

HODGE THEORY I

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We propose to give a heuristic dictionary between statements in ℓ -adic cohomology and statements in Hodge theory. This dictionary draws in particular on [3] and on Grothendieck's conjectural theory of motives [2]. So far it has mainly served to formulate conjectures in Hodge theory, and has sometimes suggested proofs of them.

Definition 1.1. A mixed Hodge structure H consists of

- (a) a finitely generated $\mathbb{Z}$ -module $H_{\mathbb{Z}}$ (the “integral lattice”);
- (b) a finite increasing filtration W on $H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes \mathbb{Q}$ (the “weight filtration”);
- (c) a finite decreasing filtration F on $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C}$ (the “Hodge filtration”).

These data are subject to the axiom:

There exists on $\mathrm{Gr}_W(H_{\mathbb{C}})$ a (unique) bigrading by subspaces $H^{p,q}$ such that

(i)

$$\mathrm{Gr}_W^n(H_{\mathbb{C}}) = \bigoplus_{p+q=n} H^{p,q};$$

(ii) the filtration F induces on $\mathrm{Gr}_W(H_{\mathbb{C}})$ the filtration

$$\mathrm{Gr}_W(F)^p = \bigoplus_{p' \geq p} H^{p',q'};$$

(iii) $\overline{H^{p,q}} = H^{q,p}$.

A morphism $f : H \rightarrow H'$ is a homomorphism $f_{\mathbb{Z}} : H_{\mathbb{Z}} \rightarrow H'_{\mathbb{Z}}$ such that $f_{\mathbb{Q}} : H_{\mathbb{Q}} \rightarrow H'_{\mathbb{Q}}$ and $f_{\mathbb{C}} : H_{\mathbb{C}} \rightarrow H'_{\mathbb{C}}$ are respectively compatible with the filtrations W and F .

The Hodge numbers of H are the integers

$$h^{p,q} = \dim H^{p,q} = h^{q,p}. \quad (1.2)$$

One says that H is pure of weight n if $h^{p,q} = 0$ for $p + q \neq n$ (i.e. if $\mathrm{Gr}_W^i(H) = 0$ for $i \neq n$). One also says that H is a Hodge structure of weight n .

The Tate Hodge structure $\mathbb{Z}(1)$ is the Hodge structure of weight -2 , purely of type $(-1, -1)$, for which $\mathbb{Z}(1)_{\mathbb{C}} = \mathbb{C}$ and $\mathbb{Z}(1)_{\mathbb{Z}} = 2\pi i\mathbb{Z} = \mathrm{Ker}(\exp : \mathbb{C} \rightarrow \mathbb{C}^*) \subset \mathbb{C}$. Put $\mathbb{Z}(n) = \mathbb{Z}(1)^{\otimes n}$.

One can show that mixed Hodge structures form an abelian category. If $f : H \rightarrow H'$ is a morphism, then $f_{\mathbb{Q}}$ and $f_{\mathbb{C}}$ are strictly compatible with the filtrations W and F ([1], 2.3.5).

2. Let A be a normal integral domain of finite type over $\mathbb{Z}$, let K be its field of fractions, and let $\overline{K}$ be an algebraic closure of K . Let K_{nr} be the largest subextension of $\overline{K}$ unramified at every prime ideal of A . It is known—or one may take this as the definition—that

$$\pi_1(\mathrm{Spec}(A), \overline{K}) = \mathrm{Gal}(K_{nr}/K).$$

For every closed point x of $\mathrm{Spec}(A)$, defined by a maximal ideal m_x of A , the residue field $k_x = A/m_x$ is finite; the point x defines a conjugacy class of “Frobenius substitutions” $\varphi_x \in \pi_1(\mathrm{Spec}(A), \overline{K})$. Put $q_x = \#k_x$ and $F_x = \varphi_x^{-1}$.

Let K be a field of finite type over the prime field of characteristic p , let $\overline{K}$ be an algebraic closure of K , let ℓ be a prime number $\neq p$, and let H be a finitely generated $\mathbb{Z}_{\ell}$ -module (or $\mathbb{Q}_{\ell}$ -module) endowed with a continuous

action ρ of $\text{Gal}(\overline{K}/K)$. In what follows we shall always suppose that there exists A as above, with ℓ invertible in A , such that ρ factors through

$$\pi_1(\text{Spec}(A), \overline{K}) = \text{Gal}(K_{nr}/K).$$

We shall say that H is pure of weight n if, for every closed point x of a nonempty open subset of $\text{Spec}(A)$, the eigenvalues α of F_x acting on H are algebraic integers all of whose complex conjugates have absolute value

$$|\alpha| = q_x^{n/2}.$$

Principle 2.1. If the Galois module H “comes from algebraic geometry”, there exists on $H_{\mathbb{Q}_\ell} = H \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ a (unique) increasing filtration W (the “weight filtration”), invariant under Galois, such that $\text{Gr}_n^W(H)$ is pure of weight n .

One may think that $\text{Gr}_n^W(H)$ is moreover semisimple.

When resolution of singularities is available, one can often give a conjectural definition of W , whose correctness follows from the Weil conjectures [5] (cf. 6).

Let μ be the subgroup of $\overline{K}^*$ formed by the roots of unity. The Tate module $\mathbb{Z}_\ell(1)$, defined by

$$\mathbb{Z}_\ell(1) = \text{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, \mu),$$

is pure of weight -2 . Put $\mathbb{Z}_\ell(n) = \mathbb{Z}_\ell(1)^{\otimes n}$.

It is trivial that every morphism $f : H \rightarrow H'$ is strictly compatible with the weight filtration.

Principle 2.1 agrees with the fact that every extension of $\mathbb{G}_m$ (“weight -2 ”) by an abelian variety (“weight $-1 > -2$ ”) is trivial.

3. Translation. The Galois modules which appear in ℓ -adic cohomology have, over $\mathbb{C}$, mixed Hodge structures as analogues. In addition one has the dictionary

pure module of weight n	Hodge structure of weight n
weight filtration	weight filtration
Galois-equivariant homomorphism	morphism
Tate module $\mathbb{Z}_\ell(1)$	Tate Hodge structure $\mathbb{Z}(1)$.

4. Let X be a complex algebraic variety (= a scheme of finite type over $\mathbb{C}$, which we shall suppose separated). There exists a subfield K of $\mathbb{C}$, of finite type over $\mathbb{Q}$, such that X can be defined over K (i.e. arises by extension of scalars from K to $\mathbb{C}$ from a K -scheme X'). Let $\overline{K}$ be the algebraic closure of K in $\mathbb{C}$. The Galois group $\text{Gal}(\overline{K}/K)$ then acts on the ℓ -adic cohomology groups $H^*(X, \mathbb{Z}_\ell)$; one has

$$H^*(X(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{Z}_\ell = H^*(X, \mathbb{Z}_\ell) = H^*(X'_{\overline{K}}, \mathbb{Z}_\ell).$$

According to 3, one should expect the cohomology groups $H^n(X(\mathbb{C}), \mathbb{Z})$ to carry natural mixed Hodge structures. This can be proved (see [1], 3.2.5, for the case where X is smooth; the proof is algebraic, starting from classical Hodge theory [6]). For X projective and smooth, the Weil conjectures imply that $H^n(X, \mathbb{Z}_\ell)$ is pure of weight n , while classical Hodge theory endows $H^n(X, \mathbb{Z})$ with a Hodge structure of weight n . For every morphism $f : X \rightarrow Y$ and for K large enough, $f^* : H^*(Y, \mathbb{Z}_\ell) \rightarrow H^*(X, \mathbb{Z}_\ell)$ commutes with Galois (by transport of structure); likewise, $f^* : H^*(Y, \mathbb{Z}) \rightarrow H^*(X, \mathbb{Z})$ is a morphism of mixed Hodge structures. For X smooth, the cohomology class in $H^{2n}(X, \mathbb{Z}_\ell(n))$ of an algebraic cycle of codimension n , Z , defined over K , is Galois-invariant, i.e. defines

$$c(Z) \in \text{Hom}_{\text{Gal}}(\mathbb{Z}_\ell(-n), H^{2n}(X, \mathbb{Z}_\ell)).$$

Similarly, the cohomology class $c(Z) \in H^{2n}(X(\mathbb{C}), \mathbb{Z})$ is purely of type (n, n) , i.e. corresponds to

$$c(Z) \in \text{Hom}_{\text{H.M.}}(\mathbb{Z}(-n), H^{2n}(X(\mathbb{C}), \mathbb{Z})).$$

5. If $f : H \rightarrow H'$ is a Galois-compatible morphism between $\mathbb{Q}_\ell$ -vector spaces of different weights, then $f = 0$. Likewise, if $f : H \rightarrow H'$ is a morphism of pure mixed Hodge structures of different weights, then f is torsion. A more useful remark is the following.

Scholium 5.1. Let H and H' be Hodge structures of weights n and n' , with $n > n'$. Let $f : H_{\mathbb{Q}} \rightarrow H'_{\mathbb{Q}}$ be a homomorphism such that $f : H_{\mathbb{C}} \rightarrow H'_{\mathbb{C}}$ respects F . Then $f = 0$.

6. Let X be a smooth projective variety over $\mathbb{C}$, let $D = \sum_{i=1}^n D_i$ be a normal-crossings divisor in X , a sum of smooth divisors, and let j be the inclusion into X of $U = X - D$. For $Q \subset [1, n]$, put

$$D_Q = \bigcap_{i \in Q} D_i.$$

In ℓ -adic cohomology, one has canonically

$$R^q j_* \mathbb{Z}_{\ell} = \bigoplus_{\#Q=q} \mathbb{Z}_{\ell}(-q)_{D_Q}, \quad (6.1)$$

and the Leray spectral sequence for j is

$$E_2^{pq} = \bigoplus_{\#Q=q} H^p(D_Q, \mathbb{Q}_{\ell}) \otimes \mathbb{Z}_{\ell}(-q) \Rightarrow H^{p+q}(U, \mathbb{Q}_{\ell}). \quad (6.2)$$

According to the Weil conjectures [5], $H^p(D_Q, \mathbb{Q}_{\ell})$ is pure of weight p , so that E_2^{pq} is pure of weight $p + 2q$. As a quotient of a subobject of E_2^{pq} , E_r^{pq} is also pure of weight $p + 2q$. By 5, $d_r = 0$ for $r \geq 3$, since the weights $p + 2q$ and $p + 2q - r + 2$ of E_r^{pq} and $E_r^{p+r, q-r+1}$ are different. Hence $E_3^{pq} = E_{\infty}^{pq}$. Up to reindexing, the weight filtration of $H^*(U, \mathbb{Q}_{\ell})$ is the abutment of (6.2):

$$\mathrm{Gr}_n^W(H^k(U, \mathbb{Q}_{\ell})) = E_3^{2k-n, n-k}. \quad (6.3)$$

7. In integral cohomology, for the usual topology, the Leray spectral sequence for j is

$$'E_2^{pq} = \bigoplus_{\#Q=q} H^p(D_Q, \mathbb{Z}) \Rightarrow H^{p+q}(U, \mathbb{Z}). \quad (7.1)$$

Since every D_Q is a nonsingular projective variety, $'E_2^{pq}$ is endowed with a Hodge structure of weight p . Put $E_2^{pq} = 'E_2^{pq} \otimes \mathbb{Z}(-q)$ (a Hodge structure of weight $p + 2q$). As an abelian group, $E_2^{pq} = 'E_2^{pq}$; it is useful to regard (7.1) as a spectral sequence with initial term E_2^{pq} . According to 3, one should expect

$$d_2 : E_2^{pq} \rightarrow E_2^{p+2, q-1}$$

to be a morphism of Hodge structures. This is proved by interpreting d_2 as a Gysin morphism. Thus E_3^{pq} is endowed with a Hodge structure of weight $p + 2q$. According to 3, one expects that, modulo torsion, the spectral sequence (6.4)¹ degenerates at E_3 ($E_3 = E_{\infty}$), and that the vanishing of the d_r ($r \geq 3$) is an application of 5.1. This program is carried out in [1] 3.2. There the weight filtration of $H^*(U, \mathbb{Q})$ is defined as the abutment of (7.1), up to the reindexing (6.3).

In fact, in order to endow cohomology groups H^* with a mixed Hodge structure, the key point up to now has always been to find a spectral sequence E abutting to H^* such that the ℓ -adic analogue of E_2^{pq} is conjecturally pure (of weight $p + 2q$); E_2^{pq} must then carry a natural Hodge structure (of weight $p + 2q$), and the filtration W is the abutment of E .

8. Let $\mathrm{Spec}(V)$ be the spectrum of a henselian discrete valuation ring (a henselian trait), with fraction field K and residue field k of finite type over the prime field of characteristic p . Let $\bar{K}$ be an algebraic closure of K and let H be a finite-dimensional vector space over $\mathbb{Q}_{\ell}$ ($\ell \neq p$), on which $\mathrm{Gal}(\bar{K}/K)$ acts continuously. According to Grothendieck, it is known ([4], appendix) that a finite-index subgroup of the inertia group I acts unipotently. Replacing V by a finite extension, one reduces to the case in which the action of all of I is unipotent (the semistable case); it then factors through the largest pro- ℓ quotient I_{ℓ} of I , canonically isomorphic to $\mathbb{Z}_{\ell}(1)$.

¹The printed text refers to (6.4); the intended reference appears to be (7.1).

Principle 8.1. In the semistable case, if the Galois module H “comes from algebraic geometry”, there exists a (unique) increasing filtration W of H (the “weight filtration”) such that I acts trivially on $\mathrm{Gr}_n^W(H)$ and such that $\mathrm{Gr}_n^W(H)$, as a Galois module under $\mathrm{Gal}(\bar{k}/k) \simeq \mathrm{Gal}(\bar{K}/K)/I$, is pure of weight n .

Compare 2.1 and the appendix of [4].

When resolution of singularities is available, one can sometimes give a conjectural definition of W , whose validity follows from the Weil conjectures. With the help of resolution and Weil, it is often easy to show that in any case H has a filtration whose successive quotients are pure Galois modules under $\mathrm{Gal}(\bar{k}/k)$.

Suppose H is semistable. For $T \in I_\ell$, define $\log T$ as the finite sum

$$-\sum_{n>0} (\mathrm{Id} - T)^n / n.$$

The map $(T, x) \mapsto \log T(x)$ is identified with a homomorphism

$$M : \mathbb{Z}_\ell(1) \otimes H \longrightarrow H. \quad (8.2)$$

Since $\mathbb{Z}_\ell(1)$ has weight -2 , one necessarily has (cf. 5)

$$M(\mathbb{Z}_\ell(1) \otimes W_n(H)) \subset W_{n-2}(H) \quad (8.3)$$

and M induces

$$\mathrm{Gr}(M) : \mathbb{Z}_\ell(1) \otimes \mathrm{Gr}_n^W(H) \longrightarrow \mathrm{Gr}_{n-2}^W(H). \quad (8.4)$$

8.5. If X is a nonsingular projective variety over an algebraically closed field k_0 , define

$$L : \mathbb{Z}_\ell(-1) \otimes H^*(X, \mathbb{Z}_\ell) \longrightarrow H^*(X, \mathbb{Z}_\ell)$$

to be cup product with the cohomology class of a hyperplane section. One will note a formal analogy between L and M : just as M is defined by an action of $\mathbb{Z}_\ell(1)$, one may regard L as defined by an action of $\mathbb{Z}_\ell(-1)$; L raises degree by 2, and $\mathrm{Gr} M$ (8.4) lowers it by 2.

9. Let D be the unit disk, $D^* = D - \{0\}$, and let X

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}^r(\mathbb{C}) \times D \\ f \searrow & & \swarrow \mathrm{pr}_2 \\ D & & \end{array}$$

be a family of projective varieties parametrized by D , with f proper and $f|_{D^*}$ smooth. Keep the notation of 8, and recall that in the analogy between a henselian trait and a small neighborhood of 0 in the complex line, one has the following dictionary (note that the spectrum of the ring of germs of holomorphic functions at 0 is a henselian trait):

D	$\mathrm{Spec}(V)$	(9.1)
D^*	$\mathrm{Spec}(K)$	
a universal covering $\tilde{D}^*$ of D^*	$\mathrm{Spec}(\bar{K})$	
fundamental group $\pi_1(D^*)$	inertia group I	
(with $\pi_1(D^*) = \mathbb{Z} \simeq \mathbb{Z}(1)_\mathbb{Z}$)	(with $I_\ell = \mathbb{Z}_\ell(1)$)	
X	projective scheme X over $\mathrm{Spec}(V)$	
$X^* = f^{-1}(D^*)$	X_K	
$\tilde{X} = X \times_D \tilde{D}^*$	$X_{\bar{K}}$	
local system $R^i f_* \mathbb{Z} _{D^*}$	Galois module $H^i(X_{\bar{K}}, \mathbb{Z}_\ell)$	
$H^i(\tilde{X}, \mathbb{Z})$	$H^i(X_{\bar{K}}, \mathbb{Z}_\ell)$.	

Note that $\tilde{X}$ is homotopy equivalent to each of the fibers $X_t = f^{-1}(t)$ ($t \in D^*$): $H^i(X_{\bar{K}}, \mathbb{Z}_\ell)$ again has $H^i(X_t, \mathbb{Z})$ as analogue, and the action of I corresponds to the monodromy transformation T .

Here again, it is known that a finite-index subgroup of $\pi_1(D^*)$ acts unipotently on $H^i(\tilde{X}, \mathbb{Q}) = H^i(X_t, \mathbb{Q})$. Assume we are in the semistable case where all of $\pi_1(D^*)$ acts unipotently (this amounts to replacing D by a finite covering), and let T be the action of the canonical generator of $\pi_1(D^*)$.

By 3 and 8, one expects $H^i(\tilde{X}, \mathbb{Q}) \simeq H^i(X_t, \mathbb{Q})$ to be endowed with an increasing filtration W , $\text{Gr}_n^W(H^i(\tilde{X}, \mathbb{Q}))$ to be endowed with a Hodge structure of weight n , $\log T(W_n) \subset W_{n-2}$, and $\log T$ to induce a morphism of Hodge structures

$$M_n : \mathbb{Z}(-1) \otimes \text{Gr}_n^W(H^i) \longrightarrow \text{Gr}_{n-2}^W(H^i).$$

One would moreover like (8.2), and not only (8.3) and (8.4), to have an analogue.

In fact, for each vector u in the tangent space to D at 0, one can define a mixed Hodge structure $\mathcal{H}_u$ on $H^i(\tilde{X}, \mathbb{Z})$. The filtration W and the Hodge structures on the $\text{Gr}_n^W(H^i)$ are independent of u , and the dependence of $\mathcal{H}_u$ on u is expressed simply in terms of T . In analogy with (8.2), one finds that, for every u , $\log T$ induces a homomorphism of mixed Hodge structures

$$M : \mathbb{Z}(1) \otimes H^i(\tilde{X}, \mathbb{Z}) \longrightarrow H^i(\tilde{X}, \mathbb{Z}).$$

Finally, the analogy 8.5 is not misleading (but here the fact that $f|D^*$ is supposed proper and smooth is no doubt essential). One proves that

$$(\log T)^k : \text{Gr}_{n+k}^W(H^n(\tilde{X}, \mathbb{Q})) \longrightarrow \text{Gr}_{n-k}^W(H^n(\tilde{X}, \mathbb{Q}))$$

is an isomorphism for every k (cf. [6], IV 6, cor. to th. 5). This characterizes the filtration W . Up to now, an analogue of Hodge's positivity theorem (cf. [6], IV 7, cor. to th. 7) is available only in very special cases. One hopes that the mixed structures $\mathcal{H}_u$ determine, as $t \rightarrow 0$, the asymptotic behavior of the family of pure structures $H^i(X_t, \mathbb{Z})$ ($t \in D^*$).

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