in the derived category $D^+(S)$ homomorphisms

$$\delta_p : Rf_*\mathbb{Q} \longrightarrow Rf_*\mathbb{Q}$$

such that $\mathcal{H}^q(\delta_p)$ is zero for $p \neq q$, and is the identity for $p = q$. By [2], one therefore has in $D^+(S)$

$$(4.1.1.1) \quad Rf_*\mathbb{Q} \approx \sum_p R^p f_*\mathbb{Q}[-p],$$

and the Leray spectral sequence degenerates.

Corollary (4.1.2). — *Let $f : X \rightarrow S$ be a proper smooth morphism whose target is a reduced connected separated scheme S . Let $(R^n f_*\mathbb{Q})^0$ be the largest constant local subsystem of $R^n f_*\mathbb{Q}$, with fiber $H^0(S, R^n f_*\mathbb{Q})$. Then, for each $s \in S$, $(R^n f_*\mathbb{Q})_s^0$ is a Hodge substructure of $(R^n f_*\mathbb{Q})_s \simeq H^n(X_s, \mathbb{Q})$, and the Hodge structure induced on $H^0(S, R^n f_*\mathbb{Q})$ is independent of s .*

Let $s \in S$ and $X_s = f^{-1}(s)$. If S is smooth, and if $\bar{X}$ is a smooth compactification of X , then by (4.1.1) the subspace $(R^n f_*\mathbb{Q})_s^0$ of $(R^n f_*\mathbb{Q})_s$ is the image of $H^n(\bar{X}, \mathbb{Q})$. Since the restriction map

$$H^n(\bar{X}, \mathbb{Q}) \longrightarrow H^n(X_s, \mathbb{Q})$$

is a morphism of Hodge structures, its image is a Hodge substructure, and the Hodge structure induced on $H^0(S, R^n f_*\mathbb{Q})$, a quotient of that of $H^n(\bar{X}, \mathbb{Q})$, is independent of s .

In the general case, (4.1.2) also means that if a is a global section of $R^n f_*\mathbb{C}$, then its components $a^{p,q}$ of type (p, q) , *a priori* only continuous sections of the complex vector bundle defined by $R^n f_*\mathbb{C}$, are in fact locally constant, i.e. are sections of $R^n f_*\mathbb{C}$. This is so because $a^{p,q}$ is continuous and locally constant on the smooth locus (which is dense) of S , by what precedes.

(4.1.3) In the case where S is compact, a generalization of (4.1.2) is proved analytically in Griffiths [5].

As corollaries of (4.1.2), let us cite:

(4.1.3.1) (Griffiths [5]) Let $f : X \rightarrow S$ be a proper smooth morphism as in (4.1.2). If a global section a of $R^n f_*\mathbb{C}$ is of Hodge type (p, q) at one point, then a is of type (p, q) everywhere. In particular, if $n = 2$ and if, at one point s , a is the cohomology class of a divisor of X_s , then a is at every point the cohomology class of a divisor and, for S smooth, is even defined by a divisor D on X .

(4.1.3.2) (Grothendieck [7]) Let S be a reduced connected scheme of finite type over $\mathbb{C}$, and let $f_1 : X_1 \rightarrow S$ and $f_2 : X_2 \rightarrow S$ be two abelian schemes over S . If a morphism

$$u : R^1 f_{2*}\mathbb{Z} \longrightarrow R^1 f_{1*}\mathbb{Z}$$

comes at one point s of S from a morphism of abelian varieties

$$\tilde{u}_s : (X_1)_s \longrightarrow (X_2)_s,$$

then u comes from one (and only one) morphism of abelian schemes

$$\tilde{u} : X_1 \longrightarrow X_2.$$

(4.1.3.3) (Cf. Katz [10]) Let S be a smooth connected scheme, let s be a point of S , let $f : X \rightarrow S$ be a proper smooth morphism, and let P be a direct factor of the local system $R^i f_* \mathbb{Q}$ which is pointwise a Hodge substructure. The following conditions are equivalent:

- a) The Hodge structure of P is locally constant.
- b) The representation P_s of $\pi_1(S, s)$ factors through a finite quotient of $\pi_1(S, s)$.
- c) There exists a nonempty finite étale covering $u : S' \rightarrow S$ such that $u^* P$ is a constant family of Hodge structures.

4.2. The semisimplicity theorem.

(4.2.1) Let S be a topological space. A *continuous family of Hodge structures* on S consists of:

- a) A local system $H_{\mathbb{Z}}$ of finitely generated $\mathbb{Z}$ -modules on S .
- b) For every point $s \in S$, a Hodge structure on the fiber $(H_{\mathbb{Z}})_s$, this structure varying continuously with s .

A continuous family H of Hodge structures on S is said to be of *weight* n if the fibers H_s ($s \in S$) have weight n .

Similarly, a *continuous family of $\mathbb{Q}$ -Hodge structures* is defined as a local system of $\mathbb{Q}$ -vector spaces endowed at each point with a continuously varying $\mathbb{Q}$ -Hodge structure.

A *polarization* of a continuous family H of $\mathbb{Q}$ -Hodge structures of weight n is a morphism of local systems from $H_{\mathbb{Q}} \otimes H_{\mathbb{Q}}$ into the constant local system $\mathbb{Q}(-n)_{\mathbb{Q}}$, which at every point $s \in S$ defines a polarization of H_s .

(4.2.2) Suppose S is connected, and let $\mathcal{C}$ be a strictly full subcategory of the category of continuous families of $\mathbb{Q}$ -Hodge structures on S . We shall consider the following conditions:

- (4.2.2.1) $\mathcal{C}$ is stable under direct factors, direct sums, and tensor products; the constant Tate families $\mathbb{Q}(n)$ ($n \in \mathbb{Z}$) belong to $\mathcal{C}$.
- (4.2.2.2) Every homogeneous Hodge structure (= of some weight n) in $\mathcal{C}$ is polarizable.
- (4.2.2.3) For every $H \in \text{Ob } \mathcal{C}$, there exists a local system $H'_{\mathbb{Z}}$ of free $\mathbb{Z}$ -modules on S such that $H'_{\mathbb{Z}} \otimes \mathbb{Q} \approx H_{\mathbb{Q}}$.
- (4.2.2.4) For every H in $\mathcal{C}$, the largest constant local subsystem H' of H is a constant family of Hodge substructures of H .

Lemma (4.2.3). — *If $\mathcal{C}$ satisfies conditions (4.2.2.1) and (4.2.2.2), then:*

- (i) $\mathcal{C}$ is a semisimple abelian subcategory of the abelian category of continuous families of $\mathbb{Q}$ -Hodge structures on S .
- (ii) If $H \in \text{Ob } \mathcal{C}$, then its dual H^* and the $\bigwedge^p H$ belong to $\mathcal{C}$; if $H_1, H_2 \in \text{Ob } \mathcal{C}$, then $\text{Hom}(H_1, H_2) \in \text{Ob } \mathcal{C}$.

If $H \in \text{Ob } \mathcal{C}$ and H_1 is a subobject of H , in the category of continuous families of $\mathbb{Q}$ -Hodge structures, we prove that H_1 is a direct factor of H in that category. We may suppose H homogeneous. If ψ is a polarization form for H , the orthogonal complement of H_1 for ψ is indeed a subobject of H complementary to H_1 . This proves (i).

If $H \in \text{Ob } \mathcal{C}$ has weight n , a polarization of H defines an isomorphism between H^* and $H \otimes \mathbb{Q}(n)$. By (4.2.2.1), one therefore has $H^* \in \text{Ob } \mathcal{C}$. For arbitrary H in $\mathcal{C}$, decomposing H into its homogeneous components, one has

$$H^* = \bigoplus_n (H^n)^*,$$

whence again $H^* \in \text{Ob } \mathcal{C}$. Finally, $\bigwedge^p H$ is a direct factor in $\bigotimes^p H$, and $\text{Hom}(H_1, H_2) \simeq H_1^* \otimes H_2$.

Let $f : X \rightarrow S$ be a proper smooth morphism of reduced schemes. The sheaf $R^i f_* \mathbb{Q}$ is then a local system, and for $s \in S$,

$$(R^i f_* \mathbb{Q})_s \simeq H^i(X_s, \mathbb{Q})$$

is endowed with a $\mathbb{Q}$ -Hodge structure. This varies continuously with s .

Definition (4.2.4). — *Let S be a smooth connected scheme. A continuous family H of $\mathbb{Q}$ -Hodge structures on S^{an} will be called algebraic if there exist a nonempty Zariski open subset U of S , an integer k , and a projective smooth morphism $f : X \rightarrow U$ such that $H|_U$ is a direct factor of $Rf_* \mathbb{Q} \otimes \mathbb{Q}(k)$.*

Proposition (4.2.5). —

- (i) The category of algebraic continuous families of $\mathbb{Q}$ -Hodge structures on S satisfies the conditions of (4.2.2).
- (ii) If a continuous family H of Hodge structures on S is such that its restriction to a dense Zariski open subset U of S is algebraic, then H is algebraic.
- (iii) If $f : X \rightarrow S$ is proper and smooth, then $Rf_*\mathbb{Q}$ is algebraic.

Assertion (ii) is evident. We prove (i). Let $\mathcal{C}_0$ be the set of continuous families of Hodge structures on S that are of the form $Rf_*\mathbb{Q}$ (f projective and smooth). Then:

a) By the Künneth formula,

$$R(f \times g)_*\mathbb{Q} \simeq Rf_*\mathbb{Q} \otimes Rg_*\mathbb{Q},$$

$\mathcal{C}_0$ is stable under tensor product.

b) $\mathcal{C}_0$ is stable under direct sum:

$$R(f \amalg g)_*\mathbb{Q} \simeq Rf_*\mathbb{Q} \oplus Rg_*\mathbb{Q}.$$

c) The homogeneous components of every $H \in \mathcal{C}_0$ are polarizable (cf. (2.2.6)).

d) The objects $H \in \mathcal{C}_0$ satisfy (4.2.2.3), since

$$Rf_*\mathbb{Q} \simeq Rf_*\mathbb{Z} \otimes \mathbb{Q}.$$

e) The objects $H \in \mathcal{C}_0$ satisfy (4.2.2.4), by (4.1.2).

Let $\mathcal{C}_1$ be the set of direct factors of objects of $\mathcal{C}_0$. Then $\mathcal{C}_1$ is stable under tensor product, direct sum, and direct factor, and satisfies (4.2.2.2) to (4.2.2.4). Moreover $\mathbb{Q}(-1)$ belongs to $\mathcal{C}_1$, since it is of the form $R^2f_*\mathbb{Q}$ for the projective bundle $f : \mathbb{P}_S^1 \rightarrow S$. The category $\mathcal{C}_2$ of the $H \otimes \mathbb{Q}(k)$ ($k \in \mathbb{N}$) therefore satisfies (4.2.2.1) to (4.2.2.4).

To deduce (i), it suffices to note that if H is a continuous family of Hodge structures, and U is a dense Zariski open subset of S , then a subobject, a polarization, or an integral lattice of $H|_U$ extends uniquely to H .

We prove (iii). If $f : X \rightarrow S$ is projective and smooth, then there exist a dense Zariski open subset U of S and a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{p} & X|_U \\ & \searrow f' & \swarrow f|_U \\ & U & \end{array}$$

with f' projective and smooth and p birational and surjective (apply Chow's lemma and resolution of singularities to the generic fiber of f). The restriction map

$$p^* : (Rf_*\mathbb{Q})|_U \longrightarrow Rf'_*\mathbb{Q}$$

is then a split injection of continuous families of Hodge structures, with left inverse the Gysin morphism p_* ; hence $Rf_*\mathbb{Q}$ is algebraic.

Theorem (4.2.6)⁽¹⁾. — Let S be a connected, locally connected and locally simply connected topological space, endowed with a base point s . Let $\mathcal{C}$ be a category of continuous families of Hodge structures on S satisfying the conditions (4.2.2). Then, if $H \in \text{Ob } \mathcal{C}$, the representation of $\pi_1(S, s)$ on the fiber $(H_{\mathbb{Q}})_s$ is semisimple.

Let $H \in \text{Ob } \mathcal{C}$. The local system $H_{\mathbb{Q}}$ defines a complex local system $H_{\mathbb{C}} = H_{\mathbb{Q}} \otimes \mathbb{C}$. For every complex local system V on S , denote by V^c the complex vector bundle which it defines, identified with the sheaf of its continuous sections.

By definition, the group S (2.1.2) acts on $H_{\mathbb{C}}^c$. A subbundle of $H_{\mathbb{C}}^c$ will be called *horizontal*, or *locally constant*, if it is defined by a local subsystem of $H_{\mathbb{C}}$.

⁽¹⁾ (Added in proof.) Let S be a smooth connected scheme. A family of Hodge structures on S is a continuous family H of Hodge structures on S satisfying the following conditions:

- a) The Hodge filtration of $(H_{\mathbb{C}})_s$ varies holomorphically with s , i.e. corresponds to a filtration F of $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C}$.
- b) The covariant derivative ∇ satisfies $\nabla F^p \subset \Omega_S^1 \otimes F^{p-1}$.

Let $\mathcal{C}$ be the category of those continuous families of $\mathbb{Q}$ -Hodge structures on S which underlie a family of Hodge structures, and whose homogeneous direct factors are polarizable. It is clear that $\mathcal{C}$ satisfies (4.2.2.1) to (4.2.2.3). From results of W. Schmid (August 1970, unpublished) and P. A. Griffiths [5] one can deduce that $\mathcal{C}$ satisfies (4.2.2.4). This theorem, of which (4.1.2) is a corollary, makes it possible to apply (4.2.6) and its corollaries to objects of $\mathcal{C}$.

Lemma (4.2.7). — *Let V be a rank-one local subsystem of $H_{\mathbb{C}}$. Suppose that a tensor power $V^{\otimes n}$ ($n \geq 1$) of V is a trivial local system, i.e. that $\pi_1(S, s)$ acts on V_s through a finite group (necessarily cyclic). Then, for $t \in S$, tV^c is again locally constant.*

For tV^c to be locally constant, it suffices that

$$(tV^c)^{\otimes n} = tV^{c \otimes n} \subset \bigotimes^n H_{\mathbb{C}}$$

be locally constant. But $V^{\otimes n}$ is generated by a global horizontal section v , and by hypothesis tv is again horizontal (4.2.2.4).

We proceed by induction on $\dim(H_{\mathbb{Q}})_s$. We may suppose H homogeneous and nonzero. Let d be the minimal dimension of the nonzero complex local subsystems of $H_{\mathbb{C}}$. The sum W of all local subsystems of $H_{\mathbb{C}}$ of dimension d (which are automatically simple) is “defined over $\mathbb{Q}$ ”, i.e. is of the form $W_{\mathbb{Q}} \otimes \mathbb{C}$ for $W_{\mathbb{Q}}$ a local subsystem of $H_{\mathbb{Q}}$. By construction, W_s is a semisimple complex $\pi_1(S, s)$ -module, hence $(W_{\mathbb{Q}})_s$ is a semisimple $\pi_1(S, s)$ -module over $\mathbb{Q}$.

Let $H_{\mathbb{Z}}$ be a local system of free $\mathbb{Z}$ -modules such that $H_{\mathbb{Z}} \otimes \mathbb{Q} \simeq H_{\mathbb{Q}}$, and let $W_{\mathbb{Z}} = H_{\mathbb{Z}} \cap W_{\mathbb{Q}}$. If W has dimension e , the local system $(\bigwedge^e W)^{\otimes 2}$ is trivial. Indeed:

$$\bigwedge^e W \simeq \bigwedge^e W_{\mathbb{Z}} \otimes \mathbb{C},$$

and $\pi_1(S, s)$ can act on $(\bigwedge^e W_{\mathbb{Z}})_s$ only by ± 1 .

Let V be a complex local subsystem of dimension d of $H_{\mathbb{C}}$, and let V' be a complement of V in W (semisimplicity of W). One has

$$\bigwedge^e W \simeq \bigwedge^d V \otimes \bigwedge^{e-d} V'.$$

Apply (4.2.7) to the local subsystem $\bigwedge^d V \otimes \bigwedge^{e-d} V'$ of $\bigwedge^d H_{\mathbb{C}} \otimes \bigwedge^{e-d} H_{\mathbb{C}}$. We find that, for $t \in S$,

$$t(\bigwedge^d V \otimes \bigwedge^{e-d} V')^c = \bigwedge^d tV^c \otimes \bigwedge^{e-d} tV'^c \subset \bigwedge^d H_{\mathbb{C}} \otimes \bigwedge^{e-d} H_{\mathbb{C}}$$

is locally constant. Hence $\bigwedge^d tV^c \subset \bigwedge^d H_{\mathbb{C}}$ and $tV^c \subset H_{\mathbb{C}}$ are locally constant.

By definition of W , one has $tV^c \subset W^c$: W^c is stable under S , and $W_{\mathbb{Q}}$ is a Hodge substructure of H . If ψ is a polarization of H , H is therefore the direct sum of W and of its orthogonal complement, both in $\mathcal{C}$, and the induction concludes.

Corollary (4.2.8). — *Under the hypotheses of (4.2.6):*

- (i) *The algebra A of endomorphisms of the local system $H_{\mathbb{Q}}$ is semisimple. It admits one (and only one) $\mathbb{Q}$ -Hodge structure such that, at each point s , $A \otimes H_s \rightarrow H_s$ is a morphism of Hodge structures.*
- (ii) *The center of A is of type $(0, 0)$ and the composition law $\cdot : A \otimes A \rightarrow A$ is a morphism of Hodge structures.*
- (iii) *Let W be a complex local subsystem of dimension d of $H_{\mathbb{C}}$:*

- a) *For $t \in S$, $tW^c \subset H_{\mathbb{C}}$ is locally constant and defines a local subsystem tW isomorphic to W .*
- b) *A power $(\bigwedge^d W)^{\otimes n}$ ($n \geq 1$) of the local system $\bigwedge^d W$ is a trivial local system.*

The algebra A is the commutant of $\pi_1(S, s)$ in $(H_{\mathbb{Q}})_s$. Its semisimplicity therefore follows from (4.2.6) and Bourb., *Alg.*, chap. 8, § 5, no. 2, prop. 4. Moreover, A is the fiber of the largest constant local subsystem of $\text{Hom}(H, H)$. Since $\text{Hom}(H, H) \in \mathcal{C}$ ((4.2.3), (ii)), the existence of the Hodge structure in (i) follows from hypothesis (4.2.2.4). It is clear that the composition law is a morphism of Hodge structures. It follows that the center Z of A is a Hodge substructure: $Z_{\mathbb{C}}$ is bigraded, as the intersection of the kernels of the bihomogeneous morphisms $[x, \cdot]$ for x bihomogeneous. Finally Z , and hence $Z_{\mathbb{C}}$, is semisimple, and for $(p, q) \neq (0, 0)$, Z^{pq} is zero because it is nilpotent.

A complex local subsystem W of $H_{\mathbb{C}}$ is defined by a projector e of $A_{\mathbb{C}}$. For $t \in S$, tW^c is defined by the projector $t.e$, and is therefore locally constant. Since the group S is connected, to verify that tW is isomorphic

to W , it suffices to verify that for t in a sufficiently small neighborhood of 1, the $\mathbb{C}[\pi_1(S, s)]$ -module tW_s is isomorphic to W_s . Let H_i be the isotypic components of the $\mathbb{C}[\pi_1(S, s)]$ -module $(H_{\mathbb{C}})_s$. One has

$$W_s = \bigoplus_i (W_s \cap H_i)$$

and

$$tW_s = \bigoplus_i (tW_s \cap H_i).$$

Moreover, for t sufficiently close to 1, tW_s is close to W_s in the Grassmannian, and hence

$$(4.2.8.1) \quad \dim(tW_s \cap H_i) \leq \dim(W_s \cap H_i).$$

In fact, since $\dim(tW_s) = \dim(W_s)$, equality holds in (4.2.8.1). The respective isotypic components of W_s and tW_s therefore have the same length, and W_s is isomorphic to tW_s .

We prove (iii), *b*). It suffices to treat the case where H is homogeneous and W_s is a simple $\mathbb{C}[\pi_1(S, s)]$ -module. Let H_1 be the isotypic component of $(H_{\mathbb{C}})_s$ which contains W_s . By *a*), H_1 is stable under S . If W_s has dimension d and H_1 has length k , one has

$$\bigwedge^{kd} H_1 \simeq \left(\bigwedge^d W_s \right)^{\otimes k}$$

as a $\mathbb{C}[\pi_1(S, s)]$ -module. If χ is the character of $\pi_1(S, s)$ defined by $\bigwedge^d W$, the character χ_1 defined by $\bigwedge^{kd} H_1$ is therefore χ^k . Put $H_2 = H_1 + \overline{H_1}$. Since H_1 is stable under S , H_2 is defined by a real Hodge substructure of $(H_{\mathbb{R}})_s$. Every polarization form on H therefore induces on H_2 a nondegenerate bilinear form invariant under $\pi_1(S, s)$.

If $\dim(H_2) = e$, the representation $(\bigwedge^e H_2)^{\otimes 2}$ of $\pi_1(S, s)$ is therefore trivial. Thus:

- a) For H_1 real: χ^{2k} is trivial;
- b) For H_1 nonreal ($H_1 \neq \overline{H_1}$): $\chi^{2k} \cdot \bar{\chi}^{2k}$ is trivial.

In both cases one has $|\chi| = 1$.

The representation of $\pi_1(S, s)$ on $(H_{\mathbb{C}})_s$ is obtained by extension of scalars from a representation defined over $\mathbb{Q}$ (namely $(H_{\mathbb{Q}})_s$). The conjugate representations of W therefore occur in $(H_{\mathbb{C}})_s$; there are at most $N = \dim(H)$ of them, and for every automorphism σ of $\mathbb{C}$ one has

$$|\sigma(\chi)| = 1.$$

For $\gamma \in \pi_1(S, s)$, $\chi(\gamma)$ is an algebraic integer with at most N complex conjugates, all of absolute value 1; $\chi(\gamma)$ is therefore a k -th root of unity, with $k \leq N$. Hence $\chi^{N!} = 1$, which proves *b*).

Corollary (4.2.9). — *Let S be a smooth connected separated scheme, endowed with a base point $s \in S$. Let $f : X \rightarrow S$ be a morphism of schemes such that $R^n f_* \mathbb{Q}$ is a local system on S , let G be the Zariski closure of the image of $\pi_1(S, s)$ in $\text{Aut}_{\mathbb{C}}((R^n f_* \mathbb{C})_s)$, and let G^0 be the identity component of G . Then:*

- a) *If f is proper and smooth, G^0 is semisimple.*
- b) *In general, G^0 has no quotient of multiplicative type (i.e., the radical of G^0 is unipotent).*

We prove *a*). After replacing S by a finite étale covering, we reduce to the case $G = G^0$. By ((4.2.5), (iii)), the local system $R^n f_* \mathbb{Q}$ underlies an algebraic family H of Hodge structures, so (4.2.8) applies to it ((4.2.5), (i)). The group G is reductive, since it has a faithful semisimple representation, namely $(R^n f_* \mathbb{C})_s$. If G (assumed connected) were not semisimple, it would admit a nontrivial complex representation ρ of rank one; since H_s is faithful, ρ would be a direct summand of $((H_{\mathbb{C}} + H_{\mathbb{C}}^*)^{\otimes m})_s$ for some m . This contradicts (4.2.8), (iii), *b*), because $(H + H^*)^{\otimes m}$ is algebraic and no tensor power of ρ is trivial.

Let π be a group, and consider the following property of a representation ρ of π on a finite-dimensional complex vector space V :

- (*) The identity component G^0 of the Zariski closure G of $\rho(\pi)$ in $\text{Aut}(V)$ has no quotient of multiplicative type.

Lemma (4.2.10). — *Let $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$ be an exact sequence of representations of the preceding kind. Then V satisfies $(*)$ if (and only if) V' and V'' satisfy $(*)$.*

Let $G, G',$ and G'' be the groups defined by $V, V',$ and V'' . Since, like π, G leaves V' invariant, there is a morphism

$$u = (u', u'') : G \longrightarrow G' \times G'',$$

with unipotent kernel, and such that u' and u'' are surjective. Let N be the identity component of the kernel of u'' . Since u' is surjective, $u'(N)$ is normal in G' , and therefore has no quotient of multiplicative type. If χ is a character of G^0 , then χ factors through $u(G^0)$, is trivial on N , a power of χ factors through G''^0 , and χ is trivial.

We prove (4.2.9), *b*), by dévissage and induction on the relative dimension of f . We may suppose X reduced.

First suppose that X is quasi-projective. Then there exist a nonempty open subset U of S , a dense open subset X' of $X_U = f^{-1}(U)$ smooth over U , and a compactification $\bar{X}'$ of X' over U , proper and smooth over U . One can choose $\bar{X}'$ so that an open X'' of $\bar{X}'$ contains X' and is proper over X_U

$$\begin{array}{ccccc}
 X_U & \xleftarrow{p} & X'' & \xhookrightarrow{k} & \bar{X}' \\
 & \searrow i & \nearrow j & \nearrow & \nearrow \\
 & & X' & & \\
 & \nearrow f_U & \searrow f' & \searrow f'' & \searrow \bar{f}' \\
 & & & U &
 \end{array}$$

We have long exact sequences of sheaves

$$\begin{aligned}
 &\rightarrow R^i f_{U*}(i_! \mathbb{Q}) \rightarrow R^i f_{U*} \mathbb{Q} \rightarrow R^i f_{U*}(\mathbb{Q}/i_! \mathbb{Q}) \rightarrow, \\
 &\rightarrow R^i f''_*(j_! \mathbb{Q}) \rightarrow R^i f''_* \mathbb{Q} \rightarrow R^i f''_*(\mathbb{Q}/j_! \mathbb{Q}) \rightarrow, \\
 &\rightarrow R^i f'_! \mathbb{Q} \rightarrow R^i \bar{f}'_* \mathbb{Q} \rightarrow R^i \bar{f}'_*(\mathbb{Q}/k_! \mathbb{Q}) \rightarrow.
 \end{aligned}$$

For U sufficiently small, these sheaves are local systems and the induction hypothesis applies to $R^* f_{U*}(\mathbb{Q}/i_! \mathbb{Q})$, to $R^* f''_*(\mathbb{Q}/j_! \mathbb{Q})$, and to $R^* \bar{f}'_*(\mathbb{Q}/k_! \mathbb{Q})$. By *a*), condition $(*)$ is satisfied by $R^* \bar{f}'_* \mathbb{Q}$. By (4.2.10), it is therefore satisfied by $R^* f''_* \mathbb{Q}$, and by the dual local system $R^* f''_* \mathbb{Q}$ (Poincaré duality). By (4.2.10), condition $(*)$ is satisfied by $R^* f''_*(j_! \mathbb{Q})$, which is isomorphic to $R^* f_{U*}(i_! \mathbb{Q})$ because p is proper, and also by $R^* f_{U*} \mathbb{Q}$. Since the morphism $\pi_1(U) \rightarrow \pi_1(S)$ is surjective, $R^i f_* \mathbb{Q}$ satisfies (4.2.9), *b*).

To pass from the quasi-projective case to the general case, it suffices to use the Leray spectral sequence for a covering (U_i) of X by quasi-projective open subsets:

$$R^* f_* \left(\bigcap_i U_i, \mathbb{Q} \right) \Rightarrow R^* f_* \mathbb{Q},$$

and (4.2.10).

4.3. Complement to [2].

In [2], (5.4) (footnote), I assert without proof that

Proposition (4.3.1). — *Let X be a proper smooth scheme over $\mathbb{C}$. If a C^∞ form on X satisfies $d'\alpha = d''\alpha = 0$ and is cohomologous to zero, then there exists β such that $\alpha = d'd''\beta$.*

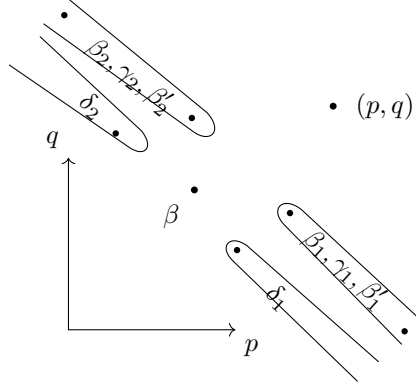
Here is the proof. As in *loc. cit.*, we reduce to the case where α is bihomogeneous, of bidegree (p, q) . Since the spectral sequence of the double complex $H^0(X, \mathcal{A}^{p,q})$ degenerates, the exterior differential d is strictly compatible with the filtration by first degree p (1.3.2). If $\alpha \sim 0$, there is therefore a form β_1 such that $d\beta_1 = \alpha$ and such that the components $\beta_1^{p',q'}$ of β_1 are zero for $p' < p$.

By complex conjugation, there similarly exists β_2 such that $d\beta_2 = \alpha$ and $\beta_2^{p',q'} = 0$ for $q' < q$. One has $d(\beta_1 - \beta_2) = 0$.

Let γ be a form satisfying $d'\gamma = d''\gamma = 0$ in the cohomology class of $\beta_1 - \beta_2$ (*loc. cit.*). Let γ_1 (resp. γ_2) be the sum of the components of γ of type (p', q') with $p' \geq p$ (resp. $q' \geq q$). Setting $\beta'_1 = \beta_1 - \gamma_1$ and $\beta'_2 = \beta_2 + \gamma_2$, one still has $d\beta'_1 = d\beta'_2 = \alpha$; moreover, $\beta'_1 - \beta'_2 = \beta_1 - \beta_2 - \gamma$ is cohomologous to zero: there exists δ such that

$$d\delta = \beta'_1 - \beta'_2.$$

Let δ_1 be the sum of the components of type (p', q') of δ with $p' < p - 1$, let δ_2 be the sum of the components for which $q' > q - 1$, and let β be the component of type $(p - 1, q - 1)$.



One has $\beta'_2 = d\delta_2 + d''\beta$ and $\alpha = d\beta'_2 = dd\delta_2 + dd''\beta = d'd''\beta$.

4.4. Homomorphisms of abelian schemes.

We propose to show how, in some cases, one can remove from (4.1.3.2) the hypothesis of algebraicity at one point.

(4.4.1) For $f : X \rightarrow S$ a proper smooth morphism, we shall denote by $R_i f_* \mathbb{Z}$ the local system on S^{an} of the homology of the fibers of f . At each point $s \in S$, $(R_i f_* \mathbb{Z})_s \simeq H_i(X_s, \mathbb{Z})$ is endowed with a Hodge structure of weight $-i$, dual to the Hodge structure of $H^i(X_s, \mathbb{Z})$. This structure varies continuously with $s \in S$.

(4.4.2) If $f : X \rightarrow S$ is an abelian scheme, the exponential defines an exact sequence of sheaves on S^{an}

$$0 \rightarrow R_1 f_* \mathbb{Z} \rightarrow \text{Lie}(X) \rightarrow X \rightarrow 0,$$

and the Hodge filtration is the short exact sequence

$$0 \rightarrow \text{Ker}(\alpha) \rightarrow R_1 f_* \mathbb{Z} \otimes \mathcal{O} \xrightarrow{\alpha} \text{Lie}(X) \rightarrow 0.$$

It is known that

Reminder (4.4.3). — *Let S be a smooth scheme of finite type over $\mathbb{C}$. The construction (4.4.2) establishes an equivalence of categories between:*

- a) *the category of abelian schemes over S ;*
- b) *the category of polarizable continuous families of Hodge structures H of type $(-1, 0) + (0, -1)$ on S , such that $H_{\mathbb{Z}}$ is torsion-free and the Hodge filtration varies holomorphically on S .*

Essential surjectivity of the functor a) $\rightarrow$ b) follows from Borel [13]; full faithfulness follows from the fact that, for X_1 and X_2 two abelian schemes over S , the S -scheme $\text{Hom}_S(X_1, X_2)$ is unramified over S , and therefore has the same algebraic and holomorphic sections.

(4.4.4) Let Z be a commutative field, H a central simple algebra over Z , and $*$ a Z -linear anti-automorphism of H . After extension of scalars, one has $H \simeq \text{End}(V)$ for a vector space V over Z , and $*$ is then transposition with respect to a bilinear form Φ on V , unique up to scalar. Suppose that $*$ is involutive; the forms $\Phi(X, Y)$ and $\Phi(Y, X)$ are then proportional: $\Phi(X, Y) = \lambda \Phi(Y, X)$, with $\lambda^2 = 1$. Put $\varepsilon_* = \lambda$. If $[H : Z] = d^2$, then

$$\text{Tr}(* : H \rightarrow H) = \varepsilon_* \cdot d.$$

Every Z -linear automorphism C of H is inner: $Cx = cxc^{-1}$. If C is involutive and commutes with $*$, the elements c^2 and cc^* are central, so that $c^* = \lambda c$ with λ central. Since $c^{**} = c$, one has $\lambda^2 = 1$. Put $\varepsilon_C = \lambda$.

One checks easily that the involution $C \circ *$ satisfies

$$\varepsilon_{C \circ *} = \varepsilon_C \cdot \varepsilon_*.$$

Reminder (4.4.5). — *Let X be a simple abelian variety over a field K and let H be the division algebra $\text{End}_K(X) \otimes \mathbb{Q}$. Every polarization of X defines a positive involution $*$ of H :*

$$\text{Tr}(hh^*) > 0 \quad \text{if } h \neq 0.$$

If Z_0 is the subfield invariant under $$ in the center Z of H , then Z_0 is totally real and one is in one of the following cases:*

- a) $Z = Z_0$; for every real place ρ of Z , $H \otimes_\rho \mathbb{R}$ is a matrix algebra over $\mathbb{R}$, and $\varepsilon_* = 1$.*
- b) $Z = Z_0$; for every real place ρ of Z , $H \otimes_\rho \mathbb{R}$ is a matrix algebra over $\mathbb{H}$, and $\varepsilon_* = -1$.*
- c) Z is a totally imaginary quadratic extension of Z_0 .*

(4.4.6) Let $f_i : X_i \rightarrow S$ ($i = 1, 2$) be two abelian schemes over a smooth scheme S . The free abelian group

$$\text{Hom}(R_1 f_{1*} \mathbb{Z}, R_1 f_{2*} \mathbb{Z}) \simeq \Gamma(S, \text{Hom}(R_1 f_{1*} \mathbb{Z}, R_1 f_{2*} \mathbb{Z}))$$

is then endowed with a Hodge structure of weight 0 (4.2.8). Moreover, by (2.1.11.1) and (4.4.3), the group $\text{Hom}_S(X_1, X_2)$ is the component of type $(0, 0)$ of this group.

(4.4.7) Let S be a nonempty smooth connected scheme with generic point η , and let $f : X \rightarrow S$ be an abelian scheme over S . The center Z of $H = \text{End}(R_1 f_* \mathbb{Z})$ is then of type $(0, 0)$ (4.2.8), and is therefore contained in $\text{End}_S(X) \simeq \text{End}_{k(\eta)}(X_\eta)$.

A polarization of X defines an involution $*$ of H . Its restriction to $\text{End}_S(X)$ is positive, so that Z is a product of totally real fields or totally imaginary quadratic extensions of totally real fields.

Let C denote the actions of $i \in S(\mathbb{R})$ on $H_{\mathbb{R}} = H \otimes \mathbb{R}$ and on the $(R_1 f_* \mathbb{R})_s$, for $s \in S$. On $H_{\mathbb{R}}$, $C^2 = 1$, and C commutes with $*$; on $(R_1 f_* \mathbb{R})_s$, $C^2 = -1$. If ψ is the alternating form on $R_1 f_* \mathbb{Z}$, with integral values, deduced from the polarization of X , then

$$\psi(hx, Cy) = \psi(x, h^* Cy) = \psi(x, C \cdot C(h^*) \cdot y).$$

It follows that

$$\text{Tr}(h \cdot C(h^*)) > 0 \quad \text{for } h \neq 0. \quad (4.4.7.1)$$

(4.4.8) Now suppose that X_η is simple, so that Z is a field. Let $\rho : Z \rightarrow \mathbb{C}$ be a complex embedding. The algebra

$$H_\rho = H \otimes_{Z, \rho} \mathbb{C}$$

is then a matrix algebra and a bigraded direct factor in $H_{\mathbb{C}} = H \otimes_{\mathbb{Q}} \mathbb{C}$. Choose an isomorphism

$$H_\rho \simeq \text{End}(V_\rho). \quad (4.4.8.1)$$

Since the infinitesimal generators of S act on H_ρ by derivations, necessarily inner, it follows that V_ρ admits a bigrading, unique up to translation, such that (4.4.8.1) is a bigraded isomorphism. Let $(R_1 f_* \mathbb{C})_\rho$ be the bigraded local system

$$(R_1 f_* \mathbb{C}) \otimes_{Z \otimes \mathbb{C}, \rho} \mathbb{C},$$

a direct factor of $R_1 f_* \mathbb{C}$. Denote by T_ρ the local system

$$T_\rho = \text{Hom}_{H_\rho}(V_\rho, (R_1 f_* \mathbb{C})_\rho).$$

One has

$$V_\rho \otimes_{\mathbb{C}} T_\rho \xrightarrow{\sim} (R_1 f_* \mathbb{C})_\rho \quad (4.4.8.2)$$

(a bigraded isomorphism). Since $(R_1 f_* \mathbb{C})_\rho$ has type $(-1, 0) + (0, -1)$, one of the following conditions is satisfied:

- a) V_ρ is bihomogeneous;
- b) T_ρ is bihomogeneous.

It is clear that condition a) (respectively b)) is simultaneously satisfied for ρ and for $\bar{\rho}$. If condition a) is satisfied at every place, then H is of type $(0,0)$. If condition b) is satisfied at every place, then the Hodge structure on $R_1 f_* \mathbb{Q}$ is locally constant.

(4.4.9) Now suppose that Z is totally real. For every real embedding φ of Z , put $\varepsilon_{H,\varphi} = +1$ if $H \otimes_{Z,\varphi} \mathbb{R}$ is a matrix algebra and $\varepsilon_{H,\varphi} = -1$ otherwise. If a complex embedding ρ induces a real embedding φ , put $\varepsilon_{H,\rho} = \varepsilon_{H,\varphi}$.

Let $\rho : Z \rightarrow \mathbb{C}$ factor through $\varphi : Z \rightarrow \mathbb{R}$. Complex conjugation on H_ρ is induced by an antilinear automorphism σ_1 of V_ρ , unique up to a real scalar factor, with scalar square. Since σ_1 commutes with σ_1^2 , σ_1^2 is real and, normalizing σ_1 , one may suppose that $\sigma_1^2 = \pm 1$. Then

$$\sigma_1^2 = \varepsilon_{H,\rho}. \quad (4.4.9.1)$$

The automorphism of complex conjugation on

$$(R_1 f_* \mathbb{C})_\rho = ((R_1 f_* \mathbb{Q}) \otimes_{Z,\varphi} \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$$

is “compatible” with (4.4.8.2) and may be written

$$\sigma = \sigma_1 \otimes \sigma_2.$$

Since $\sigma^2 = 1$, by (4.4.9.1) one has

$$\sigma_1^2 = \sigma_2^2 = \varepsilon_{H,\rho}. \quad (4.4.9.2)$$

Let Φ be a polarization form on $R_1 f_* \mathbb{Z}$. This form is necessarily of the type

$$\Phi = \text{Tr}_{Z/\mathbb{Q}}(\psi),$$

where ψ is a Z -bilinear alternating form on $R_1 f_* \mathbb{Q}$.

Let ψ_ρ be the induced form on $V_\rho \otimes_{\mathbb{C}} T_\rho$. The alternating form ψ_ρ is invariant under $\pi_1(S, s)$. Since T_ρ is irreducible, it can be written

$$\psi_\rho = A_\rho \otimes B_\rho, \quad (4.4.9.3)$$

with A_ρ on V_ρ , and B_ρ on T_ρ , invariant under $\pi_1(S, s)$.

Since T_ρ is irreducible, B_ρ is symmetric or alternating. Since ψ_ρ is alternating, A_ρ is then alternating or symmetric. Since the T_ρ are conjugate to one another, which case occurs is independent of ρ . Put

$$A_\rho(X, Y) = \varepsilon_V A_\rho(Y, X), \quad (4.4.9.4)$$

$$B_\rho(X, Y) = \varepsilon_T B_\rho(Y, X). \quad (4.4.9.5)$$

Thus

$$\varepsilon_V \varepsilon_T = -1 \quad (4.4.9.6)$$

and it is clear from (4.4.4) that if $*$ is the involution of H deduced from ψ ,

$$\varepsilon_* = \varepsilon_V. \quad (4.4.9.7)$$

In case (4.4.8), a) (respectively b)), normalize the bigradings so that V_ρ (respectively T_ρ) has bidegree $(0,0)$. Then for nonzero x, y in V_ρ and T_ρ :

$$\text{case a) } A_\rho(x, \sigma_1 x) B_\rho(y, C \sigma_2 y) > 0,$$

$$\text{case b) } A_\rho(x, C \sigma_1 x) B_\rho(y, \sigma_2 y) > 0.$$

Normalizing A_ρ and B_ρ (only their tensor product being given), we may assume that

$$\text{case a) } A_\rho(x, \sigma_1 x) > 0 \quad \text{and} \quad B_\rho(y, C \sigma_2 y) > 0;$$

$$\text{case b) } A_\rho(x, C \sigma_1 x) > 0 \quad \text{and} \quad B_\rho(y, \sigma_2 y) > 0.$$

Moreover, A_ρ and B_ρ , like Φ , are invariant under the compact subtorus $\text{Ker}(N)$ of S , and in particular under C . In case a) one has

$$0 < A_\rho(\sigma_1 x, \sigma_1 \sigma_1 x) = \varepsilon_V \cdot \varepsilon_{H,\rho} \cdot A_\rho(x, \sigma_1 x),$$

whence $\varepsilon_V \cdot \varepsilon_{H,\rho} = 1$. Similarly, in case b), one finds $\varepsilon_T \cdot \varepsilon_{H,\rho} = 1$.

Lemma (4.4.10). If $\varepsilon_T = \varepsilon_{H,\rho}$, we are in case b); if $\varepsilon_T = -\varepsilon_{H,\rho}$, we are in case a).

Proposition (4.4.11). Let S be a nonempty smooth connected scheme with generic point η , let $f : X \rightarrow S$ be an abelian scheme over S , let $H = \text{End}(R_1 f_* \mathbb{Q})$, and let Z be the center of H . Suppose that:

- (a) X_η is simple over $k(\eta)$.
- (b) X does not become constant over any finite covering of S .
- (c) One of the following conditions is satisfied (cf. (4.4.5)):
 - (c₁) Z is imaginary quadratic;
 - (c₂) Z is totally real, and for every real embedding $\varphi : Z \rightarrow \mathbb{R}$, $H \otimes_{Z,\varphi} \mathbb{R}$ is a matrix algebra over $\mathbb{R}$;
 - (c₃) Z is totally real, and for every real embedding $\varphi : Z \rightarrow \mathbb{R}$, $H \otimes_{Z,\varphi} \mathbb{R}$ is a matrix algebra over $\mathbb{H}$.

Then

$$\text{End}(X) \xrightarrow{\sim} \text{End}(R_1 f_* \mathbb{Z}).$$

We show that we are in case (4.4.8), a), or (4.4.8), b) at all places of Z simultaneously. This is clear from (4.4.8) under hypothesis (c₁), and follows from (4.4.10) under hypotheses (c₂) or (c₃), because not only ε_T but also $\varepsilon_{H,\rho}$ is then independent of ρ .

Condition (4.4.8), b) cannot hold at every place of Z , for the Hodge structure would then be locally constant, which by (4.1.3.3) and (4.4.3) contradicts b). Hence condition a) holds at every place of Z , so that H is of type (0,0) (4.4.8). In view of (4.4.3), this completes the proof.

Proposition (4.4.12). Let $f : X \rightarrow S$ be an abelian scheme over a smooth scheme S . The following conditions are equivalent:

- (i) For every abelian scheme $g : Y \rightarrow S$, one has

$$\text{Hom}_S(X, Y) \xrightarrow{\sim} \text{Hom}_S(R_1 f_* \mathbb{Z}, R_1 g_* \mathbb{Z}).$$

- (ii) Condition (i) holds for $X = Y$, and the center Z of $\text{End}_S(X) \otimes \mathbb{Q}$ has no complex place $\rho : Z \rightarrow \mathbb{C}$ such that the direct summand $R_1 f_* \mathbb{Q} \otimes_{Z,\rho} \mathbb{C}$ of $R_1 f_* \mathbb{C}$ is purely of Hodge type $(-1, 0)$.

One reduces to the case where S is nonempty and connected, with generic point η , and where X_η is a simple abelian variety over $k(\eta)$. If (i) or (ii) is satisfied, then $D = \text{End}(X_\eta) \otimes \mathbb{Q}$ is a division algebra, and, for $s \in S$, $(R_1 f_* \mathbb{Q})_s$ is an irreducible representation of $\pi_1(S, s)$.

To prove that (ii) implies (i), one may suppose that Y_η is simple and that $(R_1 g_* \mathbb{Q})_s$ is isotypic of type $(R_1 f_* \mathbb{Q})_s$. One then has an isomorphism of continuous families of $\mathbb{Q}$ -Hodge structures (with D of type (0,0))

$$R_1 g_* \mathbb{Q} \approx R_1 f_* \mathbb{Q} \otimes_D \text{Hom}(R_1 f_* \mathbb{Q}, R_1 g_* \mathbb{Q}).$$

One has $D \otimes \mathbb{C} \simeq M_n(Z \otimes \mathbb{C})$. If $e \in D \otimes \mathbb{C}$ is transformed, under such an isomorphism, into the idempotent e_{11} , one still has an isomorphism of bigraded vector spaces

$$(R_1 g_* \mathbb{C})_s \approx (R_1 f_* \mathbb{C})_s \otimes_{Z \otimes \mathbb{C}} e(\text{Hom}(R_1 f_* \mathbb{C}, R_1 g_* \mathbb{C})).$$

If condition (ii) is satisfied, and if $e(\text{Hom}(R_1 f_* \mathbb{C}, R_1 g_* \mathbb{C}))$ is not purely of type (0,0), then $(R_1 g_* \mathbb{C})_s$ could not be purely of type $(-1, 0) + (0, -1)$, which is absurd. It follows that $\text{Hom}(R_1 f_* \mathbb{Z}, R_1 g_* \mathbb{Z})$ is purely of type (0,0), and (i) follows from (4.4.3) and (2.1.11.1).

Let Z be a totally imaginary quadratic extension of a totally real number field Z^0 . If $Z \otimes \mathbb{R} \simeq \mathbb{C} \oplus \dots \oplus \mathbb{C}$, define an action of S on $Z \otimes \mathbb{R}$ by scalar multiplication, sending $s \in S(\mathbb{R}) \simeq \mathbb{C}^*$ to $(s^2 \cdot N(s)^{-1}, 1, \dots, 1)$. The pair consisting of Z and this action of S on $Z \otimes \mathbb{R}$ is a polarizable Hodge structure Z_ρ , of type $(-1, 1) + (0, 0) + (1, -1)$, depending on Z and on the chosen place $\rho : Z \rightarrow \mathbb{C}$.

Let X be an abelian scheme over S , with X_η simple. If the center Z of $\text{End}(R_1 f_* \mathbb{Q})$ is not totally real, it is a totally imaginary quadratic extension of a totally real number field. If the second hypothesis of (ii) is violated, the direct summand $R_1 f_* \mathbb{Q} \otimes_Z Z_\rho$ of the continuous family of polarizable Hodge structures $R_1 f_* \mathbb{Q} \otimes_{\mathbb{Q}} Z_\rho$ is purely of type $(-1, 0) + (0, -1)$.

By (4.4.3), the choice of an integral lattice in $R_1 f_* \mathbb{Q} \otimes_Z Z_\rho$ defines an abelian scheme $g : Y \rightarrow S$, and $\text{Hom}(R_1 f_* \mathbb{Q}, R_1 g_* \mathbb{Q})$ is not purely of type (0,0), which violates (i).

Corollary (4.4.13). — *Let S be a smooth connected scheme with generic point η , and let $f : X \rightarrow S$ be an abelian scheme over S of relative dimension $g \leq 3$. Suppose that on no finite étale covering of S does X admit a constant abelian subscheme. Then, for every abelian scheme $g : Y \rightarrow S$, one has*

$$\mathrm{Hom}(X, Y) \xrightarrow{\sim} \mathrm{Hom}(R_1 f_* \mathbb{Z}, R_1 g_* \mathbb{Z})$$

and

$$\mathrm{Hom}(Y, X) \xrightarrow{\sim} \mathrm{Hom}(R_1 g_* \mathbb{Z}, R_1 f_* \mathbb{Z}).$$

By duality, it suffices to prove the first assertion. We may also reduce to the case where X_η is simple. We verify that X satisfies one of the conditions (c) of (4.4.11). Put $H = M_c(D)$, where D is a division algebra of rank b^2 over its center Z , which is known to be either totally real or totally imaginary. Putting $a = [Z : \mathbb{Q}]$, one has

$$ab^2c \mid 2g \leq 6. \quad (4.4.13.1)$$

- 1) If Z is totally imaginary but not an imaginary quadratic field, one must have $a = 2g = 4$. For $s \in S$, the commutant of the action of Z on $(R_1 f_* \mathbb{Q})_s$ is then commutative, so the action of $\pi_1(S, s)$ is abelian and therefore factors through a finite group ((4.2.8), (iii), b) or (4.2.9)). This is absurd ((4.1.3.3) and (4.4.3)).

With the notation of (4.4.8), one could also note that the T_ρ are then of rank one, so condition (4.4.8), b) holds at every place and the Hodge structure is locally constant.

- 2) If Z is totally real and conditions (c_1) and (c_2) fail, then $a \geq 2$ and $b \geq 2$, which is absurd.

Let us verify that X satisfies the conditions of (4.4.12). If Z is imaginary quadratic and there exists $\rho : Z \rightarrow \mathbb{C}$ such that $R_1 f_* \mathbb{Q} \otimes_{Z, \rho} \mathbb{C}$ is purely of Hodge type $(-1, 0)$, then the Hodge structure on $R_1 f_* \mathbb{Q}$ is locally constant, which is absurd. This proves (4.4.13).

(4.4.14) Let $f : X \rightarrow S$ be a polarized abelian scheme of dimension g over a smooth connected base S . If $s \in S$, the fundamental group $\pi_1(S, s)$ acts on the fiber $H^1(X_s, \mathbb{Z})$ of $R^1 f_* \mathbb{Z}$ at s . The polarization of X defines an alternating form on $H^1(X_s, \mathbb{Z})$, with values in $\mathbb{Z}(-1)$, nondegenerate over $\mathbb{Q}$, and the action of $\pi_1(S, s)$ respects this form. Consider the hypothesis:

(4.4.14.1) The morphism defined by X from S into one of the irreducible components of the corresponding moduli scheme of polarized abelian varieties is dominant.

The following result was suggested to me by J.-P. Serre.

Corollary (4.4.15). — *Let S be a smooth connected scheme and let $f : X \rightarrow S$ be a polarized abelian scheme over S satisfying (4.4.14.1). Then, for every abelian scheme $g : Y \rightarrow S$, one has*

$$\mathrm{Hom}(X, Y) \xrightarrow{\sim} \mathrm{Hom}(R_1 f_* \mathbb{Z}, R_1 g_* \mathbb{Z}).$$

Let $s \in S$. By (4.4.11) and (4.4.12), it suffices to show that the representation of $\pi_1(S, s)$ on $(R_1 f_* \mathbb{Q})_s$ is absolutely irreducible, so that $H = \mathbb{Q}$. This follows from the following lemma.

Lemma (4.4.16). — *If X satisfies (4.4.14.1), then the image of $\pi_1(S, s)$ in the group of symplectic automorphisms of $H^1(X_s, \mathbb{Z})$ has finite index.*

Let n be an integer. After replacing S by a surjective étale covering corresponding to a subgroup of finite index in $\pi_1(S, s)$, we may assume that the local system ${}_n X = \mathrm{Ker}(n \cdot \mathrm{id} : X \rightarrow X)$ is trivial on S . Replacing X by an isogenous abelian scheme, we may further assume that X is principally polarized. Suppose $n \geq 3$.

The abelian scheme X then defines a dominant morphism x from S to the “Jacobi” moduli scheme M_n of principally polarized abelian schemes equipped with a symplectic isomorphism between ${}_n X$ and $(\mathbb{Z}/n)^{2g}$; moreover, X is the pullback via x of the universal abelian scheme over M_n . It is known that $\pi_1(M_n, x(s))$ maps isomorphically onto the subgroup of $\mathrm{Sp}(H^1(X_s, \mathbb{Z}))$ which is the kernel of reduction modulo n .

It therefore remains to establish the following:

Lemma (4.4.17). — Let $p : S \rightarrow M$ be a dominant morphism of smooth connected schemes. The subgroup $p(\pi_1(S, s))$ of $\pi_1(M, p(s))$ then has finite index.

Let t be a closed point of the generic fiber of p , and let T' be its closure in S . There exists a dense Zariski open subset U of M such that $T = p^{-1}(U) \cap T'$ is an étale cover of U .

There is no loss of generality in taking $s \in T$. In the diagram

$$\begin{array}{ccc} \pi_1(T, s) & \longrightarrow & \pi_1(S, s) \\ \downarrow a & & \downarrow c \\ \pi_1(U, p(s)) & \xrightarrow{b} & \pi_1(M, p(s)), \end{array}$$

the arrow a has image of finite index, because T is a finite étale cover, and the arrow b is surjective, because $M - U$ has real codimension ≥ 2 . Thus the arrow c has image of finite index.

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