ON THE CONVERGENCE OF THE DERIVED SERIES OF FOURIER SERIES

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[Read June 8th, 1916.—Received October 12th, 1916.]

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$$\int_x^y f(x) \begin{matrix}\cos\\ \sin\end{matrix} nx\, dx \to 0,$$

uniformly, as $n \to \infty$, *and* $a < x < y < b$.

LEMMA 2.—*If in addition* $g(x)$ *have bounded variation, then*

$$\int_a^b f(x+u)\, g(u) \sin nu\, du \to 0,$$

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16. COROLLARY.—*If two functions agree in a completely open interval, then if one of the first derived series of the corresponding Fourier series*

(i) *converges uniformly* $(C1)$,

(ii) *converges boundedly* $(C1)$,

(iii) *oscillates uniformly above, or below* $(C1)$,

(iv) *oscillates boundedly* $(C1)$,

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$$g(u) = \tfrac{1}{4}[f(x+2u)+f(x-2u)] \operatorname{cosec} u.$$

The former series converges (Cp), if the latter converges $(C, p-1)$.

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1. The derived series of Fourier series may themselves be Fourier series. They then require no special consideration. In the contrary case the series of which they are the derived series may be associated with a particular type of function, *e.g.* with a function of bounded variation; or the function of which we have the Fourier series may not be known to have any particular properties, except in the neighbourhood of a particular point.

I have already had occasion to point out that the derived series of Fourier series of functions of bounded variation possess some of the properties of Fourier series. I propose to illustrate the interest of these series further.

But besides these series, which possess such properties in the whole interval of periodicity, there are trigonometrical series, scarcely less important, which possess the properties in question, or closely analogous ones, if not in the whole interval, at least in a portion or portions of it. These series are got by differentiating the Fourier series of functions which in such portion or portions of the interval of periodicity are more or less highly specialised, while they may be perfectly general in the rest of the interval.

It is easy to give examples. The derived series even of such Fourier series as $\Sigma n^{-1}\cos nx$ and $\Sigma n^{-1}\sin nx$, do not converge ordinarily, but do so when summed by the method of Cesàro, index unity, throughout the whole of the completely open interval of periodicity from which the origin is excluded, and converge in the manner in question precisely to the differential coefficients of the functions with which these Fourier series are associated.

We notice further that, if we continue to differentiate these Fourier series we obtain trigonometrical series which, in the completely open interval in question, always converge to the correspondingly higher differential coefficient provided only the index of the Cesàro summation employed increases by unity at each successive differentiation.

These are comparatively trivial illustrations of a general theory. Two main cases present themselves, according as we may, with propriety, regard the derived series considered as associated, in the sub-interval or intervals to which it is restricted to converge, with a function, or not. If the Fourier series from which our series is got by derivation p times is associated with a function which in the interval in question is a p-th integral, and accordingly possesses almost everywhere in that interval a p-th differential coefficient, this p-th differential coefficient may with propriety be regarded as associated with the p-th derived series. We might then conveniently call such a p-th derived series a *restricted Fourier series*. *Mutatis mutandis* we are then able to enunciate for restricted Fourier series almost all the theorems which hold good for Fourier series; the chief difference is that we must employ Cesàro convergence index p, if the process of derivation has been employed p times, and that we must be careful to restrict their application to the sub-interval or intervals for which the p-th differential coefficient exists.

But besides these derived series there are derived series which cannot be said in the same sense to have a function associated with them. We already had a simple example of such a series. It is the series whose general term is $\cos nx$, and is accordingly the derived series of the Fourier series of a function of bounded variation, whose differential coefficient is everywhere negative unity, except at the origin, where it is $+\infty$; but it is not convenient to regard this differential coefficient as associated with the series. More generally there may be no p-th differential coefficient, and yet the p-th derived series may, in some sub-interval, possess many of the properties of Fourier series.

It is convenient then to consider separately these two classes of derived series, and I propose to do so in the present communication.

In the course of the argument, which is based on reasoning analogous to that I have already employed in a previous communication, I require certain theorems with regard to the convergence of the Fourier series, and I take the opportunity of giving new tests for the convergence of a Fourier series.* These tests enable me at the same time to give a more general form to the tests for the Cesàro convergence of a Fourier series, also contained in the previous communication to this Society just referred to.

The main object of the paper is, as the title states, the convergence of the derived series of a Fourier series. This includes also the discussion of its uniform convergence, and occasion is taken to prove for ordinary Fourier series one or two elementary theorems in a somewhat simpler manner than usual. As an illustration of the use of uniform convergence, certain theorems which I have given in earlier communications are then extended to restricted Fourier series.

One of the fundamental results obtained is the property that the convergence of a Fourier series at a point, and the Cesàro convergence,† index p, of its p-th derived series at a point, depend merely on the form of the function in the neighbourhood of the point, and that this property only just holds in general. In other words, if we replace the convergence of the Fourier series by Cesàro convergence $k-1$, where $k<1$, or the Cesàro convergence (Cp), by Cesàro convergence $(C, p-1+k)$, the independence in question ceases in general to exist. I have thought it worth while in this connection to prove one or two additional theorems. In the case of the Fourier series, if the function has bounded variation elsewhere than at the point at least, and in the case of the p-th derived series, if a corresponding restriction holds, the index of convergence may be reduced.

It appears, however, that the primary question for investigation with respect to the convergence of the p-th derived series of a Fourier series is its convergence in the Cesàro manner index p. In fact it is for this index and no lower one that the convergence at the point x depends only on the nature of the function in the neighbourhood of the point considered. Furthermore, it is for this index, and no lower one, that tests for convergence of the usual type exist. We may almost say that the concept of

* The most important case of these tests is that in which the function of bounded variation which occurs in them is an integral. The test, in the form which it then takes, has been already published in a communication I have recently made to the Académie des Sciences.

† See the second footnote to § 7, below.

Cesàro convergence, index p, is naturally associated with trigonometrical series which are the p-th derived series of Fourier series.

Again, when such a series does not converge (Cp), it may still, in certain cases, be employed just as if it did do so.

The question naturally suggests itself whether, if instead of differentiating, we integrate, there are analogous results. I have thought it relevant to the matter in hand to give the simple and easily obtained theorems in this connection. It will be seen that, if we integrate a Fourier series p times, the trigonometrical series thus obtained necessarily converges in the Cesàro manner, index $-p$ (minus p), everywhere.

What happens if, instead of starting with proper Fourier series, and differentiating and integrating them, we start with the derived series of Fourier series of functions of bounded variation? As already pointed out, these more general series behave in some respects *exactly* like Fourier series; from our present point of view, however, they behave *almost* like Fourier series, the difference being precisely analogous to that between a closed and an open interval. We find that the same theorems hold, provided only the index of Cesàro summation be increased by a positive quantity k, as small as we please. Here, as elsewhere in our theorems, we mean by $(C, p+k)$ not merely that the index is $p+k$, in the ordinary sense; it is only necessary to suppose that any kind of Cesàro convergence of positive type, logarithmic or otherwise, is superposed to the (Cp) convergence, for the theorems to hold. Indeed all that is necessary is that the superposed operation should be such that, when performed on S_n, the typical term of a sequence for which ΣS_n is bounded, the result is zero.

This result illustrates the fact that, if the p-th derived series (in particular if the first derived series), of the Fourier series of $f(x)$ is to converge at the point x with a lower index of convergence than the normal p (or in the particular case unity), the function $f(x)$ must have special properties elsewhere than in the neighbourhood of the point x; in the case of the theorems just considered the reduction of the index is *almost unity*.

The question naturally arises as to whether the reduction unity itself may not be secured by a suitable condition with regard to the behaviour of $f(x)$, and in particular the important problem suggests itself as to whether the first derived series of a Fourier series may not in certain cases converge ordinarily, without being a Fourier series. The answer to this question is certainly in the affirmative. We can indeed give three important classes of Fourier series for which such ordinary convergence is possible, though not customary. These are

(1) *The allied series of Fourier series, not themselves Fourier-series.*

(2) *Pseudo-Fourier series, that is trigonometrical series whose coefficients can be expressed in the Fourier form as absolutely convergent integrals, without the associated function possessing an absolutely convergent integral.*

(3) *More generally trigonometrical series whose coefficients can be expressed in the Fourier form as non-absolutely convergent integrals, such as Harnack-Lebesgue integrals, or what I have called Y-integrals.*

As to the first of these classes, I have on various occasions given tests for the ordinary convergence of an allied series. These tests merely involve assumptions with respect to the nature of the associated function of the Fourier series in a sub-interval of the interval of periodicity, and by no means presuppose this function to have such properties in the whole interval as to secure that the allied series is also a Fourier series, whatever restrictions we impose with regard to a particular sub-interval.

As an instance of the second class, a pseudo-Fourier series which at the same time comes under the head of the first class, we have

$$\Sigma (\sin nx)/\log n,$$

which is the derived series, as well as the allied series, of a Fourier series, and converges everywhere without exception although not itself a Fourier series.*

Conditions of space and time have prevented me on the present occasion from discussing the derived series of the allied series of a Fourier series, but it should be noted that these are derived series of Fourier series. In fact the allied series itself of a Fourier series, though not in general a Fourier series, is as remarked above always the first derived series of a Fourier series, namely that of the function

$$\frac{1}{\pi}\int_0^\pi [F(x+u)-F(x-u)] \cot \tfrac{1}{2}u\, du,$$

where $F(x)$ is an integral of the function $f(x)$, associated with the Fourier series, to which our series is allied. Consequently the allied series of the Fourier series of a function $f(x)$ may be treated by the methods of this paper.

We thus see, for example, that this allied series converges $(C1)$ at the

* Its coefficients b_n are expressible in the Fourier form, the coefficients a_n are all zero and are only expressible in the Fourier form if Cauchy's principal values be utilised.

point x if the function

$$g(u) = \int_0^\pi \frac{1}{uv}\left[F(x+u+v)-F(x-u+v)-F(x+u-v)+F(x-u-v)\right] dv$$

has bounded variation.

Again, the allied series of the Fourier series of a function which in a sub-interval of the interval of periodicity has its square summable, is a restricted Fourier series of the first class, whose associated function has its square summable in its region of existence. Or yet again, the allied series of a restricted Fourier series of Class 1 is a restricted Fourier series of a similar type, but of Class 2.

It may be remarked in the same connection not only that the allied series of the Fourier series of an integral is itself a Fourier series, from which our result that the allied series of a Fourier series is the first derived series of a Fourier series was deduced, but that the same is true of the allied series of the Fourier series of a function of bounded variation, so that the allied series of the first derived series of the Fourier series of a function of bounded variation is itself the first derived series of the Fourier series of a function of bounded variation.

Moreover, the allied series of the Fourier series of a function of bounded variation converges almost everywhere, not merely in the ordinary way, but even in the Cesàro manner with any negative fractional index $k > -1$. On the other hand, the allied series of the Fourier series of an integral converges almost everywhere $(C, -1)$.

However, besides these theorems so obtained we have others precisely analogous to the theorems of the present paper. They are related to them in the same way as the tests for the convergence of the allied series are to the corresponding tests for that of the Fourier series. I have, however, as already stated, thought it advisable not to enter into these matters on the present occasion.

2. We shall first consider various conditions for the convergence of a Fourier series. (§§ 2–9.)

We shall require the following formula :—

$$\tfrac{1}{2}\pi C(na_n) \equiv (a_1+2a_2+\ldots+na_n)/n$$

$$= \frac{1}{2n}\int_0^\pi \frac{d}{dx}\left[\cot \tfrac{1}{2}x(1-\cos nx)\right] f(x)\, dx - \tfrac{1}{2}a_n, \quad (1)$$

$f(x)$ being the even function associated with the Fourier series $\Sigma a_n \cos nx$.

This is easily proved as follows:—

$$\tfrac{1}{2}\pi C(na_n) = C\left(\int_0^\pi n\cos nx\right) f(x)\,dx$$

$$= \int_0^\pi C(n\cos nx)\, f(x)\,dx$$

$$= \int_0^\pi C\left(\frac{d}{dx}\sin nx\right) f(x)\,dx$$

$$= \int_0^\pi \frac{d}{dx}\, C(\sin nx)\, f(x)\,dx$$

$$= \frac{1}{2n}\int_0^\pi \frac{d}{dx}\left[\cot\tfrac{1}{2}x - \operatorname{cosec}\tfrac{1}{2}x\cos(n+\tfrac{1}{2})x\right] f(x)\,dx$$

$$= \frac{1}{2n}\int_0^\pi \frac{d}{dx}\left(\cot\tfrac{1}{2}x - \cot\tfrac{1}{2}x\cos nx - \sin nx\right) f(x)\,dx,$$

which becomes at once the right-hand side of (1).

Since by the Theorem of Riemann-Lebesgue a_n has the unique limit zero, it follows from (1) that, in considering the limits of $C(na_n)$, it is only the integral on the right-hand side of the formula (1) which need be retained.

Hence it immediately follows that *the limits of $C(na_n)$ are independent of the form of the function $f(x)$ except in the immediate neighbourhood of the origin*, since, e being any small positive quantity,

$$(1-\cos nx)\,\frac{d}{dx}\cot\tfrac{1}{2}x \quad \text{and} \quad \cot\tfrac{1}{2}x$$

lie between fixed finite bounds when x lies in the interval (e, π) and n increases indefinitely, so that our integral from e to π is the sum of two integrals, of which the first

$$\frac{1}{2n}\int_e^\pi (1-\cos nx)\, f(x)\,\frac{d}{dx}\cot\tfrac{1}{2}x\,dx$$

vanishes in virtue of the factor $1/2n$, and the second

$$\int_e^\pi \tfrac{1}{2}\cot\tfrac{1}{2}x\, f(x)\sin nx\,dx$$

vanishes by the theorem of Riemann-Lebesgue.

3. In considering the limits of $C(na_n)$, we may further change $\cot\tfrac{1}{2}x$

into $1/\frac{1}{2}x$ in the formula (1), which accordingly becomes

$$\underset{n\to\infty}{\mathrm{Lt}}\ \tfrac{1}{2}\pi C(na_n) = \underset{n\to\infty}{\mathrm{Lt}}\ \frac{1}{n}\int_0^e \frac{d}{dx}\left(\frac{1-\cos nx}{x}\right) f(x)\,dx, \tag{2}$$

where e is as small a positive quantity as we please, independent of n.

Indeed since

$$\frac{1}{n}\,\frac{d}{dx}\left(\cot\tfrac{1}{2}x - \frac{1}{\frac{1}{2}x}\right)(1-\cos nx)$$

$$= \frac{1}{n}(1-\cos nx)\,\frac{d}{dx}\left(\cot\tfrac{1}{2}x - \frac{1}{\frac{1}{2}x}\right) + \sin nx\left(\cot\tfrac{1}{2}x - \frac{1}{\frac{1}{2}x}\right),$$

which is the sum of two terms, the first of which is the product of $1/n$ into a bounded function of x and n, and the second the product of $\sin nx$ into a bounded function of x, for values of x in the closed interval $(0, e)$, we see that the difference of the two integrals in (1) and (2) is the sum of two integrals, of which the first vanishes uniformly in virtue of the factor $1/n$ in the integrand, and the second by the theorem of Riemann-Lebesgue, when n increases indefinitely. This proves the formula (2).

4. We can now prove the following theorem :—

THEOREM.—*If* $\Sigma a_n \cos nx$ *is the Fourier series of an even function* $f(x)$, *then*

$$\underset{n\to\infty}{\mathrm{Lt}}\ C(na_n) = 0,$$

provided

(i) $f(x)$ *is simply discontinuous at the origin, and*

(ii) $\frac{1}{x}\int_0^x |\,d[xf(x)]\,|$ *is a bounded function of* x *in a certain neighbourhood of the origin.*

In fact, in virtue of the condition (i), we may change the lower limit of integration in the formula (2) from zero to $p = 2P\pi/n$, where P is as large an integer as we please, n being correspondingly larger. To prove this let us write t/n for x; $f(t/n)$ will then be bounded and have $f(+0)$ for limit as n increases indefinitely. We may therefore multiply by the bounded function $\frac{d}{dx}\left(\frac{1-\cos t}{t}\right)$, and integrate term-by-term. This gives

$$\underset{n\to\infty}{\mathrm{Lt}}\ \frac{1}{n}\int_0^p \frac{d}{dx}\left(\frac{1-\cos nx}{x}\right) f(x)\,dx = \underset{n\to\infty}{\mathrm{Lt}}\int_0^{2P\pi} f(t/n)\,\frac{d}{dt}\left(\frac{1-\cos t}{t}\right) dt$$

$$= \int_0^{2P\pi} f(+0)\,\frac{d}{dt}\left(\frac{1-\cos t}{t}\right) dt = 0.$$

We thus have

$$\operatorname*{Lt}_{n\to\infty} \tfrac{1}{2}\pi C(na_n) = \operatorname*{Lt}_{n\to\infty} \int_p^e f(x)\frac{d}{dx}\left(\frac{1-\cos nx}{nx}\right)dx = \operatorname*{Lt}_{n\to\infty} \int_p^e xf(x)\frac{d}{dx}q_n(x)\,dx,$$

where $$q_n(x) = \frac{1-\cos nx}{nx^2} - \frac{1}{n}\int_x^e \frac{1-\cos nx}{x^3}\,dx.$$

Now, by the condition (ii), $xf(x)$ must be a function of bounded variation; for $\int |d[xf(x)]|$ is the total variation of $xf(x)$, and is, by the condition (ii), bounded.

We may therefore integrate by parts, and write

$$\operatorname*{Lt}_{n\to\infty} \tfrac{1}{2}\pi C(na_n) = \operatorname*{Lt}_{n\to\infty} \Big[xf(x)\,q_n(x)\Big]_p^e - \int_p^e q_n(x)\,d[xf(x)].$$

Now $$|q_n(x)| \leqslant \frac{2}{nx^2} + \frac{1}{n}\int_x^e \frac{2}{x^3}\,dx = \frac{3}{nx^2}.$$

Therefore

$$\left|\operatorname*{Lt}_{n\to\infty} \tfrac{1}{6}\pi C(na_n)\right| \leqslant \operatorname*{Lt}_{n\to\infty} \frac{1}{n}\left\{\left|\left[\frac{f(x)}{x}\right]_p^e\right| + \int_p^e \frac{|d[xf(x)]|}{x^2}\right.$$

$$= f(p)/2P\pi + \operatorname*{Lt}_{n\to\infty} \frac{1}{n}\left(\int_p^e \frac{d\phi(x)}{x^2}\right),$$

where $$\phi(x) = \int_0^x |d[xf(x)]|.$$

But, integrating by parts,

$$\frac{1}{n}\int_p^e \frac{d\phi(x)}{x^2} = \frac{1}{n}\left[\frac{\phi(x)}{x^2}\right]_p^e + \frac{2}{n}\int_p^e \frac{\phi(x)}{x^3}\,dx$$

$$\leqslant \phi(e)/ne^2 + B/2P\pi + \frac{2B}{n}\int_p^e \frac{dx}{x^2},$$

where B denotes the upper bound of $\frac{1}{x}\int_0^x |d[xf(x)]|$, which, by the condition (ii), is finite. Letting n increase indefinitely, the right-hand side of the last relation approaches $3B/2P\pi$. Hence

$$\left|\operatorname*{Lt}_{n\to\infty} \tfrac{1}{6}\pi C(na_n)\right| \leqslant \frac{f(p)+3B}{2P\pi}.$$

Since, by (1), $f(p)$ remains bounded as n increases indefinitely, and P is as large as we please, the right-hand side of the last relation is as small as we please, which proves the theorem.

5. Hence we have the following theorem on the convergence of a Fourier series.

THEOREM. — *If, as h approaches zero, $\frac{1}{2}[f(x+h)+f(x-h)]$ has a unique limit C, and $\frac{1}{h}\int^{h} |d[h\{f(x+h)+f(x-h)\}]|$ is a bounded function of h in a certain neighbourhood surrounding the point $h=0$, however small this neighbourhood may be, then the Fourier series of $f(x)$ converges to C at the point x.**

Since

$$\tfrac{1}{2}[f(x+u)+f(x-u)] \sim \Sigma(a_r \cos rx + b_r \sin rx)\cos ru,$$

we only have to replace the function $f(x)$ by the even function of u, $\frac{1}{2}[f(x+u)+f(x-u)]$, to reduce the problem to that of an even function having a unique limit at the origin, and to replace the point x by the origin. We shall therefore only need to prove the theorem for the case $x=0$, f being an even function.

Now writing

$$s_n = a_1+a_2+\ldots+a_n, \qquad C(na_n) = (a_1+2a_2+\ldots+na_n)/n,$$

we have

$$s_n - C(na_n) = (s_1+s_2+\ldots+s_{n-1})/n. \tag{3}$$

Now, by the preceding theorem (§ 4), $C(na_n)$ has the unique limit zero, when n increases indefinitely. Also, since $f(x)$ is bounded in the neighbourhood of the origin, by hypothesis, and approaches a unique limit $f(+0)$, the Fourier series converges when summed in the Cesàro manner, index unity, so that the right-hand side of (3) also approaches a unique limit. Thus, by (3), s_n approaches the same unique limit, that is, the Fourier series converges at the origin. This proves the theorem.

COR. 1.—*If $\frac{1}{2}[f(x+h)+f(x-h)] \to C$, as $h\to 0$, and in a certain neighbourhood of $h=0$ we have*

$$\tfrac{1}{2}[f(x+h)+f(x-h)] = \frac{1}{h}\int^{h} g(t)\,dt,$$

where $\frac{1}{h}\int_0^{h} |g(t)|\,dt$ is a bounded function of h, the Fourier series of $f(x)$ converges to C at the point x.†

* The enunciation of this theorem was made by me to the Société Helvétique des Sciences Naturelles, in August of the present year, at Schuls in the Engadine.

† This was stated and proved by me in a recent note in the *Comptes rendus*.

COR. 2.—*If* $\frac{1}{2}[f(x+u)+f(x-u)] \to C$, *as* $u \to 0$, *and, in a certain neighbourhood of* $u = 0$, *the function of* u

$$\tfrac{1}{2}u\,[f(x+u)+f(x-u)]$$

has bounded derivates, then the Fourier series of $f(x)$ *converges to* C *at the point* x.

For in this case the conditions of Cor. 1 are satisfied, $g(t)$ being any one of the derivates of $\frac{1}{2}u\,[f(x+u)+f(x-u)]$. This corollary is, in fact, identical with the first corollary in the case in which $g(t)$ is a bounded function of t.

The first condition imposed shows that the second condition of this corollary is equivalent to the hypothesis that *the derivates of*

$$\tfrac{1}{2}[f(x+u)+f(x-u)],$$

with respect to u, *when multiplied by* u, *should be bounded functions of* u *in a certain neighbourhood of* $u = 0$.

Using the more general theorem that a function whose derivates are absolutely integrable (summable), and finite except at a countable set of points, is the integral of its derivates, we have the following also:—

COR. 3.—*If* $\frac{1}{2}[f(x+u)+f(x-u)] \to C$, *as* $u \to 0$, *and, in a certain neighbourhood of* $u = 0$, *the derivates of the function of* u

$$\tfrac{1}{2}[f(x+u)+f(x-u)]$$

are, when multiplied by u, *absolutely integrable, and, except possibly at a countable set of points, finite, then the Fourier series of* $f(x)$ *converges to* C *at the point* x.

6. Remembering that the Fourier series of $f(x)$ converges (Cp) at the point x, if that of

$$\phi_1(u) = \frac{1}{u}\int_0^u \tfrac{1}{2}[f(x+t)+f(x-t)\,dt = \int_0^1 \tfrac{1}{2}[f(x+ut)+f(x-ut)]\,dt$$

converges $(C, p-1)$, we get the following additional corollaries:—

COR. 4.—*The Fourier series of* $f(x)$ *converges* $(C1)$ *at the point* x *if*

$\phi_1(+0)$ *exists** *and*

$$\frac{1}{u}\int_0^u |d[t\phi_1(t)]| \equiv \frac{1}{u}\int_0^u \tfrac{1}{2}\,|f(x+t)+f(x-t)|\,dt$$

$$= \int_0^1 \tfrac{1}{2}\,|f(x+ut)+f(x-ut)|\,dt$$

is a bounded function of u in some interval containing $u=0$. *The Cesàro sum of the series is then* $\phi_1(+0)$.

COR. 5.—*The Fourier series of* $f(x)$ *converges* ($C2$) *at the point* x, *if* $\phi_2(+0)$ *exists, where*

$$\phi_2(u) \equiv \frac{1}{u}\int_0^u \phi_1(t)\,dt = \int_0^1\int_0^1 \tfrac{1}{2}[f(x+ut_1t_2)+f(x-ut_1t_2)]\,dt_1\,dt_2,$$

and

$$\frac{1}{u}\int_0^u |d[t\phi_2(t)]| = \frac{1}{u}\int_0^u |\phi_1(t)dt|$$

$$= \int_0^1 \left|\int_0^1 \tfrac{1}{2}[f(x+ut_1t_2)+f(x-ut_1t_2)]\,dt_1\right| dt_2$$

is a bounded function of u in some interval containing $u=0$. *The second Cesàro sum of the series is then* $\phi_2(+0)$.

COR. 6.—*The Fourier series of* $f(x)$ *converges* (Cp) *at the point* x, *if* $\phi_p(+0)$ *exists, where*

$$\phi_p(u) = \frac{1}{u}\int_0^u \phi_{p-1}(t)\,dt$$

$$= \int_0^1\int_0^1\cdots\int_0^1 \tfrac{1}{2}[f(x+ut_1t_2\ldots t_p)+f(x-ut_1t_2\ldots t_p)]\,dt_1\,dt_2\ldots dt_p,$$

* The condition that $\phi_1(+0)$ should exist is, of course, the same as that

$$\int_0^u \tfrac{1}{2}[f(x+t)+f(x-t)]\,dt$$

should have a differential coefficient at $u=0$, and the same remark applies to $\phi_p(+0)$ and $\int_0^u \phi_{p-1}(t)\,dt$. These conditions given above may be compared with the less general conditions of similar form given on p. 267 of *The Convergence of a Fourier Series and its Allied Series*, in the second line of which "index p" should read "index $(p+1)$."

and if $\frac{1}{u}\int_0^u |\phi_{p-1}(u)du|$, *that is*

$$\int_0^1 \left| \int_0^1 \cdots \int_0^1 \mid \tfrac{1}{2}[f(x+ut_1t_2\ldots t_p)+f(x-ut_1t_2\ldots t_p)]\,dt_1dt_2\ldots dt_{p-1}\right| dt_p,$$

is a bounded function of u in some interval containing $u = 0$. *The p-th Cesàro sum of the Fourier series of* $f(x)$ *is then* $\phi_p(+0)$.

7. With regard to the preceding theorem (§ 5), it should be noticed that a considerable extension is possible if we use the notion of *Cesàro summation with negative index.*

Denoting the ordinary partial summation of a Fourier series by s_n and the Cesàro partial summation, index minus one, by t_n, we have by a definition which naturally suggests itself

$$ns_n = t_1+t_2+\ldots+t_n, \tag{1}$$

and therefore $t_n = ns_n-(n-1)\,s_{n-1} = s_{n-1}+n\,(s_n-s_{n-1})$

$$= s_{n-1}+n\,(a_n\cos nx+b_n\sin nx)\,; \tag{2}$$

where $f(x) \sim \tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$,

and therefore

$$\tfrac{1}{2}[f(x+u)+f(x-u)] \sim \tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\cos nu.$$

We see therefore, from (2), that t_n and s_n have the same limit at the point x, or the same limits, provided the coefficients of the derived series of the Fourier series of the even function

$$\tfrac{1}{2}[f(x+u)+f(x-u)]$$

converge to zero.

In particular, using the theorem of Riemann-Lebesgue, and the generalisation of it which I have recently given,* we have the following

* This theorem was given in its original form by Riemann, "Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe", 1854, *Ges. Werke*, p. 254, and states that, if $f(x)$ has a Riemann integral, the coefficients of its Fourier series a_n and b_n converge to zero, as $n\to\infty$. The extension to any summable function was given by Lebesgue, 1903, *Annales sc. de l'école normale sup.*, Sér. 3, Vol. xx, quoted in Hobson's *Theory of Functions of a Real Variable*, p. 674.

In other words, *the theorem of Riemann-Lebesgue states that the coefficients of the derived* results :—

The Fourier series of an integral converges $(C, -1)$ *everywhere to that integral.*

The Fourier series of a function of bounded variation converges everywhere $(C, k-1)$,* *where k is any positive quantity whatever.*

8. We can then assert that (1) *the question of the convergence* $(C, k-1)$, *where* $0 < k$, *depends only on the nature of the function in the neighbourhood of the point considered, provided the function be known to be a function of bounded variation outside this interval;* also (2) the argument we have used, involving as it does such quantities as $1/n$ to the first power only, and therefore still applicable when the index is k, where $0 < k$, shows that, so far as the neighbourhood $(0, e)$ is concerned, $C(na_n)$, that is $(C1)(na_n)$, may be replaced by $(Ck)(na_n)$, where $0 < k \leqslant 1$. Thus we have the theorem that *the conditions of the theorem of* § 5, *or of any of*

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series of the Fourier series of an integral converge to zero. The generalisation of this theorem which I have given recently in a paper "On the Order of Magnitude of the Coefficients of a Fourier Series", presented to the Royal Society, is as follows :—*The coefficients of the derived series of the Fourier series of an even function of bounded variation converge to zero, when the convergence is taken in the Cesàro manner, index unity, and those of an odd function of bounded variation converge in the same manner to* $1/\pi$ *times the jump of the function at the origin.*

* The definition of Cesàro convergence, index $k-1$, here adopted, may, in the first instance, be taken to be that explained in my paper, already cited, from these *Proceedings*, Vol. 10, p. 264, the Cesàro summations with negative integral indices being first defined. The reader will have no difficulty with these latter ; he merely has to repeat the process involved in equation (1) above. The definition so obtained is necessarily equivalent, for values of the index between 0 and -1, to that devised by Knopp and Chapman for such values, in this sense, that, if a series converges $(C, k-1)$ in their sense for all such values of the index, it will converge $(C, k-1)$ in my sense, and conversely.

With regard to Cesàro convergence of positive index, I assume tacitly here as elsewhere the known equivalence of the various definitions hitherto proposed for such convergence, and employ them indifferently, as the circumstances of the demonstration may render convenient.

Two remarks should be made about the theorem in the text :—In the first place, this theorem is an immediate consequence of the classical result that the coefficients of the derived series of the Fourier series of a function of bounded variation are bounded, provided we make use of a general theorem in the arithmetic theory of series (which is, however, more difficult than would be otherwise necessary), namely one given by Hardy and Littlewood in these *Proceedings*, Ser. 2, Vol. 11, p. 462, Theorem 37. In the second place it should be remarked that we may assert the truth of the theorem when the expression "converges $(C, k-1)$" is interpreted in the still larger sense in which Cesàro convergence of any positive type is superimposed, in the manner explained in the footnote to my paper already quoted, on the Cesàro convergence, index -1.

its corollaries, ensure not only the convergence of the Fourier series, but its convergence $(C, k-1)$, *provided only* $f(x)$ *is in addition a function of bounded variation in the part of the interval of periodicity outside the neighbourhood of the origin mentioned in the conditions.*

We have thus two important cases in which the Fourier series converges $(C, k-1)$, where $0 < k$.

(1) *When* $f(x)$ *has bounded variation in the whole interval of periodicity.*

(2) *When* (i) $f(x)$ *has bounded variation in every interval not containing the point at which the convergence is considered, while at the point itself* $f(x)$ *has a unique limit, or at least* $f(x+u)+f(x-u)$ *has a unique limit, as* $u \to 0$, *and in addition* (ii)

$$\frac{1}{u}\int^{u} | d\{u[f(x+u)+f(x-u)]\} |$$

is a bounded function of u *in the neighbourhood of* $u = 0$.

This is a remarkable extension for $(C, k-1)$ of Dirichlet's theorem, properly generalised, and bears the same relation to this generalisation that de la Vallée Poussin's condition does to Dirichlet's original test.

If we require the condition (ii) to hold throughout the interval of periodicity, we have corresponding results. For, in an interval not containing $u = 0$, the condition (ii) only has a meaning when $f(x)$ has bounded variation; and conversely, if $f(x)$ has bounded variation, our condition fulfils itself.

9. The following test for the convergence of the Fourier series of $f(x)$ is slightly more general than that commonly employed.* It requires that

$$\frac{f(x+2u)-f(x-2u)-K}{u}$$

should be absolutely integrable in an interval containing the origin, K *being a suitable constant.*

In fact, denoting by s_n the n-th partial summation of the Fourier series of $f(x)$, we have, by the usual formula,

$$s_n = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} [f(x+2u)+f(x-2u)] \operatorname{cosec} u \sin(2n+1) u\, du.$$

* Hobson's *Theory of Functions of a Real Variable*, pp. 680, 681; where, however, it is assumed that $f(x+2u)+f(x-2u)$ has, as $u \to 0$, a unique limit which is the constant K.

Also $$\operatorname*{Lt}_{n\to\infty} \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} K \operatorname{cosec} u \sin(2n+1)u\,du = \tfrac{1}{2}K.$$

Now in consequence of our hypothesis,

$$\frac{f(x+2u)+f(x-2u)-K}{u}\,\frac{u}{\sin u}$$

is absolutely integrable in some interval containing the origin, and is therefore absolutely integrable from 0 to $\frac{1}{2}\pi$. Therefore, by the theorem of Riemann-Lebesgue, when we multiply by $\sin(2n+1)u$ and integrate from 0 to $\frac{1}{2}\pi$, we get a value which vanishes as $n\to\infty$. That is, by the above,

$$\operatorname*{Lt}_{n\to\infty} s_n - \tfrac{1}{2}K = 0,$$

which proves that the Fourier series of $f(x)$ converges to $\frac{1}{2}K$.

10. It is well known that the convergence, or mode of oscillation, or more precisely the nature of the upper and lower functions, of a Fourier series at a point depend only on the nature of the function in a portion of the interval of periodicity as small as we please containing the point. Corresponding to this fact in the ordinary theory, we have the following theorem.

THEOREM.—*The upper and lower functions of the p-th derived series of a Fourier series at a particular point depend only on the nature of the function associated with the Fourier series in a neighbourhood enclosing the point, as small as we please, provided the summation of the series is performed in the Cesàro manner, index p.*

We will first prove this theorem for the first derived series.

Writing

$$u_n(t) = \tfrac{1}{2}+\cos t+\ldots+\cos(n-1)t = \tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}t\sin(n-\tfrac{1}{2})t,$$

and $$C(u_n) = (u_1+u_2+\ldots+u_n)/n = (1-\cos nt)/2n(1-\cos t),$$

we have evidently $$\frac{d}{dt}C(u_n) = C\left(\frac{d}{dt}(u_n)\right),$$

that is,

$$\frac{d}{dt}\left[(1-\cos nt)/2n(1-\cos t)\right] = C\left[-\sin t-2\sin 2t-\ldots-(n-1)\sin(n-1)t\right]. \tag{1}$$

Now $$\tfrac{1}{2}\left[f(x+t)-f(x-t)\right] \sim \Sigma(b_r\cos rx - a_r\sin rx)\sin rt. \tag{2}$$

We may therefore multiply both sides of the relation (2) by the continuous function $\frac{d}{dt}C(u_n)$, and integrate term-by-term. Thus

$$\tfrac{1}{2}\int_{-\pi}^{\pi} [f(x+t)-f(x-t)] \frac{d}{dt}[(1-\cos nt)/2n(1-\cos t)]\,dt$$

$$= \Sigma \int_{-\pi}^{\pi} (b_r \cos rx - a_r \sin rx) \sin rt \left[C(-\sin t - 2\sin 2t - \ldots - (n-1)\sin (n-1)\,t\right] dt.$$

Now $\int_{-\pi}^{\pi} \sin rt \sin st = 0$, unless $r = s$, in which case it is equal to π. Therefore the right-hand side of the last equation becomes

$$-\pi C\left\{(b_1 \cos x - a_1 \sin x) + 2\,(b_2 \cos 2x - a_2 \sin 2x) + \ldots + (n-1)[b_{n-1} \cos (n+1)\,x - a_{n-1} \sin (n-1)x]\right\}.$$

Thus, denoting by $v_n(x)$ the n-th Cesàro partial summation of the derived series of the Fourier series of $f(x)$, we have

$$v_n(x) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x+t)-f(x-t)]\,\frac{d}{dt}[(1-\cos nt)/2n(1-\cos t)]\,dt. \quad (3)$$

The integrand in this last integral is an even function of t, so that the integral is twice the same integral from 0 to π. Hence, in order to prove the theorem enunciated in § 9, it is only necessary to prove that the same integral from e to π, where e is any small positive quantity, vanishes when n increases indefinitely.

$$\text{Now}\quad \frac{d}{dt}[(1-\cos nt)/(1-\cos t)] = \frac{n \sin nt}{1-\cos t} - \frac{(1-\cos nt)\sin t}{(1-\cos t)^2}. \quad (4)$$

Thus our integral is the difference of the two integrals obtained by multiplying the two terms on the right-hand side of (4) by $[f(x+t)-f(x-t)]/n$ and integrating from e to π. The first of these two integrals is zero by the theorem of Riemann-Lebesgue, since

$$[f(x+t)-f(x-t)]/(1-\cos t)$$

is absolutely integrable in the interval considered. The second term on the right of (4) is numerically less than $2/(1-\cos e)^2$, so that the second integral vanishes in virtue of the factor $1/n$.

This proves the theorem for the first derived series.

11. When p is greater than unity, we proceed as follows to prove the theorem of § 10.

Using the generalisations of the sine and cosine discussed by me in an earlier paper,* so that

$$C_p(t) = \frac{t^p}{\Gamma(p+1)}\left[1-t^2/(p+1)(p+2)+t^4/(p+1)(p+2)(p+3)(p+4)-\dots\right], \tag{1}$$

we have,† denoting by $(Cp)[S_n(x)]$ the Cesàro n-th partial summation of index p of our series,

$$\begin{aligned}(Cp)[S_n(x)] &= \tfrac{1}{2}a_0 + \sum_{m=1}^{n} A_m(1-m/n)^p \\ &= \frac{\Gamma(p+1)}{\pi}\int_0^\infty t^{-p-1}\, C_{p+1}(t)\,[f(x+t/n)+f(x-t/n)]\,dt \\ &= \Gamma(p+1)\, I/\pi, \end{aligned} \tag{2}$$

where, integrating by parts,

$$I = n\Big[t^{-p-1}C_{p+1}(t)[F(x+t/n)-F(x-t/n)]\Big]_0^\infty$$
$$-n\int_0^\infty \frac{d}{dt}[t^{-p-1}C_{p+1}(t)]\,[F(x+t/n)-F(x-t/n)]\,dt. \tag{3}$$

Now $t^{-p-1}C_{p+1}$ vanishes when t is infinite, and $[F(x+t/n)-F(x-t/n)]$ is zero when t is zero. Therefore the square bracket expression disappears, and we are left with the integral on the right alone.

Differentiating with respect to x, we may differentiate under the sign of integration, as will be seen by the following reasoning. Let us write $G(x)$ for $\int_0^x F(x)\,dx$. Then, since $F(x)$ is continuous, it is the differential co-

* "On Infinite Integrals involving a Generalisation of the Sine and Cosine Functions," 1912, *Quarterly Journal*, Vol. 43, pp. 161–177

† *Loc. cit.*, p. 177.

efficient of $G(x)$. Hence, as $h \to 0$,

$$\frac{1}{h}\int_0^t [F(x+h+t/n)-F(x+t/n)]\,dt$$

$$= n\int_x^{x+t/n} \frac{1}{h}[F(\xi+h)-F(\xi)]\,d\xi$$

$$= n\,[G(x+h+t/n)-G(x+t/n)-G(x+h)+G(x)]/h$$

$$\to n\,[F(x+t/n)-F(x)]$$

$$= n\int_x^{x+t/n} f(\xi)\,d\xi = \int_0^t f(x+t/n)\,dt,$$

and the convergence is bounded, since the bounds of the incrementary ratio of $G(x)$ are the same as those of $F(x)$.

By a known theorem,* therefore, if $g(t)$ is any function of bounded variation in the infinite interval, with zero as its unique limit when $t \to \infty$, we may introduce the factor $g(t)$ on each side under the sign of integration, and replace the upper limit of integration by infinity. Thus we have

$$\operatorname*{Lt}_{h\to 0} \frac{1}{h}\int_0^\infty g(t)\,[F(x+h+t/n)-F(x+t/n)]\,dt = \int_0^\infty g(t)\,f(x+t/n)\,dt,$$

that is
$$\frac{d}{dx}\int_0^\infty g(t)\,[F(x+t/n)]\,dt = \int_0^\infty g(t)\,f(x+t/n)\,dt,$$

and hence also

$$\frac{d}{dx}\int_0^\infty g(t)\,[F(x+t/n)-F(x-t/n)]\,dt = \int_0^\infty g(t)\,[f(x+t/n)-f(x-t/n)]\,dt.$$

It remains therefore only to show that if we put

$$g(t) = \frac{d}{dt}\,[t^{-p-1}C_{p+1}(t)] = -(p+1)\,t^{-p-2}C_{p+1}(t)+t^{-p-1}C_p(t),$$

$g(t)$ vanishes at infinity, and is a function of bounded variation in the whole infinite interval, to justify the statement that we may differentiate under the sign of integration in (1).

That $g(t)$ vanishes at infinity is at once evident from its expression in terms of C_p and C_{p+1}; that it is a function of bounded variation in the

* W. H. Young, "The Application of Expansions to Definite Integrals", 1910, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, § 10, pp. 475–477.

whole infinite interval may be, for instance, proved by differentiating it, which gives

$$g'(t) = (p+1)(p+2)\,t^{-p-3}\,C_{p+1}(t) - 2(p+1)\,t^{-p-2}\,C_p(t) + t^{-p-1}\,C_{p-1}(t),$$

and shows at once that $g'(t)$ is summable in the infinite interval, whence $g(t)$ is a function of bounded variation in the whole infinite interval, provided that $p \geqslant 1$.

We have therefore, from (3),

$$\frac{dI}{dx} = -\int_0^\infty n\,[f(x+t/n) - f(x-t/n)]\,\frac{d}{dt}(t^{-p-1}C_{p+1})\,dt \quad (p \geqslant 1), \quad (4)$$

which is of the same form as I, $n\dfrac{d}{dt}(t^{-p-1}C_{p+1})$ taking the place of $t^{-p-1}C_{p+1}$ as the multiplier of $f(x+t/n)$ and $f(x-t/n)$ in the integrand.

We may therefore repeat the process, provided p is sufficiently large; for one differentiation we had to have $p \geqslant 1$, for two we must have $p \geqslant 2$, and so on. Thus, finally,

$$\frac{d^rI}{dx^r} = (-)^r \int_0^\infty n^r\,[f(x+t/n) + (-)^r f(x-t/n)]\,\frac{d^r}{dt^r}\,[t^{-p-1}\,C_{p+1}(t)]\,dt \quad (r \leqslant p). \quad (5)$$

Using Leibnitz's theorem for the expansion of

$$\frac{d^r}{dt^r}\,[t^{-p-1}\,C_{p+1}(t)],$$

we get, since, apart from a numerical factor,

$$\frac{d^r}{dt^r}\,t^{-p-1}\left(\frac{d^{p-r}}{dt^{p-r}}\,C_{p+1}\right) = t^{-p-r-1}\,C_{r+1},$$

$$\frac{d^pI}{dx^p} = \int_0^\infty n^r\,[f(x+t/n) + (-)^p f(x-t/n)]\,\sum_{r=0}^{p} A_r\,t^{-p-r-1}\,C_{r+1}(t)\,dt, \quad (6)$$

or, say,

$$\frac{d^pI}{dx^p} = \sum_{r=0}^{p} A_r \int_0^\infty K_r(t)\,dt, \quad (6')$$

where, since $\qquad C_{r+1} = -C_{r-1} + t^{r-1}/(r-1)!,$

$$K_r(t) = -t^{-2}K_{r-2}(t) + n^p[f(x+t/n) + (-)^p f(x-t/n]\,t^{-p-2}/(r-1)!. \quad (7)$$

Now what we have to show, in order to prove our theorem, is that

$\dfrac{d^pI}{dx^p}$ is independent of the form of $f(u)$, except for a small interval of values of u enclosing the point $u = x$. That is, we have to show that, if we change the lower limit of integration in (6), or (6′), from zero to ne, the integral vanishes in the limit, when $n \to \infty$; it will therefore be sufficient to show that this is the case with the individual integrals $\int_{ne}^{\infty} K_r(t)\,dt$.

Now, by (7), this will be the case, provided it is true for $\int_{ne}^{\infty} t^{-2}K_{r-2}(t)\,dt$, and for

$$\int_{ne}^{\infty} n^p[f(x+t/n)+(-)^p f(x-t/n)]\,t^{-p-2}\,dt$$

$$= \frac{1}{n}\int_{e}^{\infty} [f(x+u)+(-)^p f(x-u)]\,u^{-p-2}du. \tag{8}$$

The latter integral vanishes in virtue of the factor $1/n$, and the former integral will do so also, provided $\int_{ne}^{\infty} K_{r-2}(t)\,dt$ does so. Thus, by induction, the theorem is true, provided $\int_{ne}^{\infty} K_2(t)\,dt$ and $\int_{ne}^{\infty} K_1(t)\,dt$ vanish in the limit when $n \to \infty$.

$$\text{Now } \int_{ne}^{\infty} K_1(t)\,dt = \int_{ne}^{\infty} n^p[f(x+t/n)+(-)^p f(x-t/n)]\,t^{-p-1}\sin t\,dt$$

$$= \int_{e}^{\infty} [f(x+u)+(-)^p f(x-u)]\,u^{-p-1}\sin nu\,du,$$

which vanishes by the theorem of Riemann-Lebesgue; also

$$\int_{ne}^{\infty} K_2(t)\,dt = \int_{ne}^{\infty} n^p[f(x+t/n)+(-)^p f(x-t/n)]\,t^{-p-2}(1-\cos t)\,dt$$

$$= \frac{1}{n}\int_{e}^{\infty} [f(x+u)+(-)^p f(x-u)]\,u^{-p-2}(1-\cos nu)\,du,$$

which vanishes when $n \to \infty$, since it is the sum of the integral in (8) and an integral which vanishes by the theorem of Riemann-Lebesgue.

This therefore proves the theorem.

12. An immediate consequence of the preceding theorem is the following generalisation of a theorem I have given previously.*

THEOREM.—*If $f(x)$ be throughout a certain interval (a, b) a function of bounded variation, then the derived series of the Fourier series of $f(x)$ converges $(C1)$, almost everywhere in the interval in question, to the differential coefficient of the function.*

In fact the convergence $(C1)$ in the interval in question does not depend on the nature of the function outside that interval, and is therefore the same as if $f(x)$ had everywhere bounded variation. But in this latter case the series in question converges (Ck), where $0 < k$, and therefore certainly converges $(C1)$ almost everywhere. This proves the theorem.

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* W. H. Young, "The Usual Convergence of a Class of Trigonometrical Series", 1913, *Proc. London Math. Soc.*, Ser. 2, Vol. 13, pp. 21–23. The reader will have noticed that by a clerical error (Ck) has been omitted in the enunciation of the theorem, which should be as follows:—"The derived series of the Fourier series of a function of bounded variation converges (Ck), $0 < k$, almost everywhere to the differential coefficient of the function." In the course of the proof another clerical error has crept in, which I take this opportunity of correcting. In line 17, p. 22, *delete*

$$\text{const. } n^{-k} \int_1^\infty |\, df(x+t) \,| \, ;$$

line 18 should then read:—"this last integral existing, since $f(x)$ is a periodic function of bounded variation throughout the whole infinite interval. Thus this term is of the order n^{-k}."

The argument is here perhaps unduly condensed. That the fact quoted does ensure the existence of the integral is, for instance, easily seen if we integrate by parts from 1 to B. We then have

$$\int_1^B t^{-1-k} \,|\, df(x+t) \,| = \int_1^B t^{-1-k} \, dF(t) = \Big[(1+t)^{-1-k} F(t) \Big]_1^B + (1+k) \int_1^B t^{-2-k} F(t)\, dt,$$

where, since f is periodic, we have, writing

$$1 + 2r\pi \leqslant t < 1 + 2(r+1)\pi,$$

$$F(t) = \int_1^t |\, df(x+t) \,| \leqslant \int_1^{1+2(r+1)\pi} |\, df(x+t) \,| = (r+1) K \leqslant \frac{K}{2\pi}(t - 1 + 2\pi),$$

where
$$K = \int_1^{1+2\pi} |\, df(x+t) \,| .$$

Hence
$$\Big[(1+t)^{-1-k} F(t) \Big]_1^B \to 0,$$

as $B \to \infty$, and $\int_1^\infty t^{-2-k} F(t)\, dt$ exists, which proves the existence of the integral under consideration.

13. Again we immediately have the following theorem, which is a particular case of a more general theorem to be proved later (§ 22, Theorem C):—

THEOREM.—*If $f(x)$ has a derivate $f_1(x)$ which is continuous at the point x,* then the derived series of the Fourier series of $f(x)$ converges $(C1)$ at the point x to $f_1(x)$.*

In fact, since, as we have seen, the convergence of the series in question is independent of the form of $f(x)$ except in the neighbourhood of the point x considered, we may assume that $f(x)$ is an integral except in an interval containing the point x in which $f_1(x)$ is bounded; this latter hypothesis is allowable, since $f_1(x)$ is continuous at the point x.

Since a function with a bounded derivate is an integral, $f(x)$ is now an integral in the whole closed interval of periodicity, and therefore the derived series of the Fourier series of $f(x)$ is a Fourier series, and its associated function is any one of the derivates of $f(x)$, and may therefore be taken to be $f_1(x)$. Since $f_1(x)$ is continuous at the point x, the Fourier series of $f_1(x)$ converges $(C1)$ to $f_1(x)$ at the point x. That is, the derived series of the Fourier series of the modified function $f(x)$ converges $(C1)$ to $f_1(x)$ at the point x.

Since the modification of $f(x)$ did not affect the convergence of the derived series, this proves the theorem.

COR.—*It is sufficient if $\frac{1}{2}[f(x+u)-f(x-u)]$ has a derivate with respect to u which is continuous at $u=0$; the derived series of the Fourier series of $f(x)$ then converges $(C1)$ to the value of this derivate at $u=0$.*

For $$\tfrac{1}{2}[f(x+u)-f(x-u)] \sim \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \sin nu,$$

x being here a constant, and u the variable. By the above theorem the derived series (with respect to u) of the above Fourier series converges $(C1)$ at $u=0$ to the derivate in question. That is

$$\left(\frac{d}{du}\, \tfrac{1}{2}[f(x+u)-f(x-u)]\right)_{u=0} = \sum_{n=1}^{\infty} (nb_n \cos nx - na_n \sin nx),$$

which proves the corollary.

* All the derivates then coincide at the point x, so that there is a differential coefficient and all of them are continuous.

14. We now turn to a theorem of a different character, from which we shall be able to draw a variety of important consequences.

THEOREM.—*If two functions agree in a certain completely open interval, the difference of the n-th partial summations (C1) of the derived series of their Fourier series converges uniformly to zero in any closed interval inside the completely open interval.*

To prove this we require two Lemmas:—

LEMMA 1.*—*If $f(x)$ is a summable function in an interval (a, b), then, for any pair of points x and y of this interval,*

$$\operatorname*{Lt}_{n\to\infty} \int_x^y f(x) \begin{matrix}\cos\\ \sin\end{matrix} nx\, dx = 0,$$

the convergence being uniform, as x and y vary in (a, b).

If $f(x)$ is a constant, say P,

$$\left| \int_x^y P \begin{matrix}\cos\\ \sin\end{matrix} nx\, dx \right| = \left| \frac{1}{n} P \left[\begin{matrix}\sin\\ \cos\end{matrix} nx\right]_x^y \right| \leqslant 2P/n,$$

from which the result is obvious.

Hence also it is obvious if $f(x)$ is a simple l- or u-function. Next, let $f_1, f_2, \ldots$ be functions none of which exceed $\leqslant f(x)$, and whose integrals have for limit the integral of $f(x)$. This is, for instance, the case if the functions f_r form a monotone ascending sequence with $f(x)$ as limit, or if they are properly chosen u-functions less than $f(x)$. Then, since $f-f_r \geqslant 0$,

$$\left| \int_x^y (f-f_r) \begin{matrix}\cos\\ \sin\end{matrix} nx\, dx \right| \leqslant \int_x^y (f-f_r)\, dx \leqslant \int_a^b (f-f_r)\, dx \leqslant e,$$

if r is chosen greater than a certain fixed integer r_e, depending only on e.

This integral converges therefore uniformly to zero, when x and y vary as we please in (a, b) and $n \to \infty$. But

$$\int_x^y f(x) \begin{matrix}\cos\\ \sin\end{matrix} nx\, dx = \int_x^y (f-f_r) \begin{matrix}\cos\\ \sin\end{matrix} nx\, dx + \int_x^y f_r(x) \begin{matrix}\cos\\ \sin\end{matrix} nx\, dx,$$

and each of the integrals on the right converges uniformly to zero, so that the integral on the left does so. This proves the theorem for $f(x)$.

Similarly the theorem holds for $f(x)$, if

$$f_r(x) \geqslant f(x).$$

* This includes the result to which I habitually refer as the theorem of Riemann-Lebesgue.

Hence the theorem is true for semi-continuous functions, since these are the limits of monotone sequences of simple l- and u-functions; and therefore it is true for any bounded function $f(x)$, since the integral of $f(x)$ is the upper (lower) bound of the integrals of upper (lower) semi-continuous functions less (greater) than $f(x)$. Hence it is true for any positive summable function $f(x)$, as the limit of a monotone sequence of bounded functions. Finally, it is true of any summable function, as the difference of two positive summable functions.

This proves the lemma.

LEMMA 2.—*If $f(x)$ be summable in (a, b), and $g(x)$ have bounded variation, then $\int_a^b f(x+u)\, g(u) \sin nu\, du$ approaches zero uniformly as $n \to \infty$.*

Put $F(u) = \int_0^u f(x+t) \sin nt\, dt$: then, integrating by parts,

$$\int_a^b f(x+u)\, g(u) \sin nu\, du = F(b)\, g(b) - F(a)\, g(a) - \int_a^b F(u)\, dg(u). \tag{1}$$

Now $$F(u) = \cos nx \int_x^{x+u} f(t) \sin nt\, dt - \sin nx \int_x^{x+u} f(t) \cos nt\, dt,$$

and therefore approaches zero uniformly as $n \to \infty$, since $|\sin nx|$ and $|\cos nx|$ do not exceed 1, and the two integrals approach zero uniformly, by Lemma 1. Integrating therefore with respect to the function $g(u)$ of bounded variation, $\int_a^b F(u)\, dg(u)$ approaches zero uniformly. Since also $F(b)\, g(b)$ and $F(a)\, g(a)$ approach zero, by the theorem of Riemann-Lebesgue, the required result follows from (1).

15. We now return to the theorem enunciated above (§ 14), and proceed to prove it.

Let (a, b) be the completely open interval in which f_1 and f_2 agree, and let us consider any point x of the closed interval (a', b') inside (a, b), where the distances of the end-points of the latter from those of the former interval are all greater than e. Then in the interval $(x-e, x+e)$ the functions f_1 and f_2 coincide.

Let $S'_{n,1}$ and $S'_{n,2}$ denote the differential coefficients of the n-th partial summations of the corresponding Fourier series, each taken in the Cesàro manner, index unity. Then, by the formula (3) of § 10,

$$[S'_{n,1}(x) - S'_{n,2}(x)] = -\int_0^\pi [\phi(x+t) - \phi(x-t)] \frac{d}{dt}[(1-\cos nt)/2n(1-\cos t)]\,dt,$$

where $$\phi(x) = f_1(x) - f_2(x)$$

is zero in the given interval. Hence

$$[S'_{n,1}(x) - S'_{n,2}(x)] = \tfrac{1}{2}\int_e^\pi \frac{\phi(x+t)-\phi(x+t)}{1-\cos t}\sin nt\,dt$$

$$-\frac{1}{2n}\int_e^\pi \frac{\phi(x+t)-\phi(x-t)}{(1-\cos t)^2}(1-\cos nt)\sin t\,dt.$$

The first of these integrals is the sum of two integrals, one involving $\phi(x+t)$ and the other $\phi(x-t)$, each of which converges uniformly to zero, by Lemma 2, as x moves about in the interval (a', b') and $n \to \infty$, since the quantity e does not then depend upon x. The second of the integrals on the right of the last equation also approaches zero uniformly; for, in virtue of the inequality

$$\left| \int_e^\pi \frac{\phi(x+t)-\phi(x-t)}{(1-\cos t)^2}(1-\cos nt)\sin t\,dt \right|$$

$$\leqslant 2\int_e^\pi |\phi(x+t)-\phi(x-t)|\,dt \Big/ (1-\cos e)^2$$

$$\leqslant 2\left\{\int_{e+x}^{\pi+x} |\phi(t)|\,dt + \int_{x-\pi}^{x-e} |\phi(t)|\,dt\right\} \Big/ (1-\cos e)^2$$

$$\leqslant 4\int_{a-\pi}^{b+\pi} |\phi(t)|\,dt/(1-\cos e)^2,$$

the integral in question is numerically less than $K/2n$, where K is independent of both n and x, as long as x remains in (a', b').

Thus $S'_{n,1} - S'_{n,2}$ also converges uniformly to zero, when x moves about in the closed interval (a', b') and $n \to \infty$.

This proves the theorem.

16. From the theorem just proved we can deduce a number of important consequences. The following is an immediate corollary :—

COR. :—

(i) *If two functions agree in a certain completely open interval, then if one of the derived series of the corresponding Fourier series converges uniformly in a closed interval or at a point of that open interval, so does the other.*

(ii) *If one series converges boundedly, so does the other.*

(iii) *If one series oscillates uniformly above or below, so does the other.*

(iv) *If one series oscillates boundedly, so does the other.*

It is supposed that in all four cases the convergence is taken C1.

17. But further the theorem gives us numerous theorems which correspond to the theorems in my paper "On the Integration of Fourier Series."* The following three theorems are given as applications of the principles, and as showing the use we can make of the derived series of Fourier series. These theorems all refer to a certain sub-interval of the interval of periodicity, and show that, in such an interval, these series are as useful as Fourier series.

THEOREM.—*If $f(x)$ is a function which, in a certain sub-interval of the interval of periodicity, is the integral of a function whose $(1+p)$-th power is summable, and $g(x)$ of a function whose $(1+1/p)$-th power is summable, then, if we multiply the derived series of the Fourier series of $f(x)$ by $g(x)$, we may integrate term-by-term over the sub-interval, provided we sum in the Cesàro manner index unity; and the result will be the integral of $g(x)\, df/dx$ over the sub-interval.*

Let $f_1(x)$ agree with $f(x)$ in the sub-interval, say (a, b), and outside (a, b) let $f_1(x)$ be the integral of a function whose $(1+p)$-th power is summable. Then the theorem is true for $f_1(x)$ and $g(x)$.

But in (a, b)
$$\frac{df}{dx} = \frac{df_1}{dx},$$

so that
$$\int_a^b g(x)\frac{df}{dx}\,dx = \operatorname*{Lt}_{n\to\infty}\int_a^b g(x)\,v_{n,1}(x)\,dx,$$

writing $v_n(x)$ and $v_{n,1}(x)$ for the n-th Cesàro partial summations of the derived series of the Fourier series of $f(x)$ and $f_1(x)$ respectively. But, by the theorem of § 14, $[v_n(x)-v_{n,1}(x)]$ converges uniformly to zero, as $n\to\infty$, x moving as we please in (a, b). Also $g(x)$ is summable, so that

$$\int_a^b g(x)\,[v_n(x)-v_{n,1}(x)]\,dx \to 0$$

* *Proc. London Math. Soc.*, 1910, Ser. 2, Vol. 9, pp. 449–462.

as $n \to \infty$; whence

$$\int_a^b g(x) \frac{df}{dx} dx = \underset{n\to\infty}{\text{Lt}} \int_a^b g(x)\, v_n(x)\, dx,$$

which proves the theorem.

18. The argument is the same in the proof of each of the two following theorems :—

THEOREM.—*If $f(x)$ is an absolutely convergent integral in a certain interval (a, b), and $g(x)$ any bounded function, the same result holds as in the preceding theorem, viz.,*

$$\int_a^b g(x) \frac{df}{dx} dx = \underset{n\to\infty}{\text{Lt}} \int_a^b g(x)\, v_n(x)\, dx,$$

where $v_n(x)$ is the Cesàro partial summation, index unity, of the derived series of the Fourier series of $f(x)$.

And also the following :—

THEOREM.—*If $f(x)$ be the integral of a bounded function in a certain interval (a, b), and $g(x)$ any summable function, the same result holds as in the last two theorems, viz.,*

$$\int_a^b g(x) \frac{df}{dx} dx = \underset{n\to\infty}{\text{Lt}} \int_a^b g(x)\, v_n(x)\, dx.$$

It may be remarked that in the first of these theorems there is no simplification when $p = 1$, nor in the second when $g(x)$ is a function of bounded variation, since the theorem respecting the uniformity (§ 14), refers only to the Cesàro summations, and not to the ordinary summations.

19. For reasons which will immediately appear, it will be convenient to introduce the term "restricted Fourier series", using this expression in the following sense :—

The p-th derived series of the Fourier series of $f(x)$ is said to be a restricted Fourier series of the p-th class, and to be restricted to one or more intervals (a, b), if, throughout each such completely open interval (a, b), $f(x)$ is a p-th integral.

The interval (a, b) may be a sub-interval of the interval $(-\pi, \pi)$ of periodicity, but it may also coincide with it without the restricted Fourier series becoming itself a Fourier series; in fact, in order that a trigonometrical series may be a Fourier series it is necessary and sufficient that it should be the derived series of a function of period 2π which throughout the closed interval of periodicity is an integral.

The p-th differential coefficient of $f(x)$ in the interval (a, b) is called the function associated with the restricted Fourier series.

Instead of saying that a series is a restricted Fourier series, restricted to the interval (a, b), we may, for brevity, say it is *a Fourier series restricted to the interval (a, b)*.

With this definition the theorem of § 16 leads to the following important result:—

THEOREM.—*The condition that a restricted Fourier series of the p-th class should converge (Cp), at the point x of the interval to which it is restricted, is precisely of the same form as the condition that a Fourier series should converge (Cp) at that point, the function associated with the restricted Fourier series taking the place of that associated with the Fourier series.*

On the other hand, it will be noticed that there are for such a series no theorems relating to convergence (Cq), where $q < p$.

To prove this theorem consider, for simplicity, the case when $p = 1$, and let $f(x)$ be the function associated with the given restricted Fourier series, and v_n the n-th Cesàro partial summation of that series. On the other hand, let $f_1(x)$ be a function of period 2π, equal to $f(x)$ in the completely open interval (a, b), and an integral at the remaining points of the interval of periodicity, and let $v_{n,1}$ denote the n-th Cesàro partial summation of the Fourier series of $f_1(x)$. Then, by the theorem of § 16, $v_n(x) - v_{n,1}(x)$ converges uniformly to zero. Therefore, if any condition is satisfied which ensures the convergence of $v_{n,1}(x)$ to a unique and finite limit, $v_n(x)$ will also converge to the same limit. This condition will, of course, involve the function $f_1(x)$, but, since it is independent of the form of $f_1(x)$ except in a certain neighbourhood of the point x as small as we please, it will be unaltered* if we change $f_1(x)$ into $f(x)$, the point x being a point of the completely open interval (a, b).

* In all the conditions which actually are used, an arbitrary small neighbourhood of the point x appears explicitly as the range of the variable. It is hardly necessary to point out that, if the condition should involve formally values of $f_1(x)$ outside such an arbitrary small neighbourhood of the point, $f(x)$ may, when substituted, be supposed to have any convenient values at points outside (a, b), or, if we prefer, the interval (a, b) may in the condition be substituted for $(-\pi, \pi)$.

The condition being therefore fulfilled when $f(x)$ is substituted for $f_1(x)$, $v_n(x)$ converges. This proves the theorem.

In particular we have the condition corresponding to de la Vallée Poussin's condition for the convergence of a Fourier series:—

COR. 1.—*The restricted Fourier series of $f(x)$ converges (Cp) at the point x of the interval to which it is restricted, if*

$$\phi_{p+1}(u) = \int_0^1\int_0^1 \dots \int_0^1 \tfrac{1}{2}\left[f(x+ut_1t_2\dots t_{p+1})+f(x-ut_1t_2\dots t_{p+1})\right]dt_1\,dt_2\dots dt_{p+1}$$

is a function of u of bounded variation in some interval containing $u = 0$. The Cesàro sum of the series, index p, is then $\phi_{p+1}(+0)$.

We have also the condition corresponding to my own condition for the convergence of a Fourier series, given above (§ 5):—

COR. 2.—*The restricted Fourier series of $f(x)$ converges (Cp) at the point x of the interval to which it is restricted, if $\phi_p(+0)$ exists, where*

$$\phi_p(u) = \frac{1}{u}\int_0^u \phi_{p-1}(t)\,dt$$

$$= \int_0^1\int_0^1 \dots \int_0^1 \tfrac{1}{2}\left[f(x+ut_1t_2\dots t_p)+f(x-ut_1t_2\dots t_p)\right]dt_1\,dt_2\dots dt_p,$$

and if $$\frac{1}{u}\int_0^u |dt\phi_p(t)| = \int_0^1\left|\int_0^1 \dots \int_0^1 \tfrac{1}{2}\left[f(x+ut_1t_2\dots t_p) + f(x-ut_1t_2\dots t_p)\right]dt_1\,dt_2\dots\right|dt_p,$$

is a bounded function of u in some interval containing $u = 0$. The p-th Cesàro sum of the restricted Fourier series of $f(x)$ is then $\phi_p(+0)$.

20. Among the derived series of Fourier series, restricted Fourier series play an important part, owing largely to the existence of a definite function with which such a series may be associated. The theorem just given, interesting as it is, is only a special case of more general results. In order that the r-th derived series of a Fourier series should converge (Cr) at a point, or in a certain interval, it is by no means necessary that the series should be a restricted Fourier series. We proceed to give theorems of this more general nature, confining our attention in the first instance to the case in which $r = 1$. Special attention will be called to the cases when, in consequence of the hypotheses made, we are in effect dealing with restricted Fourier series. Before doing so, however, one general remark should be made. Our theory reveals the importance of the

condition for the convergence (Cr)* of a Fourier series. Whereas a Fourier series may be treated in general without going beyond $r = 1$, a restricted Fourier series cannot.

21. We shall require the following formulæ, which I had already given in a previous communication† :—

If
$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx), \tag{1}$$

and
$$g(u) \sim \sum_{n=1} A_n \cos (2n-1)\, u, \tag{2}$$

where
$$g(u) = \tfrac{1}{4}\,\{f(x+2u) - f(x-2u)\}\, \operatorname{cosec} u, \tag{3}$$

$g(u)$ being supposed to be absolutely integrable (summable), we have

$$\begin{aligned} A_n - A_{n+1} &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\,[\cos(2n-1)\,u - \cos(2n+1)\,u]\,du \\ &= \frac{2}{\pi}\int_{-\pi}^{\pi} g(u) \sin u \sin 2nu\, du \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \tfrac{1}{2}\,\{f(x+2u) - f(x-2u)\} \sin 2nu\, du \\ &= b_n \sin nx - a_n \sin nx. \end{aligned} \tag{4}$$

Now let us write s_n for the n-th partial summation‡ of the derived series of (1), so that
$$s_n = \sum_{r=1}^{n-1} (rb_r \cos rx - ra_r \sin rx),$$

and S_n for the sum of the first n quantities A_r, so that
$$S_n = \sum_{r=1}^{n} A_r.$$

We then have, multiplying both sides of (4) by n, and summing,
$$s_n = S_n - nA_n. \tag{I}$$

Thus, since
$$C(s_n) = (s_1 + s_2 + \ldots + s_n)/(n+1),$$

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* Sufficient conditions are, of course, obtained by replacing r by any smaller value, including zero.

† W. H. Young, "The Convergence of a Fourier Series and its Allied Series", 1911, *Proc. London Math. Soc.*, Ser. 2, Vol. 10, pp. 262, 263, § 4. The notation is altered here, $f(x)$, which is not an integral, taking the place of the integral $F(x)$.

‡ The term corresponding to $r = 0$ is, of course, zero.

and $$C(S_n) = (S_1+S_2+\ldots+S_n)/(n+1) = S_n - C(nA_n),$$

we have $$C(s_n) = 2C(S_n) - S_n. \qquad \text{(II)}$$

Hence, repeating the process of taking the mean any number p of times,

$$(Cp)(s_n) = 2\,(Cp)(S_n) - (C,\,p-1)\,S_n. \qquad \text{(III)}$$

In all cases therefore in which the Fourier series of

$$g(u) = \tfrac{1}{4}[f(x+2u)-f(x-2u)]\ \text{cosec}\ u$$

converges for $u = 0$, the derived series of the Fourier series of $f(x)$ converges in the Cesàro manner, index unity, to the same value as the former series. And when the former series converges $(C,\,p-1)$, the latter series converges (Cp).

22. There are accordingly four cases to be specially mentioned in which the derived series of the Fourier series of $f(x)$ converges $(C1)$ at the point x.

(A) *If $[f(x+t)-f(x-t)]/t$ is a function of t which has bounded variation in an interval containing the origin $t = 0$.*

In this case

$$g(u) - [f(x+2u)-f(x-2u)]\,/\,4u = \tfrac{1}{2}\left(\frac{u-\sin u}{\sin u}\right)[f(x+2u)-f(x-2u)]\,/\,2u,$$

and, being the product of a function of u which is an integral by a function of bounded variation, is a function of bounded variation in the interval mentioned, and has the unique limit zero at $u = 0$. Therefore the left-hand side is a function of u whose Fourier series converges at $u = 0$ to zero, so that the Fourier series of $g(u)$ converges to the same value as that of $[f(x+2u)-f(x-2u)]\,/\,4u$, the convergence of the latter series being ensured by the hypothesis made, and its sum being

$$\underset{u\to 0}{\text{Lt}}\ [f(x+2u)-f(x-2u)]/4u,$$

which is the generalised differential coefficient of $f(x)$ at the point x.

Thus, if $[f(x+t)-f(x-t)]/t$ has bounded variation in an interval containing the origin $t = 0$, the derived series of the Fourier series of $f(x)$ converges to the generalised differential coefficient of $f(x)$.

(B) *If for some value of the constant K the function (of u)*

$$[f(x+u)-f(x-u)-2Ku]\,/\,u^2$$

is absolutely integrable (summable) in a certain interval containing $u = 0$.

In this case

$$\frac{g(u)-K}{u} = \frac{f(x+2u)-f(x-2u)-4K\sin u}{4u\sin u}$$

$$= \frac{f(x+2u)-f(x-2u)-4Ku}{4u^2}\,\frac{u}{\sin u}+K\,\frac{u-\sin u}{u\sin u},$$

and is absolutely integrable in the interval mentioned. Therefore, by the condition of § 9, the Fourier series of $g(u)$ converges at $u = 0$ to the value K.

Thus if, for some value of the constant K,

$$[f(x+u)-f(x-u)-2Ku]/u^2$$

is absolutely integrable in a certain interval containing $u = 0$, the derived series of the Fourier series of $f(x)$ converges at the point x to the value K.

(C) *If $f(x)$ has a generalised differential coefficient at the point x, and*

$$\frac{1}{h}\int_0^h |d[f(x+u)-f(x-u)]|$$

is a bounded function of h in some interval containing $h = 0$.

Putting $\phi(u) = [f(x+2u)-f(x-2u)]/4u,$

$\phi(u)$ has a unique limit when $u \to 0$, namely, the generalised differential coefficient of $f(x)$. Also

$$\frac{1}{h}\int_0^h |d[u\phi(u)]| = \frac{1}{4h}\int_0^h |d[f(x+2u)-f(x-2u)]|$$

$$= \frac{1}{4h}\int_0^{2h} |d[f(x+u)-f(x-u)]|,$$

and is therefore a bounded function of h in some interval containing $h = 0$.

Thus both the conditions involved in my test (§ 5) for the convergence of the Fourier series of $\phi(u)$ at $u = 0$ are satisfied. It follows that this series converges at $u = 0$ to the generalised differential coefficient of $f(x)$ at the origin.

But $$g(u)-\phi(u)=\tfrac{1}{4}\left(\frac{u-\sin u}{u\sin u}\right)[f(x+2u)-f(x-2u)],$$

which is a function of u having the unique limit zero at the origin, and is also a function of bounded variation in an interval containing the origin, since $[f(x+2u)-f(x-2u)]$ is a function of bounded variation in virtue of the second of the given conditions, and $\frac{u-\sin u}{u\sin u}$ is an integral. Thus the Fourier series of $g(u)-\phi(u)$ converges to zero at the origin. It follows that the Fourier series of $g(u)$ converges at the origin to the generalised differential coefficient of $f(x)$. *Thus in this case the derived series of the Fourier series of $f(x)$ converges at the point x to the generalised differential coefficient, that is to*

$$\underset{h\to 0}{\mathrm{Lt}}\ [f(x+h)-f(x-h)]/2h.$$

As an immediate corollary from the result (C), we have, in particular, the two following results, which bear the same relation to (C) that Cor. 2 and Cor. 3 of § 5 do to the theorem of that article:—

If $f(x)$ has a generalised differential coefficient at the origin, and one of the derivates of $f(x)$ [or more generally those of $\{f(x+u)-f(x-u)\}$ with respect to u], is known to be, in a certain neighbourhood of the point,

(a) bounded,

or (b) summable, and, except possibly at a countable set of points, finite,

then the derived series of the Fourier series of $f(x)$ converges at the point x to the generalised differential coefficient of $f(x)$.

As a particular case we have the theorem of § 13.

(D) *If* $\frac{1}{u}\int_0^u \frac{f(x+u)-f(x-u)}{u}\,du$ *is a function of u of bounded variation in an interval containing the origin.*

In this case, writing

$$\psi(u)=[f(x+2u)-f(x-2u)]/4u,$$

we have $$\frac{1}{u}\int_0^u \psi(u)\,du=\frac{1}{2u}\int_0^{2u}\frac{f(x+u)-f(x-u)}{u}\,du,$$

a function of u of bounded variation. Thus, by de la Vallée Poussin's test, the Fourier series of $\phi(u)$ converges at $u=0$ to $\operatorname{Lt}_{u\to 0} \frac{1}{u}\int_0^u \phi(u)\,du$.

In order therefore to prove that the Fourier series of $g(u)$ converges to the same value, it is only necessary to prove that

$$\frac{1}{u}\int_0^u [g(u)-\phi(u)]\,du$$

is a function of bounded variation whose limit when $u=0$ is zero.

Now

$$\frac{1}{u}\int_0^u [g(u)-\psi(u)]\,du = \frac{1}{4u}\int_0^u [f(x+2u)-f(x-2u)]\left(\frac{u-\sin u}{u\sin u}\right)du$$

$$= \frac{1}{2u}\int_0^u q(u)\,d[F(x+2u)+F(x-2u)],$$

where $F(x)$ is the integral of $f(x)$, and $q(u)=\dfrac{u-\sin u}{u\sin u}$ is a function such that $q(u)/u$, as well as the differential coefficient $q'(u)$, is an integral.

Integrating by parts, we get now

$$\frac{1}{2u}q(u)\,[F(x+2u)+F(x-2u)]-\frac{1}{2u}\int_0^u q'(u)\,[F(x+2u)+F(x-2u)]\,du,$$

an expression of which the first term is the product of two integrals, and is therefore a function of bounded variation, and the second term is also a function of bounded variation, since the integrand is a function of bounded variation, so that the integral divided by u is also a function of bounded variation.

Also, when $u\to 0$, the first term of the expression vanishes, since $q(u)$ has zero as limit, and the second term approaches half the value at $u=0$ of the integrand, since the integrand is continuous; but $q'(0)=0$, so that this second term also approaches zero. Thus we have proved that $\frac{1}{u}\int_0^u [g(u)-\phi(u)]\,du$ is a function of bounded variation whose limit when $u=0$ is zero. This proves that *in this case the derived series of the Fourier series of $f(x)$ converges to*

$$\operatorname{Lt}_{u\to 0}\frac{1}{u}\int_0^u \frac{f(x+u)-f(x-u)}{u}\,du.$$

COR.—*If $F(x)$ be the integral of $f(x)$, it is sufficient for the conver-*

gence (C1) of the first derived series of the Fourier series of $f(x)$ that $[F(x+u)+F(x-u)-2F(x)]/u^2$ should be a function of bounded variation.

23. The formula (III) gives us corresponding conditions for the convergence (Cp) of the derived series of the Fourier series of $f(x)$. We merely have to write down conditions that the Fourier series of

$$g(u) = \tfrac{1}{4}[f(x+2u)-f(x-2u)]\operatorname{cosec} u,$$

should converge $(C, p-1)$ at $u = 0$.

Thus, for instance, using the conditions for the Cesàro convergence of a Fourier series contained in § 6 from my new condition for the convergence of a Fourier series, we have the following conditions :—

The first derived series of the Fourier series of $f(x)$ converges $(C2)$ at the point x, if $\phi_2(+0)$ exists, where

$$\phi_2(u) = \frac{1}{u}\int_0^u \phi_1(t)\,dt = \frac{1}{u}\int_0^u \frac{1}{2t}[f(x+t)-f(x-t)]\,dt$$

$$= \frac{1}{u}\int_0^1 \tfrac{1}{2}[f(x+ut)-f(x-ut)]\,dt,$$

and if

$$\frac{1}{u}\int_0^u |d[t\phi_2(t)]| = \frac{1}{u}\int_0^u |\phi_1(t)\,dt| = \frac{1}{u}\int_0^u \left|\frac{1}{2t}[f(x+t)-f(x-t)]\,dt\right|$$

is a bounded function of u in a certain interval containing $u = 0$.

Indeed if these conditions are satisfied, they are also satisfied when we change u into $2u$, so that the Fourier series of $\psi_1(2u)$ converges $(C1)$.

But, as we saw in § 22, $g(u)-\psi_1(2u)$ is a function of bounded variation of u which is continuous at the origin and has there the value zero, so that its Fourier series converges at $u = 0$ to zero. Hence the Fourier series of $g(u)$ converges $(C1)$ at $u = 0$, whence the required result is also true.

Similarly, we have the more general condition, as follows :—

The first derived series of the Fourier series of $f(x)$ converges (Cp) at the point x, if $\psi_p(+0)$ exists, where

$$\psi_p(u) = \frac{1}{u}\int_0^1\int_0^1 \dots \tfrac{1}{2}[f(x+ut_1t_2\dots t_{p-1})-f(x-ut_1t_2\dots t_{p-1})]\,dt_1\,dt_2\dots dt_{p-1},$$

and if $\dfrac{1}{u}\displaystyle\int_0^u |\psi_{p-1}(t)dt|$ *is a bounded function of u in a certain interval containing* $u = 0$.

24. The discussion of the convergence $(C2)$ of the second derived series, and the convergence (Cp) of the p-th derived series, may be undertaken on the same lines as have been followed in §§ 20–23 for the first derived series.

Let $f(x)$ be a periodic summable function of x, with period 2π, and let its Fourier series be written in the following form :—

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)/n^2 = \tfrac{1}{2}a_0 - \sum_{n=1} \alpha_n/n^2.$$

Now let us write

$$g(u) = [f(x+2u)+f(x-2u)-2f(x)]/4\sin^2 u.$$

Then $g(u)$ is an even function of u, such that $g(\pi+u) = g(u)$, and therefore its Fourier series has no sine terms, and the cosines of the odd multiples of u are absent. We may therefore write

$$g(u) \sim \tfrac{1}{2}A_0 + \sum_{n=1} A_n \cos 2nu.$$

We then have, for $n \geqslant 0$,

$$\tfrac{1}{2}\pi(A_n - A_{n+1}) = \int_{-\pi}^{\pi} g(u)\,\tfrac{1}{2}[\cos 2nu - \cos 2(n+1)u]\,du$$

$$= \int_0^{\pi} 2g(u)\sin u \sin(2n+1)\,u\,du\,;$$

and therefore, for $n \geqslant 1$,

$$\tfrac{1}{2}\pi(A_{n-1} - 2A_n + A_{n+1})$$

$$= \int_0^{\pi} 2g(u)\sin u\,[\sin(2n-1)\,u - \sin(2n+1)\,u]\,du$$

$$= -\int_0^{\pi} [f(x+2u)+f(x-2u)-2f(x)]\cos 2nu\,du$$

$$= \sum_{m=1} \frac{1}{m^2}\int_0^{\pi} 2(a_m \cos mx + b_m \sin mx)(1-\cos 2mu)\cos nu\,du.$$

Hence $\quad A_{n-1} - 2A_n + A_{n+1} = -2(a_n \cos nx + b_n \sin nx)/n^2 = 2\alpha_n.$

We have then

$$\sum_{r=1}^{n} r^2(A_{r-1} - 2A_r + A_{r+1}) = 2\sum_{r=1}^{n} \alpha_r = 2s_n,$$

say. On the left the coefficient of A_0 is 1, of A_1 is 2, of A_n is

$$[(n-1)^2-n^2] = (n+1)^2-2,$$

and of A_{n+1} is n^2; and that of each of the remaining quantities A_r is

$$(r-1)^2-2r^2+(r+1)^2 = 2.$$

Thus our equation becomes, when we write

$$S_n = \tfrac{1}{2}A_0+A_1+\ldots+A_n,$$

$$2S_r-(r+1)^2 A_r+r^2 A_{r+1} = 2s_r. \tag{IV}$$

Adding the equations (IV) for all values of r from 1 to n, we evidently get

$$2(S_1+S_2+\ldots+S_n)-4 \sum_{r=1}^{n} rA_r+n^2A_{n+1} = 2(s_1+s_2+\ldots+s_n).$$

Since, however,

$$A_1+2A_2+\ldots+nA_n = (S_1-S_0)+2(S_2-S_1)+\ldots+n(S_n-S_{n-1})$$

$$= (n+1) S_n - \sum_{r=0}^{n} S_r,$$

we get, writing

$$C(S_n) = (S_1+S_2+\ldots+S_n)/n, \qquad C(s_n) = (s_1+s_2+\ldots+s_n)/n,$$

$$6C(S_n)+4S_0/n-4(n+1)S_n/n+nA_{n+1} = 2C(s_n). \tag{V}$$

Proceeding to the limit, this becomes

$$\operatorname*{Lt}_{n\to\infty} [6C(S_n)-4S_n+nA_n] = 2 \operatorname*{Lt}_{n\to\infty} C(s_n); \tag{V$'$}$$

and, operating on (V) in the Cesàro manner and proceeding to the limit, we get

$$\operatorname*{Lt}_{n\to\infty} [6(C2)(S_n)-5C(S_n)+S_n] = 2 \operatorname*{Lt}_{n\to\infty} (C2)(s_n). \tag{VI}$$

Generally*

$$\operatorname*{Lt}_{n\to\infty} [6(Cp)(S_n)-5(C, p-1)(S_n)+(C, p-2)(S_n)] = 2 \operatorname*{Lt}_{n\to\infty} (Cp)(S_n); \tag{VII}$$

that is, denoting the Cesàro summation, index p, of the second derived series of the Fourier series of $f(x)$ by t_p, and that of $g(u)$ for $u=0$ by T_p,

$$6T_p-5T_{p-1}+T_{p-2} = 2t_p. \tag{VII$'$}$$

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* As on p. 264, footnote, of "The Convergence of a Fourier Series and its Allied Series", this relation holds in the first instance for positive integral values of p, but is true when p is fractional also.

The corresponding form of (V′), is the following, in which the index zero denotes an ordinary summation :—

$$6T_1 - 4T_0 + \operatorname*{Lt}_{n\to\infty} nA_n = t_0. \tag{V″}$$

In all cases therefore in which the Fourier series of

$$g_2(u) = [f(x+2u)+f(x-2u)-2f(x)]/4\sin^2 u$$

converges for $u = 0$, *the second derived series of the Fourier series of* $f(x)$ *converges* ($C2$) *to the same sum; also when the former series converges* $(C, p-2)$, *the latter series converges* (Cp).

Hence, as is evident, we have tests for the convergence ($C2$) of the second derived series of the Fourier series of $f(x)$ at the point x exactly corresponding to those given in § 21 for the convergence ($C1$) of the first derived series, and we have tests for the convergence $(C, p+2)$ of the second derived series, corresponding to the tests for the convergence $(C, p+1)$ of the first derived series.

25. The treatment of the third and higher derived series is precisely similar; we only have to replace the function $g_2(u)$ used in the discussion of the second derived series by

$$g_3(u) = [f(x+2u)-f(x-2u)-\psi_1(x)\sin 2u]/\sin^3 u$$

in dealing with the third derived series, and by

$$g_4(u) = [f(x+2u)+f(x-2u)-2f(x)-\phi_1(x)\sin^2 u]/\sin^4 u$$

in dealing with the fourth derived series. Here ψ_1 and ϕ_1 are in the first instance such functions of x as render g_3 and g_4 summable functions of u. In consequence of the conditions introduced in the tests employed, these functions of x come to be—to a numerical factor *près*—the generalised first and second differential coefficient of $f(x)$, the generalised differential coefficients being accordingly required to exist.

More generally the auxiliary functions to be employed are the following :—

$$g_{2n+1}(u) = [f(x+2u)-f(x-2u)-\psi_1(x)\sin 2u-\psi_2(x)\sin^3 2u-\psi_3(x)\sin^5 2u - \ldots - \psi_n(x)\sin^{2n-1} 2u]/\sin^{2n+1} u,$$

and

$$g_{2n}(u) = [f(x+2u)+f(x-2u)-2f(x)-\phi_1(x)\sin^2 u-\phi_2(x)\sin^4 u - \ldots - \phi_{n-1}(x)\sin^{2n-2} u]/\sin^{2n} u.$$

26. Taking, for example, the case of the fourth derived series, we have, g_2 and g_4 having the meanings assigned to them in the preceding article,

$$g_4(u)\sin^2 u = g_2(u) - \tfrac{1}{4}\phi_1(x).$$

Hence, if
$$g_{2r}(u) \sim \tfrac{1}{2}A_0^{2r} + \sum_{n=1}^{\infty} A_n^{2r}\cos 2nu,$$

we have
$$A_{n-1}^{(4)} - 2A_n^{(4)} + A_{n+1}^{(4)} = -\frac{2}{\pi}\int_0^{\pi} 4g_4(u)\sin^2 u\cos 2nu\,du$$
$$= -\frac{2}{\pi}\int_0^{\pi} 4\left[g_2(u) - \tfrac{1}{4}\phi_1(x)\right]\cos 2nu\,du$$
$$= -\frac{2}{\pi}\int_0^{\pi} 4g_2(u)\cos 2nu\,du \quad (n \geqslant 1)$$
$$= -4A_n^{(2)}.$$

Since
$$A_{n-1}^{(2)} - 2A_n^{(2)} + A_{n+1}^{(2)} = \frac{2}{n^4}(a_n\cos nx + b_n\sin nx) \quad (n \geqslant 1)$$

we get

$$A_{n-2}^{(4)} - 4A_{n-1}^{(4)} + 6A_n^{(4)} - 4A_{n+1}^{(4)} + A_{n+2}^{(4)} = -8(a_n\cos nx + b_n\sin nx)/n^4 \quad (n \geqslant 2).$$

In the general case in which $g_p(u)$ takes the place of $g_4(u)$, we have the corresponding formula

$$A_{n-p+1}^{(p)} - pA_{n-p+2}^{(p)} + \binom{p}{2}A_{n-p+3}^{(p)} - \ldots + (-)^r\binom{p}{r}A_{n-p+r+1}^{(p)} + \ldots$$
$$= M(a_n\cos nx + b_n\sin nx)/n^p,$$

the coefficients $\binom{p}{r}$ being the binomial coefficients, and M a numerical factor.

Multiplying up by n^p and making use of a process exactly analogous to that already used for $p = 1$ or $p = 2$, we find that the convergence (Cp) of the p-th derived series follows from the ordinary convergence of the Fourier series of $g_p(u)$ at $u = 0$, whenever this occurs.