

SHIMURA'S WORK^(*)

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Let X/Γ be the quotient of a Hermitian symmetric domain X by a discrete arithmetic group Γ (assumed to be defined by congruence conditions; see 1.7). By Baily and Borel, X/Γ is a complex algebraic variety. In papers [10] through [14], Shimura shows, among other things, that many of these varieties can be defined over number fields which he explicitly identifies. This is the subject of the present report.

To obtain clean statements, one is forced to use adelic language. This leads one to consider, for a given "congruence condition", a scheme over $\mathbb{C}$, a disjoint union of finitely many varieties of the type X/Γ (see §§ 1 and 2). In many cases, this scheme can be defined over a number field E , independent of the congruence condition under consideration. Its geometric connected components, however, will be defined over class fields of E . This phenomenon, although essential, will be neglected in the rest of this introduction.

In a small number of cases, X/Γ can be interpreted as the set of isomorphism classes of complex abelian varieties, endowed with some additional algebraic structures (polarizations, endomorphisms, structures on the points of order n). The solution, over $\mathbb{Q}$, of a corresponding moduli problem then provides a scheme M over an explicit number field F , whose complex points are X/Γ . We shall call M a "model" of X/Γ . The fundamental case is that of polarized abelian varieties of the principal series, endowed with a level- n structure (1.6, 1.11, 4.16, 4.17 and 4.21).

In other cases, X/Γ can be interpreted as the set of isomorphism classes of complex abelian varieties A endowed with some structures as above, and with some integral cohomology classes of type (p, p) $a_i \in H^{2p}(A \times \cdots \times A, \mathbb{C})$ (cf. Mumford [7]). In this case, the idea will be to construct a model M of X/Γ as a subscheme of a model M' already constructed for a variety X'/Γ' . Roughly speaking, one first constructs in M' the points of M corresponding to abelian varieties of C.M. type (= maximal complex multiplication). One then defines M as the closure of the set of these points. This construction rests on the detailed knowledge of abelian varieties of C.M. type provided by [16].

In the cases considered above, one has much information on the fields of definition of the points of M corresponding to abelian varieties of C.M. type. This makes it possible to give partial solutions to Hilbert's twelfth problem (cf. the introduction of [15] and 3.15, 3.16).

Looking at what these cases have in common, one arrives at the notion of a "canonical model" (§ 3). One shows that there is at most one "canonical model" for X/Γ (5.5). Descent techniques then make it possible to construct some canonical models which seem far from being solutions of moduli problems ([12] [13] [14] and [5]). For a description of these "strange models", see §§ 5 and 6.

We shall denote by $\mathbb{Z}/n(1)$ both the group scheme μ_n of n th roots of unity and the group of n th roots of unity in a fixed algebraically closed field $\bar{k}$. For $\bar{k} = \mathbb{C}$, the exponential map identifies $\mathbb{Z}/n(1)$ with $\mathbb{Z}/n$. The $\mathbb{Z}/n(1)$ form a projective system: the transition maps are $\varphi_{n,m} : x \mapsto x^{n/m}$. We put

$$\widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z} \simeq \prod_p \mathbb{Z}_p, \quad \widehat{\mathbb{Z}}(1) = \varprojlim_n \mathbb{Z}/n\mathbb{Z}(1),$$

$$\mathbb{A}^f = \mathbb{Q} \otimes \widehat{\mathbb{Z}} \simeq \prod_p {}' \mathbb{Q}_p \quad (\text{restricted product}), \quad \mathbb{A}^f(1) = \mathbb{Q} \otimes \widehat{\mathbb{Z}}(1) = \mathbb{A}^f \otimes_{\widehat{\mathbb{Z}}} \widehat{\mathbb{Z}}(1),$$

and we denote by $\mathbb{A}$ the ring of adeles of $\mathbb{Q}$.

We shall use the following notation:

$\pi_0(X)$ (for X a topological space): the set of connected components of X . In what follows, $\pi_0(X)$, endowed with the quotient topology from X , will be discrete or compact totally disconnected.

G° : the connected component of the identity of a topological group G (or of the base point of a pointed topological space).

^(*)Text written in March 1971

$G(K)$, G_K , $G \otimes_F K$: for G a scheme over F (for instance an algebraic group over F) and K an F -algebra, $G(K)$ denotes the set of K -valued points of G , and G_K or $G \otimes_F K$ denotes the scheme over K obtained from G by extension of scalars.

E^* : for E an algebra of finite rank over a field F , E^* denotes the algebraic group over F of invertible elements of E . For E a number field and $F = \mathbb{Q}$, $E^*(\mathbb{A})/E^*(\mathbb{Q})$ is the idele class group of E .

We shall say that an algebraic group G over $\mathbb{Q}$ satisfies the Hasse principle if $H^1(\mathbb{Q}, G) \stackrel{\text{dfn}}{=} H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), G(\overline{\mathbb{Q}}))$ injects into the product, over all places, of the $H^1(\mathbb{Q}_v, G_{\mathbb{Q}_v})$. We shall use the following theorems.

(0.1) Hasse principle: a simply connected semisimple group, without a factor of type E_8 , satisfies the Hasse principle [3].

(0.2) A simply connected semisimple group G over a non-Archimedean local field K satisfies $H^1(K, G) = 0$ (a proof independent of the classification is announced in Bruhat–Tits [1]).

(0.3) Strong approximation theorem: let G be a simply connected semisimple group over a global field K . If G is K -simple, and if v is a place of K such that $G(K_v)$ is noncompact, then $G(K_v)G(K)$ is dense in $G(\mathbb{A})$ (for a proof independent of the classification, see Platonov [9]).

(0.4) Real approximation theorem: let G be a connected (linear) algebraic group over $\mathbb{Q}$. Then $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$ (to prove this, one reduces easily to the case of tori, which is a corollary of ([4] 5.1)).

§ 1. The adelic language.

1.1. Let G be a reductive group over $\mathbb{Q}$. We denote by G' the derived group of the identity component of G , by C the center of this identity component, and we put $T = G/G'$. There are exact sequences of algebraic groups

$$(1.1.1) \quad 0 \longrightarrow C \longrightarrow G \longrightarrow G/C \longrightarrow 0,$$

$$(1.1.2) \quad 0 \longrightarrow G' \longrightarrow G \xrightarrow{\nu} T \longrightarrow 0,$$

and the canonical maps from C to T and from G' to G/C are isogenies with kernel $C \cap G'$.

1.2. Let $\mathbb{S}$ be the real algebraic group of invertible elements of the $\mathbb{R}$ -algebra $\mathbb{C}$. It is also the real algebraic group obtained from the multiplicative group $\mathbb{G}_m$ over $\mathbb{C}$ by Weil restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$. One has $\mathbb{S}(\mathbb{R}) = \mathbb{C}^*$, and $\mathbb{S}_{\mathbb{C}} \simeq \mathbb{G}_{m, \mathbb{C}} \times \mathbb{G}_{m, \mathbb{C}}$. The group $\text{Hom}(\mathbb{S}_{\mathbb{C}}, \mathbb{G}_{m, \mathbb{C}})$ of characters of $\mathbb{S}_{\mathbb{C}}$ has as a basis the characters z and $\bar{z}$ such that the composite

$$\mathbb{C}^* = \mathbb{S}(\mathbb{R}) \longrightarrow \mathbb{S}(\mathbb{C}) \xrightarrow{z, \bar{z}} \mathbb{C}^*$$

is respectively the identity and complex conjugation.

Let V be a real vector space. For any representation $h : \mathbb{S} \rightarrow \text{GL}(V)$, we denote by V^{pq} the subspace of $V_{\mathbb{C}}$ on which $\mathbb{S}$ acts by the character $z^p \bar{z}^q$. The V^{pq} form a bigrading of $V_{\mathbb{C}}$, and the construction $h \mapsto V^{pq}$ identifies the representations of $\mathbb{S}$ in $\text{GL}(V)$ with Hodge bigradings of $V_{\mathbb{C}}$, i.e. with bigradings of $V_{\mathbb{C}}$ such that $V^{pq} = \overline{V^{qp}}$. We shall denote by $F(h)$ the Hodge filtration of $V_{\mathbb{C}}$:

$$F(h)^p = \bigoplus_{p' \geq p} V^{p'q'}.$$

1.3. We denote by $r : \mathbb{G}_{m, \mathbb{C}} \rightarrow \mathbb{S}_{\mathbb{C}}$ the $\mathbb{C}$ -homomorphism such that $(z^p \bar{z}^q) \circ r = (x \mapsto x^p)$, and by $w : \mathbb{G}_{m, \mathbb{R}} \rightarrow \mathbb{S}$ the $\mathbb{R}$ -homomorphism such that $(z^p \bar{z}^q) \circ w = (x \mapsto x^{p+q})$. For a representation $h : \mathbb{S} \rightarrow \text{GL}(V)$ and $v^{pq} \in V^{pq}$, one has

$$\begin{aligned} hr(x) \cdot v^{pq} &= x^p \cdot v^{pq} & \text{and} \\ hw(x) \cdot v^{pq} &= x^{p+q} \cdot v^{pq}. \end{aligned}$$

The composite $w : \mathbb{R}^* = \mathbb{G}_m(\mathbb{R}) \xrightarrow{w} \mathbb{S}(\mathbb{R}) = \mathbb{C}^*$ is the evident inclusion.

1.4. Let $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$ be a homomorphism. For any representation $\rho : G \rightarrow \text{GL}(V)$ of G on a $\mathbb{Q}$ -vector space V , ρh defines a bigrading of $V_{\mathbb{C}}$ (1.2). If V is faithful, and if G is the subgroup of $\text{GL}(V)$ fixing some tensors s_i , the construction $h \mapsto (V^{pq})$ identifies the homomorphisms h with Hodge bigradings of $V_{\mathbb{C}}$ such that the s_i are of type $(0, 0)$.

1.5. Let $h_o : \mathbb{S} \rightarrow G_{\mathbb{R}}$ be a homomorphism satisfying the following conditions (cf. [7] and [2]; (1.5.2) is motivated by the Griffiths transversality theorem).

(1.5.1) The image of $h_o w : \mathbb{G}_m \rightarrow G_{\mathbb{R}}$ is central. We shall call $h_o w$ the weight of h_o ;

(1.5.2) the Hodge bigrading on the complexification $\mathrm{Lie}(G)_{\mathbb{C}}$ of the adjoint representation space (1.2) is of type $(-1, 1) + (0, 0) + (1, -1)$;

(1.5.3) the automorphism $\mathrm{ad} h_o(i)$ of G , an involution by (1.5.1), induces a Cartan involution of G' .

Let K_{∞} be the centralizer of h_o in $G(\mathbb{R})$; it contains the center of $G(\mathbb{R})$, and $K_{\infty} \cap G'(\mathbb{R})^o$ is a maximal compact subgroup of $G'(\mathbb{R})^o$, equal to the centralizer of $h_o(i)$ in $G'(\mathbb{R})^o$. The space $K_{\infty} \backslash G(\mathbb{R})$ is identified with the set X of conjugates of h_o by elements of $G(\mathbb{R})$. One checks that it has a unique complex structure such that, for every representation $\rho : G \rightarrow \mathrm{GL}(V)$, the filtration $F(\rho h)$ of $V_{\mathbb{C}}$ depends holomorphically on $h \in X$. This structure is right invariant under $G(\mathbb{R})$, and the connected components of X are Hermitian symmetric domains. We shall denote by X^o the connected component of h_o .

1.6. Example I. Let $\mathrm{Gp}(V)$ be the group of symplectic similitudes of a real vector space V endowed with a non-degenerate alternating form ψ . There is then a unique conjugacy class X of homomorphisms $h : \mathbb{S} \rightarrow \mathrm{Gp}(V)$ satisfying (1.5.1) and having weight -1 , i.e. weight $hw : \mathbb{G}_m \rightarrow \mathrm{Gp}(V) : x \mapsto$ the homothety with ratio x^{-1} . The space X is identified with the union of two Siegel half-spaces.

Construction 1.5 identifies the elements $h \in X$ with the decompositions $V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$ of $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ such that $V^{pq} = \overline{V^{qp}}$, $V^{0,-1}$ is totally isotropic for ψ , and, if C denotes the endomorphism of V which induces multiplication by i^{p-q} on V^{pq} , the symmetric form $\psi(x, Cy) = \psi(x, h(i)y)$ on V is definite (> 0 or < 0).

1.7. Let Γ be an arithmetic subgroup of $G(\mathbb{Q})$. We are interested in the case where Γ is defined by congruence conditions. If $V_{\mathbb{Z}}$ is an integral lattice in a faithful representation V of G , this means that there exists n such that Γ contains, with finite index, the group $\Gamma_n \subset G(\mathbb{Q})$ of $\gamma \in G(\mathbb{Q})$ such that $\gamma \in G(\mathbb{R})^o$, $\gamma V_{\mathbb{Z}} = V_{\mathbb{Z}}$, and $\gamma \equiv \mathrm{Id} \pmod{n}$. By Baily and Borel, the quotient X^o/Γ carries a natural quasi-projective scheme structure over $\mathbb{C}$. Let K_n be the compact open subgroup of $G(\mathbb{A}^f)$ consisting of those $k = (k_p)$ such that $k_p \cdot V_{\mathbb{Z}} \otimes \mathbb{Z}_p = V_{\mathbb{Z}} \otimes \mathbb{Z}_p$ and $k_p \equiv 1 \pmod{n}$ for $p \mid n$. The group Γ_n is then the intersection, inside $G(\mathbb{A})$, of $G(\mathbb{Q})$ and $G(\mathbb{R})^o \times K_n$.

1.8. Let K be a compact open subgroup of $G(\mathbb{A}^f)$. The space $K_{\infty} \times K \backslash G(\mathbb{A})/G(\mathbb{Q}) = K \backslash X \times G(\mathbb{A}^f)/G(\mathbb{Q}) = X \times (K \backslash G(\mathbb{A}^f))/G(\mathbb{Q})$ is then a finite disjoint union of spaces of the type considered in 1.7. It is therefore a quasi-projective scheme over $\mathbb{C}$. We shall denote it by ${}_K M_{\mathbb{C}}(G, h_o)$ (or simply ${}_K M_{\mathbb{C}}(G)$ or ${}_K M_{\mathbb{C}}$).

As K becomes smaller, these schemes form a projective system of schemes with finite and surjective transition morphisms

$$(1.8.1) \quad {}_K M_{\mathbb{C}}(G, h_o) \quad (K \rightarrow 0).$$

We shall denote by $M_{\mathbb{C}}(G, h_o)$ (or simply $M_{\mathbb{C}}(G)$, or $M_{\mathbb{C}}$) the quasi-compact and separated scheme (not of finite type for $G \neq \{1\}$) that is the projective limit of (1.8.1)

$$(1.8.2) \quad M_{\mathbb{C}}(G, h_o) = \varprojlim_K K_{\infty} \times K \backslash G(\mathbb{A})/G(\mathbb{Q}).$$

It should be regarded as an avatar of the projective system (1.8.1). The group $G(\mathbb{A}^f)$ acts on $M_{\mathbb{C}}$, or, if one prefers, on the pro-object defined by the projective system (1.8.1), but of course not on the individual ${}_K M_{\mathbb{C}}$ (cf. 3.2). One has

$$K \backslash M_{\mathbb{C}} = {}_K M_{\mathbb{C}},$$

so that

$$(1.8.3) \quad M_{\mathbb{C}} = \varprojlim_K {}_K M_{\mathbb{C}} \quad (K \text{ compact open in } G(\mathbb{A}^f)).$$

This amounts to saying that $G(\mathbb{A}^f)$ acts continuously on $M_{\mathbb{C}}$.

The following construction will be useful for interpreting the ${}_K M_{\mathbb{C}}$ as moduli schemes.

1.9. Let H be an algebraic group over $\mathbb{Q}$ and V a faithful representation of H . We define an H -structure $\bar{t}$ on a vector space W to be an isomorphism from V to W , given modulo $H(\mathbb{Q})$: $\bar{t} \in \mathrm{Isom}(V, W)/H(\mathbb{Q})$. If $H(\mathbb{Q})$ is the group of automorphisms of a structure s of type Σ on V , then an H -structure on W is identified with a structure t of the same type on W , isomorphic to s (with $\bar{t} = \mathrm{Isom}((V, s), (W, t))$). Let W be endowed with an H -structure $\bar{t}$. The algebraic group ${}_{\bar{t}} H \subset \mathrm{GL}(W)$ ($\iota \in \bar{t}$) depends only on $\bar{t}$. We denote it by $H^{\bar{t}}$. Let A be a $\mathbb{Q}$ -algebra. The set $\mathrm{Isom}_{\bar{t}}(W \otimes A, V \otimes A)$ of permissible isomorphisms from $W \otimes A$ to $V \otimes A$ is the set of isomorphisms k such that, for $\iota \in \bar{t}$, one has $k\iota \in H(A)$.

1.10. Let (G, h_o) be as in 1.5, let V be a faithful linear representation of G , let K be a compact open subgroup of $G(\mathbb{A}^f)$, and consider objects H of the following type: H consists of

- (a) a vector space $H_{\mathbb{Q}}$ over $\mathbb{Q}$, endowed with a G -structure $\bar{\iota}$ (1.9);
- (b) a homomorphism $h : \mathbb{S} \rightarrow G_{\mathbb{R}}^{\bar{\iota}}$ such that, for $\iota \in \bar{\iota}$, $\iota^{-1}h\iota : \mathbb{S} \rightarrow G_{\mathbb{R}}$ is conjugate to h_o ;
- (c) a class $\bar{k} \in K \backslash \text{Isom}_{\bar{\iota}}(H \otimes \mathbb{A}^f, V \otimes \mathbb{A}^f)$ (1.9).

1.11. Example I (continued from 1.6). Let V be a $\mathbb{Q}$ -vector space endowed with a non-degenerate alternating form ψ , let G be the group of symplectic similitudes of V , let $V_{\mathbb{Z}}$ be an integral lattice on which ψ has integral values and discriminant 1, and let h_o be as in 1.6. Take $K = \prod_{\ell} \text{Gp}(V_{\mathbb{Z}} \otimes \mathbb{Z}_{\ell})$. A datum (a) is interpreted as a vector space H of the same dimension as V , endowed with an alternating form ψ given up to a rational factor. A datum (b) is interpreted as a Hodge bigrading of $H_{\mathbb{C}}$, of type 1.6. A datum (c) is interpreted as an integral lattice $H_{\mathbb{Z}}$ in H , on which a rational multiple of ψ has integral values and discriminant 1: k is the set of k such that, for every ℓ , $V_{\mathbb{Z}} \otimes \mathbb{Z}_{\ell} = k_{\ell}(H_{\mathbb{Z}} \otimes \mathbb{Z}_{\ell})$. We may normalize ψ by imposing the conditions that it have integral values and discriminant 1 on $H_{\mathbb{Z}}$, and that $\psi(x, Cx) > 0$ (cf. 1.6).

Let n be an integer, and let K_n be the subgroup of K consisting of those k such that $k \equiv 1 \pmod{n}$. For K_n , a datum (c) is interpreted as $H_{\mathbb{Z}}$ as above, together with a symplectic similitude $H_{\mathbb{Z}}/nH_{\mathbb{Z}} \xrightarrow{\sim} V_{\mathbb{Z}}/nV_{\mathbb{Z}}$.

1.12. Under the hypotheses of 1.10, the points of ${}_K M_{\mathbb{C}}(G, h_o)$ are in bijective correspondence with the isomorphism classes of objects H of type 1.10. Indeed, let $H = (H_{\mathbb{Q}}, \bar{\iota}, h, \bar{k})$. Given H and $\iota \in \bar{\iota}$, we associate the morphism $\iota^{-1}h\iota : \mathbb{S} \rightarrow G_{\mathbb{R}}$, an element of $X \simeq K_{\infty} \backslash G(\mathbb{R})$, and $\bar{k}\iota$ in $K \backslash G(\mathbb{A}^f)$, hence an element of $K \backslash X \times G(\mathbb{A}^f)$; to H alone we associate only an element of ${}_K M_{\mathbb{C}}(G, h_o) = K \backslash X \times G(\mathbb{A}^f)/G(\mathbb{Q}) = K_{\infty} \times K \backslash G(\mathbb{A})/G(\mathbb{Q})$, which uniquely determines the isomorphism class of H .

In certain cases (see 4.11), to give an object H as above amounts to giving an abelian variety endowed with some additional structures. One then obtains an interpretation of ${}_K M_{\mathbb{C}}(G)$ (and of $M_{\mathbb{C}}(G)$) as the coarse moduli scheme for abelian varieties of this type.

1.13. Let G^i ($i = 1, 2$) be two reductive groups, let $h^i : \mathbb{S} \rightarrow G_{\mathbb{R}}^i$ be homomorphisms satisfying the conditions of 1.5, and let X^i be the set of conjugates of h^i by an element of $G^i(\mathbb{R})$. Let $G = G^1 \times G^2$, let $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$ be the morphism (h^1, h^2) , and let $X = X^1 \times X^2$ be the set of conjugates of h . There is an evident isomorphism

$$(1.13.1) \quad M_{\mathbb{C}}(G^1, h^1) \times M_{\mathbb{C}}(G^2, h^2) = M_{\mathbb{C}}(G, h).$$

For K^i compact open in $G^i(\mathbb{A}^f)$ and $K = K^1 \times K^2$, one likewise has

$$(1.13.2) \quad {}_{K^1} M_{\mathbb{C}}(G^1, h^1) \times {}_{K^2} M_{\mathbb{C}}(G^2, h^2) = {}_K M_{\mathbb{C}}(G, h).$$

1.14. Let G^1, G^2, h^1, h^2 be as in 1.13. We shall denote by $u : (G^1, h^1) \rightarrow (G^2, h^2)$ a homomorphism $u : G^1 \rightarrow G^2$ such that $uX^1 \subset X^2$. Such a homomorphism defines

$$(1.14.1) \quad u : M_{\mathbb{C}}(G^1, h^1) \longrightarrow M_{\mathbb{C}}(G^2, h^2).$$

For K^i compact open in $G^i(\mathbb{A}^f)$, with $u(K^1) \subset K^2$, it defines

$$(1.14.2) \quad u(K^1, K^2) : {}_{K^1} M_{\mathbb{C}}(G^1, h^1) \longrightarrow {}_{K^2} M_{\mathbb{C}}(G^2, h^2).$$

Proposition 1.15. With the notation of 1.14, suppose that G^1 is a subgroup of G^2 . Then, for every compact open subgroup K^1 of $G^1(\mathbb{A}^f)$, there exists a compact open subgroup $K^2 \supset K^1$ of $G^2(\mathbb{A}^f)$ such that $u(K^1, K^2)$ identifies ${}_{K^1} M_{\mathbb{C}}(G^1, h^1)$ with a closed subscheme of ${}_{K^2} M_{\mathbb{C}}(G^2, h^2)$.

If the K^i are small enough that the map from $X^i \times (K^i \backslash G^i(\mathbb{A}^f))$ to ${}_K M_{\mathbb{C}}(G^i, h^i)$ is étale, then the map $u(K^1, K^2)$ is finite and unramified. It suffices to treat this case, and to show that $u(K^1, K^2)$ is injective for $K^2 \supset K^1$ sufficiently small. The graph of the equivalence relation $u(K^1, K^2)(x) = u(K^1, K^2)(y)$ is a closed subscheme of $({}_K M_{\mathbb{C}}(G^1, h^1))^2$, decreasing with K^2 , hence stationary for $K^2 \supset K^1$ sufficiently small. It is enough to show that

$$u(K^1) : {}_{K^1} M_{\mathbb{C}}(G^1, h^1) \longrightarrow \varprojlim_{K^2 \supset K^1} {}_{K^2} M_{\mathbb{C}}(G^2, h^2) = \text{dfn } {}_{K^1} M_{\mathbb{C}}(G^2, h^2)$$

is injective. One has

$$\begin{aligned} {}_{K^1} M_{\mathbb{C}}(G^1, h^1) &= K^1 \backslash M_{\mathbb{C}}(G^1, h^1), \\ {}_{K^1} M_{\mathbb{C}}(G^2, h^2) &= K^1 \backslash M_{\mathbb{C}}(G^2, h^2), \end{aligned}$$

so it is enough to prove the

Variant 1.15.1. $u : M_{\mathbb{C}}(G^1, h^1) \longrightarrow M_{\mathbb{C}}(G^2, h^2)$ is injective.

This assertion follows easily from the following two lemmas.

Lemma 1.15.2. Let $U^i \subset C^i(\mathbb{Q})$ be the group of units of the centre of G^i , and put $G^i(\mathbb{Q})^\wedge = \varprojlim_n (G^i(\mathbb{Q})/(U^i)^n)$.
Then

$$M_{\mathbb{C}}(G^i, h^i) = K_\infty^i \backslash G^i(\mathbb{A})/G^i(\mathbb{Q})^\wedge.$$

This is a corollary of Chevalley's theorem, according to which the $(U^i)^n$ are congruence subgroups of U^i .

Lemma 1.15.3. $u : G^1(\mathbb{A}^f)/G^1(\mathbb{Q})^\wedge \longrightarrow G^2(\mathbb{A}^f)/G^2(\mathbb{Q})^\wedge$ is injective.

By left translation, it is enough to show that $u^{-1}(\{e\}) = \{e\}$: consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & G^1(\mathbb{Q})^\wedge & \longrightarrow & G^2(\mathbb{Q})^\wedge & \longrightarrow & (G^2/G^1)(\mathbb{Q})^\wedge \\ & & \downarrow & & \downarrow & & \downarrow ! \\ 0 & \longrightarrow & G^1(\mathbb{A}^f) & \longrightarrow & G^2(\mathbb{A}^f) & \longrightarrow & (G^2/G^1)(\mathbb{A}^f) \\ & & \downarrow & & \downarrow & & \\ & & G^1(\mathbb{A}^f)/G^1(\mathbb{Q})^\wedge & \longrightarrow & G^2(\mathbb{A}^f)/G^2(\mathbb{Q})^\wedge & & \end{array}$$

where $(G^2/G^1)(\mathbb{Q})^\wedge = \varprojlim_n (G^2/G^1)(\mathbb{Q})/(U^2/U^1)^n$.

§ 2. Connected components.

2.1. Let G be a connected reductive group over $\mathbb{Q}$. We use the notation of 1.1 and assume that G' admits no non-trivial invariant subgroup H (defined over $\mathbb{Q}$) and such that $H(\mathbb{R})$ is compact. This hypothesis does not exclude the possibility that $G'_{\mathbb{R}}$ has compact factors. We denote by $\tilde{G}'$ the simply connected covering of the derived group G' .

Proposition 2.2. (i) The group $G(\mathbb{A}^f)$ acts transitively on the set $\pi_0(G(\mathbb{A})/G(\mathbb{Q}))$ of connected components of $G(\mathbb{A})/G(\mathbb{Q})$.

(ii) The group $\tilde{G}'(\mathbb{A})$ acts trivially on $\pi_0(G(\mathbb{A})/G(\mathbb{Q}))$.

(iii) The quotient of $G(\mathbb{A})$ by the image of $\tilde{G}'(\mathbb{A})$ is a commutative group.

Assertion (i) follows from (0.4): $G(\mathbb{A}) = G(\mathbb{A}^f)G(\mathbb{R})^0G(\mathbb{Q})$. Since $\tilde{G}'$ is simply connected, $\tilde{G}'(\mathbb{R})$ is connected. By (0.3), $\tilde{G}'(\mathbb{R}) \cdot \tilde{G}'(\mathbb{Q})$ is dense in $\tilde{G}'(\mathbb{A})$, so that $\tilde{G}'(\mathbb{A})/\tilde{G}'(\mathbb{Q})$ is connected. For $x \in G(\mathbb{A})$ one has $\tilde{G}'(\mathbb{A}) \cdot x = x \cdot \tilde{G}'(\mathbb{A})$; hence the image of $\tilde{G}'(\mathbb{A}) \cdot x$ in $G(\mathbb{A})/G(\mathbb{Q})$ is an image of $\tilde{G}'(\mathbb{A})/\tilde{G}'(\mathbb{Q})$: it is connected, and (ii) follows.

Let Z be the centre of $\tilde{G}'$. Assertion (iii) follows from the fact that the commutator map $xyx^{-1}y^{-1} : G \times G \longrightarrow G$ has a factorization

$$G \times G \longrightarrow G/C \times G/C \longleftarrow \tilde{G}'/Z \times \tilde{G}'/Z \longrightarrow \tilde{G}' \longrightarrow G.$$

Definition 2.3. We denote by $\pi(G)$ the commutative quotient of $G(\mathbb{A})$ (or $G(\mathbb{A}^f)$) that acts faithfully on $\pi_0(G(\mathbb{A})/G(\mathbb{Q}))$.

In other words, $\pi(G)$ is $\pi_0(G(\mathbb{A})/G(\mathbb{Q}))$, endowed with its “evident” commutative group structure. We recall the following theorem.

Theorem 2.4. Under the hypotheses of 2.1, if G' is simply connected, then the canonical homomorphism

$$(2.4.1) \quad \pi(G) = \pi_0(G(\mathbb{A})/G(\mathbb{Q})) \longrightarrow \pi_0(T(\mathbb{A})/T(\mathbb{Q}))$$

is bijective.

To prove the surjectivity of (2.4.1), by 2.2(i) applied to T it is enough to show that $G(\mathbb{A}^f)$ maps onto $T(\mathbb{A}^f)$. Since the kernel G' of $\nu : G \rightarrow T$ is connected, one knows that, for almost all p , $\nu(G(\mathbb{Z}_p)) = T(\mathbb{Z}_p)$; this formula has a meaning for almost all p , and is a corollary of Lang's theorem applied to G' reduced modulo p . It is therefore enough to prove that, for every p , $\nu(G(\mathbb{Q}_p)) = T(\mathbb{Q}_p)$, which follows from (0.2) applied to G' .

Theorem 2.4 is equivalent to the following more concrete statement.

Variant 2.5. For every compact open subgroup K of $G(\mathbb{A}^f)$, the map

$$(2.5.1) \quad \nu : \pi_0(K \backslash G(\mathbb{A})/G(\mathbb{Q})) \longrightarrow \pi_0(\nu(K) \backslash T(\mathbb{A})/T(\mathbb{Q}))$$

is bijective.

Indeed, $\pi_0(G(\mathbb{A})/G(\mathbb{Q})) = \varprojlim_K \pi_0(K \backslash G(\mathbb{A})/G(\mathbb{Q}))$, and similarly for T . Also, $\pi_0(K \backslash G(\mathbb{A})/G(\mathbb{Q})) = G(\mathbb{R})^0 \backslash K \backslash G(\mathbb{A})/G(\mathbb{Q})$. We show that if $x, y \in G(\mathbb{A})$ have the same image in the right-hand side of (2.5.1), then they have the same image in the left-hand side. Left translation by y^{-1} , which replaces K by $y^{-1}Ky$, allows us to suppose that $y = e$. Modifying x by an element of $G(\mathbb{R})^0 \times K$, we may suppose that $\nu(x) = \tau$ with $\tau \in T(\mathbb{Q})$. By the Hasse principle (0.1) for G' , the equation $\nu(y) = \tau$ has a solution in $G(\mathbb{Q})$. Correcting x by y , we may suppose that $x \in G'(\mathbb{A})$. By 2.2(ii), the image of x in $G(\mathbb{A})/G(\mathbb{Q})$ is in the neutral component, and a fortiori x and y have the same image in the left-hand side of 2.5.1.

This proof seems to use, via (0.1), that G' has no factor of type E_8 . In fact, the class in $H^1(\mathbb{Q}, G')$ which obstructs the equation $\nu(y) = \tau$ comes from the centre of G' , and the E_8 factors, being adjoint, count for nothing. Proposition 2.6. One has $K_\infty = C(\mathbb{R}) \cdot (K_\infty \cap G'(\mathbb{R})^0)$. The group $K_\infty \cap G'(\mathbb{R})^0$ is connected, and

$$\pi_0(K_\infty) \xrightarrow{\sim} \text{Im}(\pi_0(C(\mathbb{R})) \longrightarrow \pi_0(G(\mathbb{R}))),$$

Let $h' : \mathbb{S} \rightarrow G_{\mathbb{R}} \rightarrow (G/C)_{\mathbb{R}}$. The centralizer of h' in $(G/C)(\mathbb{R})$ is connected, because it is a compact group whose complexification is connected (as the centralizer of a torus). In particular, the image of K_∞ in $G/C(\mathbb{R})$ lies in the neutral component, the image of $G'(\mathbb{R})^0$, and $K_\infty = C(\mathbb{R}) \cdot (K_\infty \cap G'(\mathbb{R})^0)$. The group $K_\infty \cap G'(\mathbb{R})^0$ is connected as a maximal compact subgroup of a connected group. This proves 2.6.

2.7. The image of $\pi_0(K_\infty)$ in $\pi_0(G(\mathbb{A})/G(\mathbb{Q}))$ depends only on the conjugacy class of h , and $\pi_0(M_{\mathbb{C}}(G, h)) = \varprojlim_K \pi_0(K_\infty \times K \backslash G(\mathbb{A})/G(\mathbb{Q}))$ is identified with $\pi(G)/\pi_0(K_\infty)$.

In particular, if G' is simply connected, one has

$$(2.7.1) \quad \pi_0(M_{\mathbb{C}}(G, h)) \xrightarrow{\sim} \pi_0(T(\mathbb{A})/T(\mathbb{Q}))/\pi_0(K_\infty).$$

More concretely, for every compact open subgroup K of $G(\mathbb{A}^f)$, $\pi_0({}_K M_{\mathbb{C}}(G, h))$ is then a principal homogeneous space under $\nu(K_\infty \times K) \backslash T(\mathbb{A})/T(\mathbb{Q})$.

§ 3. Models.

Let G be a reductive group over $\mathbb{Q}$, let $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$ satisfy the conditions of 1.5, and let E be a field endowed with a complex embedding $\rho : E \rightarrow \mathbb{C}$. Put $M_{\mathbb{C}}(G) = M_{\mathbb{C}}(G, h)$.

Definition 3.1. A model over E of $M_{\mathbb{C}}(G)$ consists of

- (a) a scheme M over E , endowed with a continuous action of $G(\mathbb{A}^f)$;
- (b) an isomorphism m from $M \otimes_{E, \rho} \mathbb{C}$ to $M_{\mathbb{C}}(G)$ compatible with the action of $G(\mathbb{A}^f)$.

Let F be a finite extension of E , endowed with a complex embedding extending that of E . If $M_E(G, h)$ is a model of $M_{\mathbb{C}}(G, h)$ over E , we denote by $M_F(G, h)$ the model $M_E(G, h) \otimes_E F$ of $M_{\mathbb{C}}(G, h)$ over F .

Remark 3.2. Giving a scheme M over E , endowed with a continuous action of $G(\mathbb{A}^f)$, is equivalent to giving

- (a) for every compact open subgroup K of $G(\mathbb{A}^f)$, a scheme ${}_K M$ over E ;
- (b) for compact open subgroups K, L of $G(\mathbb{A}^f)$ and for $x \in G(\mathbb{A}^f)$ such that $xKx^{-1} \subset L$, a homomorphism $J_{L, K}(x) : {}_K M \longrightarrow {}_L M$, these homomorphisms satisfying

- (α) $J_{M, L}(y)J_{L, K}(x) = J_{M, K}(yx)$,
 - (β) $J_{K, K}(x) = \text{Id}$ if $x \in K$,
 - (γ) for K normal in L , J defines an action of L/K on ${}_K M$,
- and $J_{L, K}(e)$ defines $(L/K) \backslash {}_K M \xrightarrow{\sim} {}_L M$.

One has

$$(3.2.1) \quad M = \varprojlim_K {}_K M \quad \text{and}$$

$$(3.2.2) \quad {}_K M = K \backslash M.$$

3.3. Let $M(G)$ be a model over E of $M_{\mathbb{C}}(G)$. For K compact open in $G(\mathbb{A}^f)$, put ${}_K M(G) = K \backslash M(G)$. The ${}_K M(G)$ are quasi-projective schemes over E , not necessarily geometrically connected. The structural isomorphism m induces isomorphisms

$$(3.3.1) \quad {}_K M(G) \otimes_{E, \rho} \mathbb{C} \xrightarrow{\sim} {}_K M_{\mathbb{C}}(G).$$

3.4. Suppose that G satisfies the hypotheses of 2.1, and let (M, m) be a model of $M_{\mathbb{C}}(G, h)$ over E . Let $\bar{E}$ be the algebraic closure of E in $\mathbb{C}$. Since M is defined over E , the Galois group $\text{Gal}(\bar{E}/E)$ acts on the profinite set

$$\pi_0(M \otimes_E \bar{E}) = \varprojlim_K \pi_0(KM \otimes_E \bar{E}) \xrightarrow{m} \pi_0(M_{\mathbb{C}}(G, h)).$$

The group $G(\mathbb{A}^f)$ acts on $\pi_0(M \otimes_E \bar{E})$ through its quotient $\pi(G)/\pi_0(K_{\infty})$, and $\pi_0(M \otimes_E \bar{E})$ is a principal homogeneous space under the commutative group $\pi(G)/\pi_0(K_{\infty})$ (2.6). The action of $\text{Gal}(\bar{E}/E)$ commutes with that of $G(\mathbb{A}^f)$, hence with that of $\pi(G)/\pi_0(K_{\infty})$. The action of an element σ of $\text{Gal}(\bar{E}/E)$ can therefore only be the translation defined by an element $\lambda(\sigma)$ of $\pi(G)/\pi_0(K_{\infty})$, and λ is a homomorphism

$$(3.4.1) \quad \lambda_M : \text{Gal}(\bar{E}/E) \longrightarrow \pi(G)/\pi_0(K_{\infty}).$$

Assume that E is a number field. Class field theory identifies the largest abelian quotient of $\text{Gal}(\bar{E}/E)$ with $\pi_0(E^*(\mathbb{A})/E^*(\mathbb{Q}))$, and identifies (3.4.1) with

$$(3.4.2) \quad \lambda_M : \text{Gal}(\bar{E}/E)^{\text{ab}} = \pi_0(E^*(\mathbb{A})/E^*(\mathbb{Q})) \longrightarrow \pi(G)/\pi_0(K_{\infty}).$$

In the special case where G' is simply connected, the homomorphism (3.4.2) is identified, by (2.7.1), with

$$(3.4.3) \quad \lambda_M : \pi_0(E^*(\mathbb{A})/E^*(\mathbb{Q})) \longrightarrow \pi_0(T(\mathbb{A})/T(\mathbb{Q}))/\pi_0(K_{\infty}).$$

Definition 3.5. The homomorphisms (3.4.1), (3.4.2), and (3.4.3) are called the reciprocity law of the model M . If G' is simply connected, and if λ_M comes from a homomorphism of algebraic groups over $\mathbb{Q}$ from E^* to T , the latter will also be called the “reciprocity law” of M .

3.6. There is a rule, applicable to each of the models constructed by Shimura, for determining E and the reciprocity law λ_M from G and h . The field E , for arbitrary G , and λ_M , for simply connected G' , are described as follows.

3.7. The composite homomorphism hr (1.3)

$$hr : \mathbb{G}_m \xrightarrow{r} \mathbb{S}_{\mathbb{C}} \xrightarrow{h} G_{\mathbb{C}}$$

is a homomorphism over $\mathbb{C}$ between algebraic groups defined over $\mathbb{Q}$. The subfield E of $\mathbb{C}$ is the field of definition $E(G, h)$ of the conjugacy class of this homomorphism.

For G semisimple adjoint, the field $E(G, h)$ can be described as follows. Let Δ be the Dynkin diagram of $G_{\mathbb{C}}$, and let $|\Delta|$ be the set of vertices of Δ . The Galois group $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on Δ .

Let H be a maximal torus of $G_{\mathbb{C}}$, endowed with a system of simple roots $(\alpha_i)_{i \in |\Delta|}$. The α_i identify H with $\mathbb{G}_{m, \mathbb{C}}^{|\Delta|}$. For every homomorphism $u : \mathbb{G}_{m, \mathbb{C}} \rightarrow G_{\mathbb{C}}$, there is then a unique homomorphism $u' : \mathbb{G}_{m, \mathbb{C}} \rightarrow H = \mathbb{G}_{m, \mathbb{C}}^{\Delta}$: $x \mapsto (x^{n_i})_{i \in |\Delta|}$ with $n_i \geq 0$, which is conjugate to u . The construction $u \mapsto \eta(u) = (n_i)_{i \in |\Delta|}$ identifies the set of conjugacy classes of maps from $\mathbb{G}_{m, \mathbb{C}}$ to $G_{\mathbb{C}}$ with the set of functions from $|\Delta|$ to $\mathbb{N}$. It follows that $E(G, h)$ is defined by the subgroup of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which stabilizes $\eta(hr)$.

Proposition 3.8. Let $G/C = \prod_{i=1}^g G_i$ be the decomposition of the adjoint group G/C into $\mathbb{Q}$ -simple factors, and let h_i be the composite $h_i : \mathbb{S} \xrightarrow{h} G_{\mathbb{R}} \longrightarrow (G_i)_{\mathbb{R}}$.

- (i) The subfield $E(G, h)$ of $\mathbb{C}$ is the compositum of the subfields $E(T, \nu \circ h)$ and $E(G_i, h_i)$ ($1 \leq i \leq g$) of $\mathbb{C}$.
- (ii) If the opposition involution of the Dynkin diagram of G_i respects $\eta(h_i r)$, then $E(G_i, h_i)$ is totally real. Otherwise, $E(G_i, h_i)$ is a totally imaginary quadratic extension of a totally real number field.

Assertion (i) is easy to check, and it is enough to prove (ii) for G $\mathbb{Q}$ -simple and adjoint. Let Δ be its Dynkin diagram. The hypotheses 1.5 imply that $G_{\mathbb{R}}$ has a compact maximal torus; this implies that complex conjugation acts on Δ by the opposition involution ι . This involution is in the centre of the automorphism group of Δ .

Let E' be the Galois extension of $\mathbb{Q}$ defined by the subgroup of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which acts trivially on Δ . Then the image σ of complex conjugation is central in $\text{Gal}(E'/\mathbb{Q})$; hence E' is totally real or a totally imaginary quadratic extension of a totally real field. Finally, σ is the identity on $E(G, h) \subset E'$ if and only if ι respects $\eta(hr)$, and the assertion follows.

3.9. We use the notation of 3.6. Suppose that G' is simply connected, that E contains $E(G, h)$, and that the complex embedding of E induces that of $E(G, h)$. The composite morphism

$$r''(h) : \nu hr : \mathbb{G}_{m, \mathbb{C}} \xrightarrow{r} \mathbb{S}_{\mathbb{C}} \xrightarrow{h} G_{\mathbb{C}} \xrightarrow{\nu} T_{\mathbb{C}}$$

depends only on the conjugacy class of hr . It is therefore defined over E , i.e. it arises, by extension of scalars from E to $\mathbb{C}$, from a homomorphism of algebraic groups over E , again denoted $r''(h) : \mathbb{G}_m \longrightarrow T_E$. Apply Weil restriction of scalars, and let $r'(h) : E^* \rightarrow T$ be the composite

$$E^* = R_{E/\mathbb{Q}}(\mathbb{G}_{m,E}) \xrightarrow{R_{E/\mathbb{Q}}(r''(h))} R_{E/\mathbb{Q}}(T_E) \xrightarrow{N_{E/\mathbb{Q}}} T.$$

The reciprocity law of M is the inverse $r(G, h)$ of $r'(h)$:

$$(3.9.1) \quad r(G, h) = r'(h)^{-1} : E^* \longrightarrow T.$$

3.10. Example II. Let H be a torus and let $h : \mathbb{S} \rightarrow H_{\mathbb{R}}$. The conditions of 1.5 are automatically satisfied. The ${}_K M_{\mathbb{C}}(H, h)$ are finite sets. A finite reduced scheme over the field $E(H, h)$ (3.7) is identified with a Galois set. It follows that, up to unique isomorphism, there is a unique model of $M_{\mathbb{C}}(H, h)$ over $E(H, h)$ whose reciprocity law is given by 3.9.1. It is denoted $M(H, h)$.

3.11. Let F be a finite extension of E , endowed with a complex embedding extending that of E . The diagram

$$\begin{array}{ccc} F^* & \xrightarrow{N_{F/E}} & E^* \\ & \searrow r(G, h) & \swarrow r(G, h) \\ & T & \end{array}$$

is commutative. If $M_E(G, h)$ is a model over E of $M_{\mathbb{C}}(G, h)$ whose reciprocity law is (3.9.1), it follows that $M_F(G, h)$ (3.1) again has (3.9.1) as its reciprocity law.

3.12. Let $u : (G^1, h^1) \rightarrow (G^2, h^2)$ be as in 1.14, and let $M_{E^i}^i(G^i, h^i)$ be a model of $M_{\mathbb{C}}(G^i, h^i)$ over E^i . Let E be the compositum of the subfields E^1 and E^2 of $\mathbb{C}$. With the notation of 3.1 one has

$$M_E^i(G^i, h^i) \otimes_E \mathbb{C} = M_{\mathbb{C}}(G^i, h^i) \quad (i = 1, 2).$$

It therefore makes sense to ask whether

$$u : M_{\mathbb{C}}(G^1, h^1) \longrightarrow M_{\mathbb{C}}(G^2, h^2)$$

is defined over E .

Definition 3.13. Let G be a connected reductive group over $\mathbb{Q}$ satisfying the condition of 2.1, $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$ be a homomorphism satisfying the conditions of 1.5, and E be an extension of $E(G, h)$ (3.7) endowed with a complex embedding extending that of $E(G, h)$. A model $M_E(G, h)$ of $M_{\mathbb{C}}(G, h)$ over E is called weakly canonical if, for every torus $u : H \rightarrow G$ in G , endowed with $h' : \mathbb{S} \rightarrow H_{\mathbb{R}}$ such that uh' is the conjugate of h by an element of $G(\mathbb{R})$, the morphism (1.14)

$$M_{\mathbb{C}}(H, h') \longrightarrow M_{\mathbb{C}}(G, h)$$

is defined, in the sense of (3.10) and (3.12), over the compositum $E(H, h') \cdot E \subset \mathbb{C}$. A model $M_E(G, h)$ is called canonical if it is weakly canonical and if $E = E(G, h)$.

3.14. Translation. Let M be a canonical model of $M_{\mathbb{C}}(G, h)$, with reciprocity law λ_M . If G' is simply connected, then λ_M is given by 3.9.1 (5.6). For every compact open subgroup K of $G(\mathbb{A}^f)$, again denote by $\nu(K)$ the image of K in $\pi(G)/\pi_0(K_{\infty})$, and let $E(K)$ be the extension of $E(G, h)$ defined by the idèle class group $\lambda_M^{-1}(\nu(K))$. One verifies that $E(K)$ is the field of definition of the neutral component ${}_K M_{\mathbb{C}}^0$ of ${}_K M_{\mathbb{C}}$, considered as a subscheme of ${}_K M_{\mathbb{C}} = {}_K M \otimes_{E, \rho} \mathbb{C}$.

By definition, ${}_K M_{\mathbb{C}}^0$ is the scalar extension to $\mathbb{C}$ of a subscheme ${}_K M'$ of ${}_K M \otimes_E E(K)$. The scheme ${}_K M'/E(K)$ is geometrically connected over $E(K)$, and one can also describe ${}_K M'$ as the union of the connected components (over E) of ${}_K M$ that, after extension of scalars from E to $\mathbb{C}$, contain the origin:

$$\begin{array}{ccccc} {}_K M & \longleftarrow & {}_K M' & \longleftarrow & {}_K M_{\mathbb{C}}^0 \\ \downarrow & & \downarrow & & \downarrow \\ & & \text{Spec } E(K) & \longleftarrow & \text{Spec } \mathbb{C} \\ & & \downarrow & & \\ \text{Spec } E & = & \text{Spec } E & & \end{array}$$

Shimura usually states his results in terms of the system of varieties ${}_K M'/E(K)$ and the homomorphisms of type 2.2 between them. For a typical statement, see [14, p. 146]. For the translation, cf. [14, 2.7].

3.15. For $u : (H, h') \rightarrow (G, h)$ as in 3.13, the image of $M_{\mathbb{C}}(H, h')$ in ${}_K M_{\mathbb{C}}(G, h)$ is a finite set. The hypothesis is that this subset of ${}_K M_{\mathbb{C}}(G, h)$ be defined over $E(H, h')$, that its points be defined over an abelian extension of $E(H, h')$, and that $\text{Gal}(\overline{E(H, h')}/E(H, h'))$ act on them in the prescribed manner.

We shall call "special" the points of ${}_K M_{\mathbb{C}}(G, h)$ lying in the image of some $M_{\mathbb{C}}(H, h')$.

Suppose given a projective embedding $q : {}_K M_{\mathbb{C}}^0 \rightarrow \mathbb{P}^r$ defined over $E(K)$. It may be interpreted as $\tilde{q} : X^0 \rightarrow \mathbb{P}^r$ (cf. 1.7, 1.8). If $uh' \in X$ lies in X^0 , then the coordinates of $\tilde{q}(uh')$ generate over $E(K)E(H, h')$ an abelian extension of $E(H, h')$ that can be described explicitly as a class field.

3.16. The classical example of such a situation is provided by the moduli scheme of elliptic curves (corresponding to $G = \text{GL}_2$, $K = \text{GL}_2(\hat{\mathbb{Z}})$, h as in 1.6), and by its points corresponding to elliptic curves with complex multiplication.

Let j be the modular invariant, viewed as a function on the Poincaré upper half-plane X^0 . If $\tau \in X^0$ lies in an imaginary quadratic field K , let $L(\tau) = \mathbb{Z} \oplus \mathbb{Z}\tau \subset \mathbb{C}$, and let $\mathcal{O}(\tau) = \{k \in K \mid kL(\tau) \subset L(\tau)\}$. The field $K(j(\tau))$ is the abelian extension of K whose Galois group is the ideal class group of $\mathcal{O}(\tau)$. If σ in this Galois group corresponds to an invertible $\mathcal{O}(\tau)$ -module L , and if $L(\tau') \simeq L(\tau) \otimes_{\mathcal{O}(\tau)} L$, then $j(\tau) = j(\tau')^\sigma$.

For more exotic explicit examples, see [15], pp. 45–46.

§ 4. Abelian varieties.

4.1. Let k be an algebraically closed field of characteristic 0. Let A be an abelian variety over k . For $n \in \mathbb{N}^+$, denote by A_n the kernel of multiplication by n . The A_n form a projective system, with $\varphi_{nm,n} : A_{nm} \rightarrow A_n$ given by $x \mapsto x^m$, and we put

$$\hat{T}(A) = \varprojlim A_n, \quad \hat{V}(A) = \mathbb{A}^f \otimes_{\hat{\mathbb{Z}}} \hat{T}(A).$$

The $\hat{\mathbb{Z}}$ -module $\hat{T}(A)$ is the product of the Tate modules (= Weil representations) $T_\ell(A)$. If $k = \mathbb{C}$, then

$$(4.1.1) \quad \hat{T}(A) = \hat{\mathbb{Z}} \otimes H_1(A, \mathbb{Z}),$$

$$(4.1.2) \quad \hat{V}(A) = \mathbb{A}^f \otimes H_1(A, \mathbb{Q}).$$

4.2. The category of abelian varieties up to isogeny over k is the category whose objects are the abelian varieties over k and in which $\text{Hom}_{\text{iso}}(A, B) = \text{Hom}(A, B) \otimes \mathbb{Q}$. We denote by $A \otimes \mathbb{Q}$ the abelian variety up to isogeny "underlying" an abelian variety A . Every additive functor from the category of abelian varieties to a $\mathbb{Q}$ -linear additive category factors through the functor $A \mapsto A \otimes \mathbb{Q}$. Thus the contravariant functor "dual variety" $A \mapsto A^*$ extends to abelian varieties up to isogeny. The functor $A \mapsto \hat{V}(A)$ extends in the same way. Moreover, if A_0 is an abelian variety up to isogeny, it is equivalent to give B together with an isomorphism $B \otimes \mathbb{Q} = A_0$, or to give $\hat{T}(B) \subset \hat{V}(A_0)$.

4.3. Let A be an abelian variety over k . Let $\text{NS}(A)$ denote the group of algebraic-equivalence classes of invertible sheaves on A . An effective polarization (resp. a polarization) of A is defined to be an element of $\text{NS}(A)$, respectively of $\text{NS}(A) \otimes \mathbb{Q}$, a positive multiple of which is defined by a projective embedding of A . A homogeneous polarization is an element of $(\text{NS}(A) \otimes \mathbb{Q})/\mathbb{Q}^*$ which is the class of a polarization.

Recall that $\text{NS}(A) \otimes \mathbb{Q}$ is identified, by a bijection $p \mapsto p'$, with the set of elements of $\text{Hom}(A, A^*) \otimes \mathbb{Q}$ equal to their transpose. An isomorphism $u \in \text{Hom}(A \otimes \mathbb{Q}, B \otimes \mathbb{Q})$ induces isomorphisms from $\text{Hom}(A, A^*) \otimes \mathbb{Q}$ to $\text{Hom}(B, B^*) \otimes \mathbb{Q}$, and from $\text{NS}(A) \otimes \mathbb{Q}$ to $\text{NS}(B) \otimes \mathbb{Q}$, and carries polarizations to polarizations. This makes it possible to speak of a polarization of an abelian variety up to isogeny.

4.4. A polarization p of A defines a positive involution on $\text{End}(A) \otimes \mathbb{Q}$, $u \mapsto p'^{-1}u^*p'$, which depends only on the homogeneous polarization $\mathbb{Q}^*.p$.

Let F be a product of totally real fields, and let $\rho : F \rightarrow \text{End}(A) \otimes \mathbb{Q}$.

Let $\text{NS}_\rho(A) \subset \text{NS}(A)$ be the set of p such that $p'.\rho(f) = \rho(f)^*p'$ for $f \in F$.

The vector space $\text{NS}_\rho(A) \otimes \mathbb{Q}$ is endowed, for $f \in F$, with an action of F : $(f.p)' = p' \circ \rho(f) = \rho(f)^* \circ p'$. A weak polarization of A (relative to ρ, F) is an element of $(\text{NS}_\rho(A) \otimes \mathbb{Q})/F^*$ which is the class of a polarization. The set of weak polarizations of A depends only on $A \otimes \mathbb{Q}$. The restriction to the centralizer of F of the involution on $\text{End}(A) \otimes \mathbb{Q}$ defined by a polarization $p \in \text{NS}_\rho(A)$ depends only on the weak polarization $F^*.p$.

4.5. If A_0^* is the dual of an abelian variety up to isogeny A_0 , then $\widehat{V}(A_0)$ and $\widehat{V}(A_0^*)$ are in duality with values in $\mathbb{A}^f(1)$. If p is a polarization of A_0 , the polarization form $\psi_p(x, y) = \langle x, p'(y) \rangle$ is a nondegenerate alternating form on the free $\mathbb{A}^f$ -module $\widehat{V}(A_0)$, with values in $\mathbb{A}^f(1)$.

The facts recalled in 4.1 and 4.5 extend to abelian schemes over an arbitrary base.

4.6. Set $k = \mathbb{C}$. The additive functor $A \mapsto H_1(A, \mathbb{Q})$ extends to abelian varieties up to isogeny over $\mathbb{C}$ (4.2). For an abelian variety up to isogeny A_0 , $H_1(A_0, \mathbb{Q})$ and $H_1(A_0^*, \mathbb{Q})$ are in duality. Via 4.1.2 and the isomorphism of $\mathbb{A}^f(1)$ with $\mathbb{A}^f$ defined by the exponential, this duality is compatible with the one considered in 4.5. The alternating form $\psi_p(x, y) = \langle x, p'(y) \rangle$ on $H_1(A_0, \mathbb{Q})$ defined by a polarization p of A_0 is again called the polarization form.

The Hodge structure on $H_1(A_0, \mathbb{Q})$ is the homomorphism $h : \underline{\mathbb{S}} \rightarrow \mathrm{GL}(H_1(A_0, \mathbb{Q}))_{\mathbb{R}}$, such that $\underline{\mathbb{S}}$ acts on $H_1(A_0, \mathbb{Q}) \otimes \mathbb{C}$ through the characters z^{-1} and $\bar{z}^{-1}$, and such that $F(h)^\circ$ (1.5) is the kernel of the exponential map $H_1(A_0, \mathbb{Q}) \otimes \mathbb{C} \rightarrow \mathrm{Lie}(A)$.

Theorem 4.7. (Riemann). The constructions

$$(A, p) \mapsto (H_1(A, \mathbb{Q}), \psi_p, H_1(A, \mathbb{Z}), h),$$

$$(V, \psi, V_{\mathbb{Z}}, h) \mapsto F(h)^\circ \backslash V \otimes \mathbb{C} / V_{\mathbb{Z}}$$

establish an equivalence between

- (a) polarized abelian varieties over $\mathbb{C}$;
- (b) systems consisting of a vector space V over $\mathbb{Q}$, a nondegenerate alternating form ψ on V , an integral lattice $V_{\mathbb{Z}}$ in V and a homomorphism h from $\underline{\mathbb{S}}$ into the group $\mathrm{Gp}(V)_{\mathbb{R}}$ of symplectic similitudes of $V \otimes \mathbb{R}$, of type 1.6 and such that

$$\psi(x, h(i)x) > 0 \quad (x \in V \otimes \mathbb{R}, x \neq 0).$$

Variant 4.8. It is equivalent to give an abelian variety up to isogeny A over $\mathbb{C}$, endowed with a homogeneous polarization, or to give a vector space over $\mathbb{Q}$, endowed with a nondegenerate alternating form ψ given up to scalar, together with $h : \underline{\mathbb{S}} \rightarrow \mathrm{Gp}(V)$ of type 1.6.

4.9. Let L be a semisimple algebra with involution over $\mathbb{Q}$, and let V be a vector space over $\mathbb{Q}$, endowed with a faithful L -module structure and with a nondegenerate alternating form ψ such that

$$(4.9.1) \quad \psi(\ell x, y) = \psi(x, \ell^* y).$$

Let G denote the algebraic group over $\mathbb{Q}$ of L -linear symplectic similitudes of V . The group $G(\mathbb{Q})$ is the set of $g \in \mathrm{GL}_L(V)$ for which there exists $\mu(g) \in \mathbb{Q}^*$ satisfying

$$(4.9.2) \quad \psi(gx, gy) = \mu(g)\psi(x, y).$$

Suppose given a homomorphism $h_0 : \underline{\mathbb{S}} \rightarrow G_{\mathbb{R}}$ such that, via h_0 , $\underline{\mathbb{S}}$ acts on $V_{\mathbb{C}}$ through the characters z^{-1} and $\bar{z}^{-1}$ and such that the form $\psi(x, h_0(i)y)$ is symmetric and positive definite. The conditions (1.5.1) to (1.5.3) are then satisfied by (G, h_0) , and the involution of L is positive.

4.10. Apply (1.11) to (G, h_0) and to the representation V of G . A G -structure $\bar{\iota}$ on a vector space H may be interpreted, by 1.8, as specifying on H an L -module structure and an alternating form ψ , given up to scalar, such that H , endowed with these structures, is isomorphic to V . Let $\bar{\iota}$ be a G -structure on H . A datum 1.9 (b), $h : \underline{\mathbb{S}} \rightarrow G_{\mathbb{R}}^1$ on H , then defines, by 4.8 applied to (H, ψ, h) , an abelian variety up to isogeny A , endowed with a homogeneous polarization $\bar{p}$, such that H is $H_1(A, \mathbb{Q})$, h is the Hodge structure on $H_1(A, \mathbb{Q})$, and ψ is the polarization form. Moreover, the L -module structure on H comes from $\rho : L \rightarrow \mathrm{End}(A)$, and H , endowed with $\bar{\iota}$ and h , is determined by $(A, \bar{p}, \rho)$.

For a triple $(A, \bar{p}, \rho)$ to arise from such an H , it is necessary and sufficient that

$$(4.10.1) \quad H_1(A, \mathbb{Q}), \text{ endowed with its } L\text{-module structure and with the polarization form (given up to scalar), is isomorphic to } V;$$

$$(4.10.2) \quad \text{For an isomorphism } t : V \rightarrow H_1(A, \mathbb{Q}), t^{-1}h_1t \text{ is conjugate to } h_0.$$

Let t be the map from L to $\mathbb{C}$ given by

$$(4.10.3) \quad t(\ell) = \mathrm{Tr}(\ell; V_{\mathbb{C}}/F^0(h_0)).$$

Scholium 4.11. Let K be a compact open subgroup of $G(\mathbb{A}^f)$. The points of ${}_K M_{\mathbb{C}}(G, h_0)$ correspond bijectively to isomorphism classes of abelian varieties up to isogeny A , endowed with

(a) $\rho : L \longrightarrow \text{End}(A)$ such that

$$\text{Tr}(\rho(\ell), \text{Lie}(A)) = t(\ell) \quad (\ell \in L),$$

(b) a homogeneous polarization $\bar{p}$ that induces the given involution of L ,

(c) a class mod $K, \bar{k}$, of symplectic similitudes L -linear

$$k : \widehat{V}(A) \xrightarrow{\sim} V \otimes \mathbb{A}^f,$$

and such that

(*) the conditions (4.10.1) and (4.10.2) are satisfied.

We apply 1.11 and 4.10, observing that a datum 1.9 (c) is identified with a datum (c), and that the conditions in (b) and (a) follow from (4.10.1) and (4.10.2), respectively.

4.12. Let K be a compact open subgroup of $G(\mathbb{A}^f)$, let $V_{\mathbb{Z}}$ be an integral lattice in V such that $V_{\widehat{\mathbb{Z}}} = V_{\mathbb{Z}} \otimes \widehat{\mathbb{Z}}$ is K -invariant, let $L_{\mathbb{Z}}$ be an order in L such that $L_{\mathbb{Z}} V_{\mathbb{Z}} \subset V_{\mathbb{Z}}$, let $\psi_{\mathbb{Z}}$ be a positive rational multiple of the alternating form of V , with integral values on $V_{\mathbb{Z}}$, and let $V'_{\mathbb{Z}}$ be the largest lattice in V such that $\psi_{\mathbb{Z}}(V_{\mathbb{Z}}, V'_{\mathbb{Z}}) \subset \widehat{\mathbb{Z}}$. Let n be an integer, and put $K_n = \{g \in G(\mathbb{A}^f) \mid (g-1)V'_{\mathbb{Z}} \subset nV_{\mathbb{Z}}\}$. Assume that n is large enough that $K \supset K_n$.

If A is as in 4.11, then $k^{-1}(V_{\widehat{\mathbb{Z}}}) \subset \widehat{V}(A)$ does not depend on $k \in \bar{k}$, and defines an abelian variety B with $B \otimes \mathbb{Q} \simeq A$ and $\widehat{T}(B) = k^{-1}(V_{\widehat{\mathbb{Z}}}) \subset \widehat{V}(A)$. The action of L on A induces an action of $L_{\mathbb{Z}}$ on B . There exists a unique effective polarization p of B , with $p \in \bar{p}$, such that $k^{-1}(V'_{\widehat{\mathbb{Z}}})$ is the largest “lattice” $\widehat{T}'(B) \supset \widehat{T}(B)$ satisfying $\psi_p(\widehat{T}(B), \widehat{T}'(B)) \subset \widehat{\mathbb{Z}}(1)$. Finally, the set $\bar{k}$ of L -linear symplectic isomorphisms from $\widehat{T}(B)$ to $V_{\widehat{\mathbb{Z}}}$ is the inverse image of its image $\bar{k}_n$ in $\text{Isom}(B_n, V_{\mathbb{Z}}/nV_{\mathbb{Z}})$.

This again permits the points of ${}_K M_{\mathbb{C}}(G, h_0)$ to be interpreted as corresponding to isomorphism classes of systems $(B, p, \rho, \bar{k}_n)$ consisting of

(a) a polarized abelian variety (B, p) , with complex multiplication by $L_{\mathbb{Z}}$, satisfying 4.11 (a) and (b);

(b) a class mod $K/K_n, \bar{k}_n$, of isomorphisms $k_n : B_n \xrightarrow{\sim} V_{\mathbb{Z}}/nV_{\mathbb{Z}}$, which can be lifted to L -linear symplectic isomorphisms $k : \widehat{T}(B) \longrightarrow V_{\widehat{\mathbb{Z}}}$, and which satisfy (4.10.1) and (4.10.2).

4.13. Let F denote the subalgebra of the center of L fixed by $*$. It is a product of totally real fields. There exists a unique alternating F -bilinear form ${}_F\psi$ on V such that $\text{Tr}_{F/\mathbb{Q}} {}_F\psi = \psi$.

Let G_1 denote the group of L -linear symplectic similitudes of ${}_F\psi$: $G_1(\mathbb{Q})$ is the set of $g \in \text{GL}_L(V)$ for which there exists $\mu(g) \in F^*$ satisfying

$$(4.13.1) \quad {}_F\psi(gx, gy) = \mu(g) {}_F\psi(x, y), \quad \text{i.e.}$$

$$(4.13.2) \quad \psi(gx, gy) = \psi(\mu(g)x, y).$$

As above, we verify the

Variant 4.14. Let K be a compact open subgroup of $G_1(\mathbb{A}^f)$. The points of ${}_K M_{\mathbb{C}}(G_1, h_0)$ correspond bijectively to isomorphism classes of abelian varieties up to isogeny A , endowed with

(a) $\rho : L \longrightarrow \text{End}(A)$ as in 4.11,

(b) a weak polarization (relative to F) $\bar{p}$ which induces the given involution of L ,

(c) a class modulo K of L -linear symplectic similitudes $k : \widehat{V}(A) \xrightarrow{\sim} V \otimes \mathbb{A}^f$ (for the $F \otimes \mathbb{A}^f$ -bilinear forms on the two sides),

and such that

(*) conditions analogous to (4.10.1) and (4.10.2) are satisfied.

4.15. One can make 4.11 and 4.14 more precise by showing that ${}_K M_{\mathbb{C}}(G, h)$ is the coarse moduli scheme of the evident functor on schemes of finite type over $\mathbb{C}$.

4.16. Example I (continuation of 1.6, 1.11). Consider the special case of 4.9 in which $L = \mathbb{Q}$ and $\dim(V) = 2g$. Then $\bar{G} = \text{Gp}(V)$ (the group of symplectic similitudes). For $h_0 : \mathbb{S} \longrightarrow G_{\mathbb{R}}$ of type 1.6, one has $E(\text{Gp}, h_0) = \mathbb{Q}$.

Let $V_{\mathbb{Z}}$, n , and K_n be as in 1.11. For every scheme S , let $F(S)$ be the set of isomorphism classes of principally polarized abelian schemes A/S endowed with a symplectic similitude $k_n : A_n \xrightarrow{\sim} (V_{\mathbb{Z}}/nV_{\mathbb{Z}})_S$. For $n \geq 3$, the classified objects have no nontrivial automorphisms, and the functor F is represented by a $\mathbb{Q}$ -scheme ${}_{K_n}M$ [6]. By 1.11, 4.11, and 4.12, there is an isomorphism

$$m_n : {}_{K_n}M(\mathbb{C}) \xrightarrow{\sim} {}_{K_n}M_{\mathbb{C}}(\mathrm{Gp}, h_0).$$

In the language of 3.1, one has

Proposition 4.17. The $\mathbb{Q}$ -scheme $M(\mathrm{Gp}, h_0) = \varprojlim_{{}_{K_n}} {}_{K_n}M$ is a model over $\mathbb{Q} = E(\mathrm{Gp}, h_0)$ of $M_{\mathbb{C}}(\mathrm{Gp}, h_0)$.

4.18. Consider the special case of 4.9 in which V is a cyclic L -module. The groups G and G_1 are then contained in the center of L^* .

Let $\bar{E}$ be an algebraic closure of $E(G, h)$, and let M^+ be the set of isomorphism classes $[A, \rho, k, \bar{p}]$ of objects $(A, \rho, k, \bar{p})$ consisting of

- (a) an abelian variety up to isogeny $A/\bar{E}$, endowed with a homogeneous polarization $\bar{p}$ and with $\rho : L \rightarrow \mathrm{End}(A)$ such that, with the notation of 4.10.3, for $\ell \in L$ one has

$$\mathrm{Tr}(\rho(\ell), \mathrm{Lie}(A)) = t(\ell) \in E(G, h);$$

- (b) an $L \otimes \mathbb{A}^f$ -linear isomorphism $k : \widehat{V}(A) \xrightarrow{\sim} V \otimes \mathbb{A}^f$.

The abelian variety A is therefore of CM type. We refer to [16] for the proof of the following fundamental theorem.

Theorem 4.19. (Shimura–Taniyama). The Galois group $\mathrm{Gal}(\bar{E}/E(G, h))$ acts on M^+ through its largest abelian quotient $\pi_0(E(G, h)^*(\mathbb{A})/E(G, h)^*(\mathbb{Q}))$. For $\underline{e} \in E(G, h)^*(\mathbb{A})$, with image $\varphi(\underline{e})$ in abelianized Galois and with component $\underline{e}^f$ in $E(G, h)^*(\mathbb{A}^f)$, one has

$$(4.19.1) \quad \varphi(\underline{e})([A, \rho, k, \bar{p}]) = [A, \rho, r(G, h)(\underline{e}^f) \cdot k, \bar{p}].$$

Let τ be an embedding of $\bar{E}$ into $\mathbb{C}$ extending that of $E(G, h)$, let K be a compact open subgroup of $G(\mathbb{A}^f)$, and let ${}_KM(G, h)(\bar{E})$ be the set of isomorphism classes of objects $(A, \rho, \bar{k}, \bar{p})$ where

- (a) A, ρ , and $\bar{p}$ are as above;
- (b) $\bar{k}$ is a class modulo K of $L \otimes \mathbb{A}^f$ -linear symplectic similitudes $k : \widehat{V}(A) \xrightarrow{\sim} V \otimes \mathbb{A}^f$;
- (c) the vector space $H_1(A_{\mathbb{C}}, \mathbb{Q})$, where $A_{\mathbb{C}}$ is defined via τ , endowed with the action of L and with the polarization form given up to scalar, is isomorphic to V .

By 4.19, condition (c) is independent of the choice of τ , so that the finite set ${}_KM(G, h)(\bar{E})$ is endowed with an action of $\mathrm{Gal}(\bar{E}/E(G, h))$; as a Galois set, it is identified with the set of $\bar{E}$ -points of a finite étale $E(G, h)$ -scheme ${}_KM(G, h)$. By 4.14, ${}_KM(G, h)(\mathbb{C}) = G(\mathbb{R}) \times K \backslash G(\mathbb{A})/G(\mathbb{Q})$, and by 4.19 one has:

Proposition 4.20. The $E(G, h)$ -scheme $M(G, h) = \varprojlim_{{}_{K}} {}_KM(G, h)$ is a canonical model of $M_{\mathbb{C}}(G, h)$.

Theorem 4.21. The model $M(\mathrm{Gp}, h_0)$ constructed in 4.17 is a canonical model.

Let (V, ψ) be as in 4.16, and let $L, \rho : L \rightarrow \mathrm{End}(V)$, G , and $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$ be as in 4.18. We have $u : (G, h) \rightarrow (\mathrm{Gp}, h_0)$. Let $K \subset G(\mathbb{A}^f)$ be such that $u(K) \subset K_n$ (4.16). If $\bar{E}$ is an algebraic closure of $E(G, h)$, define

$$u : {}_KM(G, h)(\bar{E}) \longrightarrow {}_{K_n}M(\mathrm{Gp}, h_0)(\bar{E})$$

by “forgetting the complex multiplication”, associating to $[A, \rho, \bar{k}, \bar{p}]$ the triple consisting of

- (a) the abelian variety B , equipped with an isomorphism $B \otimes \mathbb{Q} \simeq A$ and such that

$$\widehat{T}(B) = k^{-1}(V_{\widehat{\mathbb{Z}}}) \subset \widehat{V}(A) \quad (k \in \bar{k});$$

- (b) the polarization $\bar{p}$ of B
- (c) the isomorphism $k : B_n \xrightarrow{\sim} V_{\mathbb{Z}}/nV_{\mathbb{Z}}$ ($k \in \bar{k}$).

The maps u define a morphism of models

$$u : M(G, h) \longrightarrow M(\mathrm{Gp}, h_0) \otimes E(G, h).$$

We complete the proof by showing ([8]) that for every $u : (H, h') \rightarrow (G, h)$ as in 3.13, there exists $v : (G, h) \rightarrow (G, h_0)$ as above (for a suitable L) such that $uh' = vh$.

Variant 4.22. Let F be a totally real number field, let $n = [F : \mathbb{Q}]$, and let $\mathrm{Gp}_{2g}(F)$ be the group of symplectic similitudes of a symplectic F -vector space of dimension $2g$. Let $h_0 : \mathbb{S} \rightarrow \mathrm{Gp}_{2g}(F)_{\mathbb{R}} \simeq \mathrm{Gp}_{2g}(\mathbb{R})^n$ be as in 1.6. One can construct a canonical model of $M_{\mathbb{C}}(\mathrm{Gp}_{2g}(F), h_0)$ from moduli of weakly polarized abelian schemes.

5. Construction techniques.

Definition 3.13 is usable only because there are “many” special points.

Theorem 5.1. Let G and h be as in 3.13. For every finite extension F of $E(G, h)$, there exists $(H, h', u : H \hookrightarrow G)$ as in 3.13 such that the extension $E(H, h')$ of $E(G, h)$ is linearly disjoint from F .

Let $Y = \underline{\mathrm{Hom}}(\mathbb{G}_m, G)/G$ be the scheme of conjugacy classes of morphisms from $\mathbb{G}_m$ to G . The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the (infinite discrete) set $Y(\mathbb{C})$ of conjugacy classes of morphisms from $\mathbb{G}_{m\mathbb{C}}$ to $G_{\mathbb{C}}$. Let Y_o be the finite subscheme of Y such that $Y_o(\mathbb{C})$ is the Galois orbit of the class of hr . It is the spectrum of $E(G, h)$.

Let $V' = \mathrm{Lie}(G)$, regarded as an affine space over $\mathbb{Q}$, and let V be the open subscheme of regular elements of the Lie algebra. For v in V , let T_v denote the centralizer of v (a maximal torus). Let W be the moduli scheme of maximal tori T of G , endowed with $s : \mathbb{G}_m \rightarrow T$ and $v \in \mathrm{Lie}(T)$ such that

- (a) v is regular in $\mathrm{Lie}(G)$;
- (b) the class of $s : \mathbb{G}_m \rightarrow G$ lies in Y_o .

The morphism $f : W \rightarrow V : (T, s, v) \mapsto v$ is finite étale and surjective (note that $T = T_v$). Let p be the morphism from W to $Y_o : (T, s, v) \mapsto (\text{the class of } s)$.

$$(5.1.1) \quad \begin{array}{ccc} W & \xrightarrow{f} & V \\ p \downarrow & & \downarrow \\ \mathrm{Spec}(E(G, h)) & \longrightarrow & \mathrm{Spec}(\mathbb{Q}) \end{array}$$

Lemma 5.1.2. The scheme W is irreducible, and the geometric fibers of p are (geometrically) irreducible.

Since Y_o is irreducible (the spectrum of an extension of $\mathbb{Q}$), it suffices to prove the second assertion. This follows from the fact that

- (a) since G is connected, the scheme over $\mathbb{C}$ of homomorphisms from $\mathbb{G}_{m\mathbb{C}}$ to $G_{\mathbb{C}}$ conjugate to a given homomorphism is irreducible;
- (b) for a given $i : \mathbb{G}_{m\mathbb{C}} \rightarrow G_{\mathbb{C}}$, by the theorem on conjugacy of maximal tori applied to the (connected) centralizer of $i(\mathbb{G}_{m\mathbb{C}})$, the scheme of maximal tori of $G_{\mathbb{C}}$ containing $i(\mathbb{G}_{m\mathbb{C}})$ is irreducible.

Let $U \subset V(\mathbb{R})$ be the set of $v \in V(\mathbb{R})$ such that $(T_v/C)(\mathbb{R})$ is compact. For the usual topology, U is open.

For $v \in U$ and $w \in W(\mathbb{C})$ such that $f(w) = v \in V(\mathbb{R}) \subset V(\mathbb{C})$, there exists a unique homomorphism $h(w) : \mathbb{S} \rightarrow T_v$ such that $h(w) \circ r = s_w$: after extension of scalars to $\mathbb{C}$

$$h(w) = s_w \circ z \cdot \bar{s}_w \circ \bar{z}.$$

Let $v \in U$. By the theorem on conjugacy of maximal compact subgroups in $G/C(\mathbb{R})^\circ$, there exists $g \in G(\mathbb{R})$ (indeed one may take $g \in G'(\mathbb{R})^\circ$) such that $gT_vg^{-1} \subset K_\infty$. Since T_v is a maximal torus, gT_vg^{-1} contains the center of K_∞ . It follows that there exists $w \in W(\mathbb{C})$ such that $f(w) = v$ and such that $h(w)$ is conjugate to h .

We conclude the proof of 5.1 by applying to diagram (5.1.1) and to U the following variant of Hilbert’s irreducibility theorem, whose proof I owe to J.-P. Serre.

Lemma 5.13. Let E be a finite extension of $\mathbb{Q}$, and let there be a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{f} & V \\ p \downarrow & & \downarrow \\ \text{Spec}(E) & \xrightarrow{\quad} & \text{Spec}(\mathbb{Q}) \end{array}$$

in which

- (i) V is a Zariski open subset of an affine space over $\mathbb{Q}$
- (ii) the geometric fibers of p are irreducible, and W is reduced.
- (iii) f is quasi-finite and dominant.

Then, for every open set $U \subset V(\mathbb{R})$ (in the usual topology) and every extension F of E , there exists $v \in V(\mathbb{Q}) \cap U$ such that $f^{-1}(v)$ is the spectrum of an extension of E linearly disjoint from F .

Let $W' = W \otimes_E F$. The hypotheses of 4.12 are again satisfied for W'/F and $f' : W' \rightarrow W \rightarrow V$, and one must construct $v \in V(\mathbb{Q}) \cap U$ such that $f'^{-1}(v) = f^{-1}(v) \otimes_E F$ is the spectrum of a field. This reduces us to the case $E = F$, which follows from Hilbert's irreducibility theorem (cf. Lang, *Diophantine Geometry*, Chap. VIII). Proposition 5.2. For $x \in M_{\mathbb{C}}(G, h)$ and K compact open in $G(\mathbb{A}^f)$, the image in ${}_K M_{\mathbb{C}}(G, h)$ of $G(\mathbb{A}^f) \cdot x \subset M_{\mathbb{C}}(G, h)$ is dense.

This is a consequence of (0.4). The reader will verify:

Proposition 5.3. Let E be a field endowed with a complex embedding ρ , X a scheme over E , and Y a reduced closed subscheme of $X_{\mathbb{C}}$. Suppose there exist finite extensions E_i of E in $\mathbb{C}$, schemes $M_{i,\alpha}/E_i$ and E_i -morphisms $v_{i,\alpha} : M_{i,\alpha} \rightarrow X \otimes_E E_i$, or $v_i : \coprod_{\alpha} M_{i,\alpha} \rightarrow X \otimes_E E_i$, such that $E = \bigcap_i E_i$, that $v_i(\coprod_{\alpha} M_{i,\alpha} \otimes_{E_i} \mathbb{C}) \subset Y$ and that $v_i(\coprod_{\alpha} M_{i,\alpha} \otimes_{E_i} \mathbb{C})$ is dense in Y . Then Y is defined over E .

Corollary 5.4. Let $u : (G^1, h^1) \rightarrow (G^2, h^2)$ be as in 1.14. Let $M_E(G^1, h^1)$ and $M_E(G^2, h^2)$ be weakly canonical models of the $M_{\mathbb{C}}(G^i, h^i)$ over a number field E ($E(G^i, h^i) \subset E \subset \mathbb{C}$). Then $v : M_{\mathbb{C}}(G^1, h^1) \rightarrow M_{\mathbb{C}}(G^2, h^2)$ is defined over E (cf. 3.12)

Let $K^i \subset G^i(\mathbb{A}^f)$ be compact open subgroups such that $v(K^1) \subset K^2$. We prove that $v(K^1, K^2) : {}_{K^1} M_{\mathbb{C}}(G^1, h^1) \rightarrow {}_{K^2} M_{\mathbb{C}}(G^2, h^2)$ is defined over E . Let $u : (H, h') \rightarrow (G^1, h^1)$ be as in 3.13 and set $F = E \cdot E(H, h')$. For $g \in G^1(\mathbb{A}^f)$, one has F -morphisms

$$\begin{aligned} a(g) : M_F(H, h') &\rightarrow M_F(G^1, h^1) \xrightarrow{g} M_F(G^1, h^1) \rightarrow {}_{K^1} M_F(G^1, h^1), \\ b(g) : M_F(H, h') &\rightarrow M_F(G^2, h^2) \xrightarrow{v(g)} M_F(G^2, h^2) \rightarrow {}_{K^2} M_F(G^2, h^2), \end{aligned}$$

and, after extension of scalars to $\mathbb{C}$, $v(K^1, K^2) \circ a(g) = b(g)$.

By 5.1 and 5.2, one can apply 5.3 to $X = {}_{K^1} M_E(G^1, h^1) \times {}_{K^2} M_E(G^2, h^2)$, to the graph Y of v , to the $E_i = E \cdot E(H, h')$ for variable (H, h', u) , and to $(a(g), b(g)) : M_{E_i}(H, h') \rightarrow X_{E_i}$: it follows that the graph of v is defined over E .

In the special case $(G^1, h^1) = (G^2, h^2)$ and $u = \text{Id}$, 5.4 says:

Corollary 5.5. Up to (unique) isomorphism, there exists at most one weakly canonical model of $M_{\mathbb{C}}(G, h)$ over E ($E(G, h) \subset E \subset \mathbb{C}$).

Corollary 5.6. If G' is simply connected, the reciprocity law of a weakly canonical model of $M_{\mathbb{C}}(G, h)$ over E ($E(G, h) \subset E \subset \mathbb{C}$) is given by 3.9.1.

The proof is the same as in the special case $v : (G, h) \rightarrow (T, vh)$ of 5.4.

Corollary 5.7. Let $v : (G^1, h^1) \hookrightarrow (G^2, h^2)$ be as in 1.15. If $M_{\mathbb{C}}(G^2, h^2)$ admits a weakly canonical model over E , with $E(G^2, h^2) \subset E(G^1, h^1) \subset E \subset \mathbb{C}$, then $M_{\mathbb{C}}(G^1, h^1)$ admits a weakly canonical model over E .

Let K^i be a compact open subgroup of $G^i(\mathbb{A}^f)$. Using 1.15, we are reduced to showing only that if $v(K^1) \subset K^2$ and ${}_{K^1} M_{\mathbb{C}}(G^1, h^1) \hookrightarrow {}_{K^2} M_{\mathbb{C}}(G^2, h^2)$, then the subscheme ${}_{K^1} M_{\mathbb{C}}(G^1, h^1)$ of ${}_{K^2} M_{\mathbb{C}}(G^2, h^2) = {}_{K^2} M_E(G^2, h^2) \otimes_E \mathbb{C}$ is defined over E .

Let $u : (H, h') \rightarrow (G^1, h^1)$ be as in 3.13, and for $g \in G^1(\mathbb{A}^f)$ let $a(g)$ be the composite homomorphism defined over $F = E \cdot E(H, h')$:

$$a(g) : M_F(H, h') \longrightarrow M_F(G^1, h^1) \xrightarrow{v(g)} M_F(G^2, h^2) \longrightarrow {}_{K^2}M_F(G^2, h^2).$$

By 5.1 and 5.2, one concludes by applying 5.3 to $X = {}_{K^2}M_E(G^2, h^2)$, to $Y = {}_{K^1}M_{\mathbb{C}}(G^1, h^1)$, to the $E_i = E \cdot E(H, h')$ for variable (H, h', u) , and to $a(g) : M_{E_i}(H, h') \longrightarrow X_{E_i}$.

Example 5.8. For (G, h_0) as in 4.9, one constructs, by 5.7 and 4.21, a canonical model $M(G^\circ, h_0) \subset M(\mathrm{Gp}(V, \psi), h_0) \otimes_{\mathbb{Q}} E(G^\circ, h^\circ)$.

Variant 5.9. Let (G_1, h_0) be such that, with the notation of 4.22, there exists $u : (G_1, h_0) \hookrightarrow (\mathrm{Gp}_{2g}^{(F)}, h_0)$ (for example (G_1°, h_0) with (G_1, h_0) as in 4.13). An application of 5.7 yields a canonical model of $M(G_1, h_0)$.

Proposition 5.10. Let (G, h) be as in 3.13, let the $E_i : E(G, h) \subset E_i \subset \mathbb{C}$ be number fields, let E be the intersection of the E_i , and let $M_{E_i}(G, h)$ be a weakly canonical model of $M_{\mathbb{C}}(G, h)$ over E_i . Then there exists a unique model $M_E(G, h)$ of $M_{\mathbb{C}}(G, h)$ over E , such that for every i the model $M_E(G, h) \otimes_E E_i$ over E_i is isomorphic to $M_{E_i}(G, h)$.

In view of the uniqueness assertion, one may suppose the E_i finite in number. Let F be the extension of E in $\mathbb{C}$ generated by them, and let $\mathfrak{E}$ be the set of subextensions of F containing one of the E_i . Using 5.5, one obtains for every $F' \in \mathfrak{E}$ a model $M_{F'}(G, h)$, and for every inclusion $F' \subset F''$ ($F', F'' \in \mathfrak{E}$) an isomorphism of models $\alpha(F'', F') : M_{F'}(G, h) \otimes_{F'} F'' \longrightarrow M_{F''}(G, h)$.

Let $u : (H, h') \rightarrow (G, h)$ be as in 3.13, with $E(H, h')$ linearly disjoint from F over $E(G, h)$ (5.1). Put $E_1 = E(H, h') \cdot E$ and $F'_1 = E_1 \cdot F' = E(H, h') \cdot F'$ ($F' \in \mathfrak{E}$).

Let $g \in G(\mathbb{A}^f)$ and let $K \subset G(\mathbb{A}^f)$ be a compact open subgroup. Since $M_{F'}(G, h)$ is weakly canonical, one has an F'_1 -morphism

$$a'(g) : M_{F'_1}(H, h') \longrightarrow M_{F'_1}(G, h) \xrightarrow{g} M_{F'_1}(G, h) \longrightarrow {}_K M_{F'_1}(G, h).$$

Since $F'_1 = E_1 \otimes_E F'$, one may “interpret” this as a morphism of F' -schemes

$$a(g) : M_{E_1}(H, h') \otimes_E F' \longrightarrow {}_K M_{F'}(G, h).$$

The $a(g)$ satisfy $\alpha(F'', F') a(g)_{F''} = a(g)$. One concludes by 5.2 and the following lemma, applied to $X^1 = \coprod_{g \in G(\mathbb{A}^f)} M_{E_1}(H, h')$. Its proof is left to the reader as an exercise.

Lemma 5.10.1. Let F/E be a finite extension of fields and let $\mathfrak{E}$ be a set of subfields of F , whose intersection is reduced to E , such that

$$F \in \mathfrak{E} \text{ and } F' \subset F'' \subset F \implies F'' \in \mathfrak{E}.$$

- (i) The functor that sends a scheme X/E to the system $\mathcal{X}$ of schemes $X_{F'}/F'$ ($F' \in \mathfrak{E}$) and the isomorphisms $(X_{F'}) \otimes_{F'} F'' \xrightarrow{\sim} X_{F''}$ ($F' \subset F''$, $F', F'' \in \mathfrak{E}$) is fully faithful.
- (ii) For a system $\mathcal{X} = \{X(F')/F' \text{ } (F' \in \mathfrak{E}); \alpha(F'', F') : X(F') \otimes_{F'} F'' \xrightarrow{\sim} X(F'') \text{ } (F' \subset F'', F', F'' \in \mathfrak{E})\}$ to come from a scheme X/E , it is enough that the $X(F')/F'$ be quasi-projective and that there exist a scheme X^1/E and a morphism $u : X_{\mathfrak{E}}^1 \longrightarrow \mathcal{X}$ such that the $u(F') : X_{F'}^1 \longrightarrow X(F')$ have dense image.

Remark 5.10.2. Under the hypotheses of 5.10, for $u : (H, h') \hookrightarrow (G, h)$ as in 3.13, the field of definition of $u : M_{\mathbb{C}}(H, h') \longrightarrow M_{\mathbb{C}}(G, h)$ is contained in the extension $\bigcap_i E(H, h') \cdot E_i$ of $E(H, h') \cdot E$.

Proposition 5.11. Let $h : \mathbb{S} \longrightarrow G_{\mathbb{R}}$ be as in 3.13 and let $\delta : \mathbb{S} \longrightarrow C_{\mathbb{R}}^\circ$ (C the center of G). If $E(G, h), E(C, \delta) \subset E \subset \mathbb{C}$ and if $M_{\mathbb{C}}(G, h)$ admits a weakly canonical model over E , then $M_{\mathbb{C}}(G, h, \delta)$ does as well.

Let K be a compact open subgroup of $G(\mathbb{A}^f)$ and let $L = K \cap C^\circ(\mathbb{A}^f)$. Over $\mathbb{C}$, the morphism

$$u : (G \times C^\circ, (h, \delta)) \longrightarrow (G, h\delta) : (g, c) \longmapsto gc$$

induces morphisms

$$u_K : {}_K M_{\mathbb{C}}(G, h) \times {}_L M_{\mathbb{C}}(C^\circ, \delta) \longrightarrow {}_K M_{\mathbb{C}}(G, h\delta).$$

Let d be the morphism $d : C^\circ \longrightarrow G \times C^\circ : c \longmapsto (c, c^{-1})$. The subgroup $d(C^\circ(\mathbb{A}^f))$ of $G \times C^\circ(\mathbb{A}^f)$ acts on $M_{\mathbb{C}}(G, h) \times M_{\mathbb{C}}(C^\circ, \delta)$ through the quotient $\pi_0(C^\circ(\mathbb{A}))$ of $C^\circ(\mathbb{A}^f)$, and u_K induces an isomorphism

$$\bar{u}_K : ({}_K M_{\mathbb{C}}(G, h) \times {}_L M_{\mathbb{C}}(C^\circ, \delta)) / (L \backslash \pi_0(C^\circ(\mathbb{A}))) \xrightarrow{\sim} {}_K M_{\mathbb{C}}(G, h\delta).$$

Define ${}_K M_E(G, h\delta)$ to be the quotient

$${}_K M_E(G, h) \times {}_L M_E(C^\circ, \delta) / (L \backslash \pi_0(C^\circ(\mathbb{A})))$$

and one verifies that this gives a weakly canonical model. By construction, one has

$$(5.11.1) \quad M_E(G, h) \times_E M_E(C^\circ, \delta) \longrightarrow M_E(G, h\delta).$$

Conjecture 5.12. For every (G, h_0) as in 3.13, there exists a canonical model of $M_{\mathbb{C}}(G, h_0)$.

Suppose—this is a typical case—that G is $\mathbb{Q}$ -simple and adjoint. Let $\tilde{G}$ be the universal covering of G and let ω be a weight of $\tilde{G}_{\mathbb{C}}$. If, with the notation of 3.7, the conjugacy class of $h_0 r$ is described by integers $(n_i)_{i \in |\Delta|}$, and if $\omega = \sum m_i \alpha_i$, put $\langle \omega, h_0 r \rangle = \sum n_i m_i$. So far, it has only been possible to attack 5.12 when $\tilde{G}$ admits a nontrivial representation V such that all irreducible components V' of $V_{\mathbb{C}}$ satisfy:

(*) as ω runs through the weights of V' , $\langle \omega, h_0 r \rangle$ takes at most two values.

Let σ be the opposition involution. Condition (*) also means that the dominant weight ω of V' satisfies $\langle \omega, h_0 r \rangle + \langle \sigma \omega, h_0 r \rangle = 0$ or 1. There is no such representation V if G is exceptional (the noncompact factors of $G_{\mathbb{R}}$ are then of type $E_{6(-14)}$ or $E_{7(-25)}$), or of triality type D_4 , nor if G is of type D_n and $G_{\mathbb{R}}$ has noncompact factors of both types $D_n^{\mathbb{H}}$ and $D_n^{\mathbb{R}, 2}$.

§ 6. Strange models.

6.1. Let F be a totally real number field, and let

$$(6.1.1) \quad {}_F \Sigma : 0 \longrightarrow {}_F G' \longrightarrow {}_F G \longrightarrow \mathbb{G}_m \longrightarrow 0$$

be a form over F of the exact sequence

$$(6.1.2) \quad {}_F \Sigma_0 : 0 \longrightarrow \mathrm{Sp}_{2n} \longrightarrow \mathrm{Gp}_{2n} \xrightarrow{\mu} \mathbb{G}_m \longrightarrow 0,$$

where Gp denotes a group of symplectic similitudes.

Assume that, for every real place τ of F , $\Sigma \otimes_{F, \tau} \mathbb{R}$ is of one of the following types:

- (a) ${}_F \Sigma \otimes_{F, \tau} \mathbb{R}$ is isomorphic to ${}_F \Sigma_0 \otimes_{F, \tau} \mathbb{R}$, or
- (b) $G' \otimes_{F, \tau} \mathbb{R}$ is compact.

Let $\tau_1, \dots, \tau_g$ denote the real places of F , and suppose that the places of type (a) are $\tau_1, \dots, \tau_r$, with $r > 0$.

6.2. The group ${}_F G$ may be described as follows:

- (a) Choose a quaternion algebra B over F , indefinite at the places τ_i for $i \leq r$, and definite for $i > r$. Let $x \mapsto \bar{x}$ denote its canonical involution.
- (b) Choose a free B -module V , endowed with a nondegenerate F -bilinear form Φ , symmetric and such that

$$\Phi(bx, y) = \Phi(x, \bar{b}y).$$

- (c) Assume that, for $i > r$, the form deduced from Φ by extension of scalars through $\tau_i : F \rightarrow \mathbb{R}$ is positive definite.
- (d) Take ${}_F G$ to be the group of B -linear similitudes of Φ ; in other words, the elements $g \in G(\mathbb{Q})$ are the $g \in \mathrm{GL}_B(V)$ for which there exists $\mu(g) \in F^*$ satisfying

$$\Phi(gx, gy) = \mu(g)\Phi(x, y).$$

6.3. Let

$$\Sigma : 0 \longrightarrow G' \longrightarrow G \longrightarrow F^* \longrightarrow 0$$

denote the exact sequence of algebraic groups over $\mathbb{Q}$ deduced from ${}_F \Sigma$ by Weil restriction of scalars from F to $\mathbb{Q}$. We have

$$(6.3.1) \quad G_{\mathbb{R}} = \prod_{\tau} {}_F G \otimes_{F, \tau} \mathbb{R} \quad (\tau \text{ a real place}).$$

A morphism $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$ is defined by its coordinates $h_i : \mathbb{S} \rightarrow {}_F G \otimes_{F, \tau_i} \mathbb{R}$. By 1.6, there exists a unique conjugacy class of morphisms h such that, for $i \leq r$, h_i is of type 1.6, and for $i > r$, h_i is trivial. Let h_0 be in this class.

The field $E(G, h_0)$ is the totally real subfield of $\mathbb{C}$ generated by the elements $\sum_{i=1}^r \tau_i(f)$ for $f \in F$.

Theorem 6.4. With the preceding notation, $M_{\mathbb{C}}(G, h_0)$ admits a canonical model $M(G, h_0)$.

Let Z be a totally imaginary quadratic extension of F . Let $V' = V \otimes_F Z$, $L = B \otimes_F Z$, and let G' be the algebraic subgroup of $\mathrm{GL}_L(V')$ generated by G and Z^* :

$$(6.4.1) \quad 0 \longrightarrow F^* \xrightarrow{(f, f^{-1})} G \times Z^* \xrightarrow{q} G' \longrightarrow 0.$$

Let $x \mapsto \tilde{x}$ be the involution of L given by the tensor product of the involutions of B and Z . Up to a factor in F , there exists on Z a unique alternating F -bilinear form ψ_Z . If ${}_F \psi$ is the alternating F -bilinear form on V' which is the tensor product of Φ and ψ_Z , then

$${}_F \psi(\ell x, y) = {}_F \psi(x, \tilde{\ell} y).$$

Put $\psi(x, y) = \mathrm{Tr}_{F/\mathbb{Q}}({}_F \psi(x, y))$.

Choose, for every real place of F , a complex embedding of Z inducing it. Then $Z^*(\mathbb{R}) \simeq \prod_{i=1}^g \mathbb{C}^*$. Let h_Z be the homomorphism from $\mathbb{S}$ into $Z_{\mathbb{R}}^*$ with coordinates $h_i : \mathbb{S}(\mathbb{R}) = \mathbb{C}^* \rightarrow \mathbb{C}^*$ ($1 \leq i \leq g$) equal to $(z \mapsto 1)$ for $i \leq r$, and to $(z \mapsto z^{-1})$ for $i > r$.

For every $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$ as in 6.3, put $h' = q \circ (h \times h_Z) : \mathbb{S} \rightarrow G'_{\mathbb{R}}$.

Lemma 6.4.2. There exists $b \in L$ such that, for a suitable choice of ψ_Z , the form

$$\psi(x, b h'_0(i) y) \quad \text{on} \quad V' \otimes_{\mathbb{Q}} \mathbb{R}$$

is symmetric and positive definite.

This is verified after extension of scalars from $\mathbb{Q}$ to $\mathbb{R}$.

This lemma and 5.9 make it possible to construct a canonical model of $M(G', h'_0)$ (since G' is contained in the group of symplectic similitudes of ${}_F \psi(x, by)$). Applying 5.11 and 5.7, one obtains a weakly canonical model of $M_{\mathbb{C}}(G, h_0)$ over the extension $E(Z^*, h_Z)$ of $E(G, h_0)$. The conclusion follows from 5.10, 5.10.2 and the following lemma, which can be deduced from 5.13.

Lemma 6.5. Let F be a totally real number field, endowed with a set S of real embeddings. Let F^* be the number field corresponding to the subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which stabilizes S . For Z a totally imaginary quadratic extension of F , endowed with a set T of complex embeddings, one above each element of S , let likewise $Z^* \supset F^*$ be the field of invariants of the stabilizer of T . Then, for every extension E of F^* , there exist Z and T such that Z^* is linearly disjoint from E .

Remark 6.6. The canonical map 1.13, 1.14

$$(6.6.1) \quad M_{\mathbb{C}}(G, h_0) \times M_{\mathbb{C}}(Z^*, h_Z) \longrightarrow M_{\mathbb{C}}(G', h'_0)$$

is interpreted as follows. Each conjugate of h defines on $V_{\mathbb{C}}$ a Hodge bigrading invariant under F :

$$V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1} \oplus V^{0,0} \quad (\overline{V^{pq}} = V^{qp}).$$

One should beware that the grading (by weight) of $V_{\mathbb{C}}$ by the $V^n = \sum_{p+q=n} V^{pq}$ is not defined over $\mathbb{Q}$ if $r \neq g$.

Similarly, $Z_{\mathbb{C}}$ is bigraded:

$$Z_{\mathbb{C}} = Z^{-1,0} \oplus Z^{0,-1} \oplus Z^{0,0}.$$

The miracle is that the $\mathbb{Q}$ -vector space $V' = V \otimes_F Z$, endowed with the tensor-product Hodge bigrading $V'_{\mathbb{C}} = V'^{-1,0}_{\mathbb{C}} \oplus V'^{0,-1}_{\mathbb{C}}$ (!), is the H_1 of an abelian variety.

BIBLIOGRAPHY

- [1] F. BRUHAT and J. TITS - Groupe algébriques simples sur un corps local, Proc. on a conf. on local fields, p. 23-26, Driebergen 1966, Springer-Verlag.
- [2] P. DELIGNE - Travaux de Griffiths, Sémin. Bourbaki, 1969/70, exposé n° 376, Lecture Notes 180, Springer-Verlag.
- [3] G. HARDER - Über die Galoiskohomologie halbeinfacher Matrizengruppen II, Math. Zeitschrift, 92 (1966), p. 396-415.
- [4] M. KNESER - Schwache Approximation in algebraischen Gruppen, Coll. sur la théorie des groupes algébriques, p. 41-52, Bruxelles 1962, CBRM.
- [5] K. MIYAKE - On models of certain automorphic function fields.
- [6] D. MUMFORD - Geometric invariant theory, Ergebnisse 34, 1965, Springer-Verlag.
- [7] D. MUMFORD - Families of abelian varieties, in Alg. groups and discontinuous subgroups, p. 347-351, Boulder 1965, AMS.
- [8] D. MUMFORD - A Note on Shimura's Paper "Discontinuous groups and Abelian Varieties". Math. Ann. 181 345-351 (1969)
- [9] V.P. PLATONOV - Le problème d'approximation forte et l'hypothèse de Kneser-Tits, Izvestia Acad. Nauk CCCP, 33 (1969), p. 1211-1219 and 34 (1970), p. 775-777.
- [10] G. SHIMURA - On analytic families of polarized abelian varieties and automorphic functions, Ann. of Math., 78 1 (1963), p. 149-192.
- [11] G. SHIMURA - On the field of definition for a field of automorphic functions I, II, III, Ann. of Math., 80 1 (1964), p. 160-189 ; 81 1 (1965), p. 124-165 and 83 2 (1966), p. 377-385.
- [12] G. SHIMURA - Construction of class fields and zeta functions of algebraic curves, Ann. of Math. 85 1 (1967), p. 58-159.
- [13] G. SHIMURA - Algebraic number fields and symplectic discontinuous groups, Ann. of Math., 86 3 (1967), p. 503-592.
- [14] G. SHIMURA - On canonical models of arithmetic quotients of bounded symmetric domains, Ann. of Math., 91 1 (1970), p. 144-222.
- [15] G. SHIMURA - Automorphic functions and number theory, Lecture Notes in Math. 54, 1968, Springer-Verlag.
- [16] G. SHIMURA and T. TANIYAMA - Complex multiplication of abelian varieties and its applications to number theory, Publ. Math. Soc. Jap., 6 (1961).