

ON A FORMULA FOR THE SUM OF A FINITE NUMBER OF
TERMS OF THE HYPERGEOMETRIC SERIES WHEN THE
FOURTH ELEMENT IS EQUAL TO UNITY

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1. The series being

$$1 + \frac{a\beta}{1\gamma} + \frac{a(a+1)\beta(\beta+1)}{2! \gamma(\gamma+1)} + \dots,$$

let

$$a_r = a(a+1) \dots (a+r-1).$$

The $(r+1)$ -th term is $\frac{a_r \beta_r}{r! \gamma_r}$. Also

$$\frac{a_r \beta_r}{r! (\gamma-1)_r} - \frac{a_r \beta_r}{r! \gamma_r} = \frac{a\beta}{(\gamma-1)\gamma} \frac{(a+1)_{r-1} (\beta+1)_{r-1}}{(r-1)! (\gamma+1)_{r-1}}.$$

Now, let $G(a, \beta, \gamma, s)$ denote the sum of the first $(s+1)$ terms of the series. Then

$$\begin{aligned} G(a, \beta, \gamma-1, s) - G(a, \beta, \gamma, s) &= \sum_{r=1}^{r=s} \left(\frac{a_r \beta_r}{r! (\gamma-1)_r} - \frac{a_r \beta_r}{r! \gamma_r} \right) \\ &= \frac{a\beta}{(\gamma-1)\gamma} \sum_{r=1}^{r=s} \frac{(a+1)_{r-1} (\beta+1)_{r-1}}{(r-1)! (\gamma+1)_{r-1}}; \end{aligned}$$

and therefore

$$G(a, \beta, \gamma-1, s) - G(a, \beta, \gamma, s) = \frac{a\beta}{(\gamma-1)\gamma} G(a+1, \beta+1, \gamma+1, s-1). \quad (\text{I.})$$

2. Consider next the expression

$$\begin{aligned} G(a, \beta, \gamma, s) - \frac{\gamma-a-\beta-1}{\gamma} G(a+1, \beta+1, \gamma+1, s-1) \\ = 1 + \frac{a\beta}{\gamma} + \frac{a_2 \beta_2}{2! \gamma_2} + \dots + \frac{a_s \beta_s}{s! \gamma_s} \\ - \frac{\gamma-a-\beta-1}{\gamma} \left(1 + \frac{(a+1)(\beta+1)}{(\gamma+1)} + \frac{(a+1)_2 (\beta+1)_2}{2! (\gamma+1)_2} + \dots \right. \\ \left. + \frac{(a+1)_{s-1} (\beta+1)_{s-1}}{(s-1)! (\gamma+1)_{s-1}} \right). \end{aligned}$$

The first and second terms of the first series, together with the first term of the second series, add up to

$$[(\alpha+1)(\beta+1)]/\gamma.$$

This together with one more term from each series add up to

$$[(\alpha+1)_2(\beta+1)_2]/2! \gamma_2.$$

This suggests that the sum of the first r terms of the first series and the first $(r-1)$ terms of the second series will add up to

$$\frac{(\alpha+1)_{r-1}(\beta+1)_{r-1}}{(r-1)! \gamma_{r-1}}.$$

Adding in one more term of each series, we have

$$\begin{aligned} \frac{(\alpha+1)_{r-1}(\beta+1)_{r-1}}{(r-1)! \gamma_{r-1}} + \frac{\alpha_r \beta_r}{r! \gamma_r} - \frac{\gamma-\alpha-\beta-1}{\gamma} \frac{(\alpha+1)_{r-1}(\beta+1)_{r-1}}{(r-1)! (\gamma+1)_{r-1}} \\ = \frac{(\alpha+1)_{r-1}(\beta+1)_{r-1}}{r! \gamma_r} [r(\gamma+r-1) + \alpha\beta - r(\gamma-\alpha-\beta-1)] \\ = \frac{(\alpha+1)_r(\beta+1)_r}{r! \gamma_r}. \end{aligned}$$

Hence the induction holds good, and therefore

$$G(\alpha, \beta, \gamma, s) - \frac{\gamma-\alpha-\beta-1}{\gamma} G(\alpha+1, \beta+1, \gamma+1, s-1) = \frac{(\alpha+1)_s(\beta+1)_s}{s! \gamma_s}. \quad (\text{II.})$$

3. It follows from (I.) and (II.) that

$$\begin{aligned} (\gamma-\alpha-1)(\gamma-\beta-1) G(\alpha, \beta, \gamma, s) - (\gamma-1)(\gamma-\alpha-\beta-1) G(\alpha, \beta, \gamma-1, s) \\ = \frac{\alpha_{s+1}\beta_{s+1}}{s! \gamma_s}. \quad (\text{III.}) \end{aligned}$$

Putting aside for the present the special cases $\gamma=1$, $\gamma=\alpha+\beta+1$, it follows that

$$\begin{aligned} G(\alpha, \beta, \gamma-1, s) \\ = \frac{(\gamma-\alpha-1)(\gamma-\beta-1)}{(\gamma-1)(\gamma-\alpha-\beta-1)} G(\alpha, \beta, \gamma, s) - \frac{\alpha_{s+1}\beta_{s+1}}{(\gamma-\alpha-\beta-1)s! (\gamma-1)_{s+1}}. \end{aligned}$$

Therefore

$$\begin{aligned}
 & G(a, \beta, \gamma, s) \\
 &= \frac{(\gamma-a)(\gamma-\beta)}{\gamma(\gamma-a-\beta)} G(a, \beta, \gamma+1, s) - \frac{a_{s+1}\beta_{s+1}}{(\gamma-a-\beta)s! \gamma_{s+1}} \\
 & G(a, \beta, \gamma+1, s) \\
 &= \frac{(\gamma-a+1)(\gamma-\beta+1)}{(\gamma+1)(\gamma-a-\beta+1)} G(a, \beta, \gamma+2, s) - \frac{a_{s+1}\beta_{s+1}}{(\gamma-a-\beta+1)s! (\gamma+1)_{s+1}} \\
 & G(a, \beta, \gamma+2, s) \\
 &= \frac{(\gamma-a+2)(\gamma-\beta+2)}{(\gamma+2)(\gamma-a-\beta+2)} G(a, \beta, \gamma+3, s) - \frac{a_{s+1}\beta_{s+1}}{(\gamma-a-\beta+2)s! (\gamma+2)_{s+1}} \\
 & \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
 & G(a, \beta, \gamma+t, s) \\
 &= \frac{(\gamma-a+t)(\gamma-\beta+t)}{(\gamma+t)(\gamma-a-\beta+t)} G(a, \beta, \gamma+t+1, s) - \frac{a_{s+1}\beta_{s+1}}{(\gamma-a-\beta+t)s! (\gamma+t)_{s+1}}
 \end{aligned}
 \tag{IV.}$$

From (IV.) it follows that

$$\begin{aligned}
 & G(a, \beta, \gamma, s) \\
 &= \frac{(\gamma-a)_{t+1}(\gamma-\beta)_{t+1}}{\gamma_{t+1}(\gamma-a-\beta)_{t+1}} G(a, \beta, \gamma+t+1, s) - \frac{a_{s+1}\beta_{s+1}}{(\gamma-a-\beta)s! \gamma_{s+1}} f(a, \beta, \gamma, s, t),
 \end{aligned}
 \tag{V.}$$

where $f(a, \beta, \gamma, s, t)$

$$\begin{aligned}
 &= 1 + \frac{(\gamma-a)(\gamma-\beta)}{(\gamma-a-\beta+1)(\gamma+s+1)} + \frac{(\gamma-a)_2(\gamma-\beta)_2}{(\gamma-a-\beta+1)_2(\gamma+s+1)_2} + \dots \\
 & \quad + \frac{(\gamma-a)_t(\gamma-\beta)_t}{(\gamma-a-\beta+1)_t(\gamma+s+1)_t}.
 \end{aligned}
 \tag{VI.}$$

The series $f(a, \beta, \gamma, s, t)$ contains $t+1$ terms, and is, when t is infinite, of the form

$$1 + \frac{ab}{cd} + \frac{a(a+1)b(b+1)}{c(c+1)d(d+1)} + \dots$$

The condition for the convergency of this series is

$$a+b-c-d+1 < 0.$$

Hence, if t be made infinite in $f(a, \beta, \gamma, s, t)$, the condition that $f(a, \beta, \gamma, s, \infty)$ shall be finite is

$$(\gamma-a) + (\gamma-\beta) - (\gamma-a-\beta+1) - (\gamma+s+1) + 1 < 0,$$

$$\text{i.e.,} \quad -(s+1) < 0,$$

which is always satisfied, since s is a positive integer.

Hence $f(a, \beta, \gamma, s, \infty)$ is always finite. If s tend to infinity, it tends to the limit unity. Thus

$$f(a, \beta, \gamma, \infty, \infty) = 1.$$

Further, when t tends to infinity $G(a, \beta, \gamma + t + 1, s)$ tends to unity. Then (V.) becomes

$$G(a, \beta, \gamma, s) = \lim_{t=\infty} \frac{(\gamma-a)_{t+1}(\gamma-\beta)_{t+1}}{\gamma_{t+1}(\gamma-a-\beta)_{t+1}} - \frac{a_{s+1}\beta_{s+1}}{(\gamma-a-\beta)s!\gamma_{s+1}} f(a, \beta, \gamma, s, \infty).$$

Adopting Gauss's notation,

$$\Pi(n, z) = \frac{n!n^z}{(z+1)(z+2)\dots(z+n)} = \frac{n^z\Gamma(n+1)\Gamma(z+1)}{\Gamma(n+z+1)}.$$

$$\frac{(\gamma-a)_{t+1}(\gamma-\beta)_{t+1}}{\gamma_{t+1}(\gamma-a-\beta)_{t+1}} = \frac{\Pi(t+1, \gamma-1)\Pi(t+1, \gamma-a-\beta-1)}{\Pi(t+1, \gamma-a-1)\Pi(t+1, \gamma-\beta-1)},$$

therefore $\lim_{t=\infty} \frac{(\gamma-a)_{t+1}(\gamma-\beta)_{t+1}}{\gamma_{t+1}(\gamma-a-\beta)_{t+1}} = \frac{\Pi(\gamma-1)\Pi(\gamma-a-\beta-1)}{\Pi(\gamma-a-1)\Pi(\gamma-\beta-1)}.$

Also $\frac{a_{s+1}\beta_{s+1}}{s!\gamma_{s+1}} = \frac{\Pi(s+1, \gamma-1)}{\Pi(s+1, a-1)\Pi(s+1, \beta-1)}(s+1)^{a+\beta-\gamma}.$

Hence (V.) gives on making t infinite

$$G(a, \beta, \gamma, s) = \frac{\Pi(\gamma-1)\Pi(\gamma-a-\beta-1)}{\Pi(\gamma-a-1)\Pi(\gamma-\beta-1)} - \frac{\Pi(s+1, \gamma-1)(s+1)^{a+\beta-\gamma}f(a, \beta, \gamma, s, \infty)}{\Pi(s+1, a-1)\Pi(s+1, \beta-1)(\gamma-a-\beta)};$$

or, writing the symbols at length,

$$1 + \frac{a\beta}{\gamma} + \frac{a(a+1)\beta(\beta+1)}{2!\gamma(\gamma+1)} + \dots + \frac{a(a+1)\dots(a+s-1)\beta(\beta+1)\dots(\beta+s-1)}{s!\gamma(\gamma+1)\dots(\gamma+s-1)} \\ = \frac{\Pi(\gamma-1)\Pi(\gamma-a-\beta-1)}{\Pi(\gamma-a-1)\Pi(\gamma-\beta-1)} - \frac{\Pi(s+1, \gamma-1)(s+1)^{a+\beta-\gamma}f(a, \beta, \gamma, s, \infty)}{(\gamma-a-\beta)\Pi(s+1, a-1)\Pi(s+1, \beta-1)}$$

where $f(a, \beta, \gamma, s, \infty)$ (VII.)

$$= \left(1 + \frac{(\gamma-a)(\gamma-\beta)}{(\gamma-a-\beta+1)(\gamma+s+1)} + \frac{(\gamma-a)(\gamma-a+1)(\gamma-\beta)(\gamma-\beta+1)}{(\gamma-a-\beta+1)(\gamma-a-\beta+2)(\gamma+s+1)(\gamma+s+2)} + \dots \text{ to } \infty \right).$$

4. From this may be inferred, if the real part of $\gamma-a-\beta$ be positive, that the series is convergent, and that it converges to the limit

$$\frac{\Pi(\gamma-1)\Pi(\gamma-a-\beta-1)}{\Pi(\gamma-a-1)\Pi(\gamma-\beta-1)},$$

which is Gauss's formula.

In this case also, if s be finite, the second term gives an approximate value for the error made by leaving out the rest of the terms of the series.

If, however, the real part of $\gamma - \alpha - \beta$ be negative, then the series is asymptotic with

$$\frac{\Pi(\gamma-1)(s+1)^{\alpha+\beta-\gamma}}{(a+\beta-\gamma)\Pi(a-1)\Pi(\beta-1)}$$

when s is large, and of course $(s+1)$ can be replaced by s in this expression.

The formula fails when $\gamma - \alpha - \beta$ is zero or a negative integer. It fails also if γ is a negative integer, but in the latter case all the terms in the series are infinite from a certain term onwards. In a subsequent communication I hope to discuss these cases of failure.

5. The following special cases may be noticed by way of verification.

(a) If $\gamma = \alpha + \beta + 1$, it follows from equation (II.) or (III.) that

$$G(a, \beta, \alpha + \beta + 1, s) = \frac{(a+1)_s (\beta+1)_s}{s! (\alpha + \beta + 1)_s}.$$

Similar, but less convenient, formulæ can be deduced from (III.) for the case $\gamma = \alpha + \beta + a$ a positive integer.

(b) If a be a negative integer ($-r$), then the series contains a finite number of terms. Taking now $s \geq r$, it follows that $a_{s+1} = 0$,

$$\frac{\Pi(\gamma - a - 1)}{\Pi(\gamma - 1)} = (\gamma - a - 1)(\gamma - a - 2) \dots (\gamma),$$

$$\frac{\Pi(\gamma - a - \beta - 1)}{\Pi(\gamma - \beta - 1)} = (\gamma - a - \beta - 1)(\gamma - a - \beta - 2) \dots (\gamma - \beta);$$

and therefore the sum of the series is

$$\frac{(\gamma - a - \beta - 1)(\gamma - a - \beta - 2) \dots (\gamma - \beta)}{(\gamma - a - 1)(\gamma - a - 2) \dots (\gamma)}$$

where a is a negative integer.

(c) If $\gamma - a$ be zero or a negative integer ($-r$), where r is a positive integer,

$$\Pi(\gamma - a - 1) = \infty, \quad (\gamma - a)_{r+1} = 0,$$

and so the series $f(a, \beta, \gamma, s, \infty)$ contains only $(r+1)$ terms.

If $\gamma = a$, the result is

$$1 + \frac{\beta}{1} + \frac{\beta(\beta+1)}{2!} + \dots + \frac{\beta(\beta+1) \dots (\beta+s-1)}{s!} = \frac{\beta(\beta+1) \dots (\beta+s)}{s!},$$

which is, of course, well known.

[I am indebted to the referee for the following remark.

Taking the case in which the real part of $\gamma - \alpha - \beta$ is positive, the equation (VII.) may, by substituting in (VII.) for

$$\frac{\Pi(\gamma-1) \Pi(\gamma-\alpha-\beta-1)}{\Pi(\gamma-\alpha-1) \Pi(\gamma-\beta-1)}$$

the series
$$1 + \frac{\alpha\beta}{1! \gamma} + \frac{\alpha_2\beta_2}{2! \gamma_2} + \dots,$$

transferring this series to the other side of the equation, and then dividing out by the factor

$$-\frac{\Pi(s+1, \gamma-1) (s+1)^{\alpha+\beta-\gamma}}{\Pi(s+1, \alpha-1) \Pi(s+1, \beta-1)},$$

be transformed into the following:—

$$\begin{aligned} & \frac{1}{s+1} \left(1 + \frac{\alpha+s+1}{s+2} \frac{\beta+s+1}{\gamma+s+1} + \frac{(\alpha+s+1)_2 (\beta+s+1)_2}{(s+2)_2 (\gamma+s+1)_2} + \dots \right) \\ &= \frac{1}{\gamma-\alpha-\beta} \left(1 + \frac{\gamma-\alpha}{\gamma-\alpha-\beta+1} \frac{\gamma-\beta}{\gamma+s+1} + \frac{(\gamma-\alpha)_2 (\gamma-\beta)_2}{(\gamma-\alpha-\beta+1)_2 (\gamma+s+1)_2} + \dots \right). \end{aligned} \quad \text{(VIII.)}$$

Equation (VIII.) has been proved on the hypothesis that $s+1$ was a positive integer. In the particular case, however, when $\beta = \gamma$, the equation becomes

$$\frac{1}{\alpha} + \frac{1}{s+1} \left(1 + \frac{\alpha+s+1}{s+2} + \frac{(\alpha+s+1)_2}{(s+2)_2} + \dots \right) = 0, \quad \text{(IX.)}$$

and the condition that the real part of $\gamma - \alpha - \beta > 0$ becomes that the real part of α is negative.

Now
$$\frac{1}{\alpha} + \frac{1}{s+1} = \frac{\alpha+s+1}{\alpha(s+1)},$$

$$\frac{1}{\alpha} + \frac{1}{s+1} \left(1 + \frac{\alpha+s+1}{s+2} \right) = \frac{(\alpha+s+1)_2}{\alpha(s+1)_2}.$$

This suggests that

$$\frac{1}{\alpha} + \frac{1}{s+1} \left(1 + \frac{\alpha+s+1}{s+2} + \dots + \frac{(\alpha+s+1)_{n-1}}{(s+2)_{n-1}} \right) = \frac{(\alpha+s+1)_n}{\alpha(s+1)_n},$$

which can be immediately verified by induction. But

$$\frac{(\alpha+s+1)_n}{\alpha(s+1)_n} = \frac{1}{\alpha} \frac{\Pi(n, s)}{\Pi(n, \alpha+s)} n^{\alpha}.$$

Hence the left-hand side of equation (IX.) is

$$\lim_{n=\infty} \frac{1}{a} \frac{\Pi(n, s)}{\Pi(n, a+s)} n^a,$$

and this vanishes, since the real part of a is negative.

Hence equation (IX.) is true independently of the integral character of $(s+1)$, and the referee suggests that equation (VIII.) may also be true independently of the integral character of $(s+1)$.

I hope to be able to include a discussion of this point in the communication promised above in regard to the cases of failure of equation (VII.).—*March 18th, 1907.*]