

LEFSCHETZ THEOREM AND DEGENERATION CRITERIA FOR SPECTRAL SEQUENCES

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1. General theorems

Let $\mathcal{A}$ be an abelian category and let $D^b(\mathcal{A})$ be the full subcategory of its derived category formed by the bounded complexes. Recall that if T is a cohomological functor (see [8] or [9], p. 10) from $D^b(\mathcal{A})$ to an abelian category $\mathcal{B}$, Verdier [8] has defined a spectral sequence, functorial in $X \in D^b(\mathcal{A})$:

$$E_2^{pq} = T(H^q(X)[p]) \implies T(X[p+q]). \quad (1.1)$$

Proposition (1.2). Let $X \in D^b(\mathcal{A})$. The following conditions are equivalent:

- (i) for every T as above, the spectral sequence (1.1) degenerates;
 - (ii) the object X is isomorphic, in $D^b(\mathcal{A})$, to $\sum_i H^i(X)[-i]$.
- (i) $\Rightarrow$ (ii). Denote by T_i the cohomological functor in K defined by

$$T_i(K) = \text{Hom}_{D^b(\mathcal{A})}(H^i(X), K).$$

For each of the T_i ($i \in \mathbb{Z}$), the spectral sequence (1.1) is written

$$E_2^{pq} = \text{Ext}^p(H^i(X), H^q(K)) \implies \text{Hom}_{D^b(\mathcal{A})}(H^i(X), K[p+q]), \quad (1.3)$$

and the homomorphism from $\text{Hom}(H^i(X), K[i])$ to

$$E_2^{0i} = \text{Hom}(H^i(X), H^i(K))$$

which follows from it is none other than the one coming from the functoriality of H^0 .

If the spectral sequences (1.3) degenerate, the maps

$$\text{Hom}(H^i(X)[-i], K) \longrightarrow \text{Hom}(H^i(X), H^i(K))$$

are surjective. By hypothesis, this is so for $K = X$; hence there exist morphisms a_i from $H^i(X)[-i]$ to X , inducing the identity on the i -th cohomology object. Moreover, since $H^i(X) = 0$ except for finitely many values of i , the a_i vanish except for finitely many i , which allows one to define

$$a = \sum_i a_i : \sum_i H^i(X)[-i] \longrightarrow X.$$

By construction, a induces the identity on cohomology, hence is an isomorphism.

(ii) $\Rightarrow$ (i). The spectral sequence (1.1) is an additive functor in X , and degenerates when X has only one nonzero cohomology object; hence it also degenerates if X is a sum of complexes having only one nonzero cohomology object.

Remark (1.4). It follows from the preceding proof that it is enough to verify condition (i) for cohomological functors of the form $\text{Hom}_{D^b(\mathcal{A})}(A, *)$, with $A \in \text{Ob}(\mathcal{A})$. By duality, it would also be enough to consider only the functors $\text{Hom}_{D^b(\mathcal{A})}(*, A)$, since condition (ii) is self-dual.

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Theorem (1.5). Let $X \in D^b(\mathcal{A})$, let n be an integer, and let u be an endomorphism of degree 2 of X ,

$$u : X \longrightarrow X[2],$$

whose iterates induce isomorphisms

$$u^i : H^{n-i}(X) \xrightarrow{\sim} H^{n+i}(X) \quad (i \geq 0).$$

Under these conditions, X is isomorphic to $\sum_i H^i(X)[-i]$ in the category $D^b(\mathcal{A})$.

We verify condition (i) of (1.2). Let T be a cohomological functor from $D^b(\mathcal{A})$ to $\mathcal{B}$, and put $T^p(K) = T(K[p])$.

The primitive part of $H^{n-i}(X)$, denoted ${}_0H^{n-i}(X)$, is the kernel of the homomorphism

$$u^{i+1} : H^{n-i}(X) \longrightarrow H^{n+i+2}(X).$$

One checks that, for $i \geq 0$, the maps

$$\begin{aligned} \bigoplus_{k \geq 0} u^k : \bigoplus_{k \geq 0} {}_0H^{n-i-2k}(X) &\xrightarrow{\sim} H^{n-i}(X), \\ \bigoplus_{k \geq 0} u^{k+i} : \bigoplus_{k \geq 0} {}_0H^{n-i-2k}(X) &\xrightarrow{\sim} H^{n+i}(X) \end{aligned} \quad (1.6)$$

are isomorphisms.

Since the spectral sequence (1.1) is functorial in X and compatible with translations of degrees, the endomorphism u of X defines an endomorphism of degree 2 of the spectral sequence,

$$u : E_r^{pq} \longrightarrow E_r^{p,q+2},$$

commuting with the differentials d_r ($r \geq 2$).

When $r = 2$, the endomorphism u of the spectral sequence

$$u : T^p(H^q(X)) \longrightarrow T^p(H^{q+2}(X))$$

is obtained from the morphism $u : H^q(X) \rightarrow H^{q+2}(X)$ by functoriality of T^p .

Let $i \geq 0$. Denote by ${}_0E_r^{p,n-i}$ the kernel of the homomorphism

$$u^{i+1} : E_r^{p,n-i} \longrightarrow E_r^{p,n+i+2}$$

(the primitive part of the spectral sequence). For $r = 2$, the decompositions (1.6) induce, by functoriality of each T^p , decompositions

$$\begin{aligned} \bigoplus_{k \geq 0} u^k : \bigoplus_{k \geq 0} {}_0E_2^{p,n-i-2k} &\xrightarrow{\sim} E_2^{p,n-i}, \\ \bigoplus_{k \geq 0} u^{k+i} : \bigoplus_{k \geq 0} {}_0E_2^{p,n-i-2k} &\xrightarrow{\sim} E_2^{p,n+i}. \end{aligned} \quad (1.7)$$

We prove by induction on r that the differentials d_r ($r \geq 2$) are zero. The induction hypothesis implies that $E_2 = E_r$, so that the decomposition (1.7) applies to the term E_r of the spectral sequence. Since u commutes with the d_r , it will be enough to prove that d_r vanishes on the primitive part of E_r .

The following diagram is commutative:

$$\begin{array}{ccc} {}_0E_r^{p,n-i} & \xrightarrow{d_r} & E_r^{p+r,n-i-r+1} \\ \downarrow u^{i+1} & & \downarrow u^{i+1} \\ E_r^{p,n+i+2} & \xrightarrow{d_r} & E_r^{p+r,n+i-r+3}. \end{array}$$

The left vertical arrow is zero by definition of primitivity. The right vertical arrow is a monomorphism, because the arrow

$$u^{i+r-1} : E_r^{p+r,n-(i+r-1)} \longrightarrow E_r^{p+r,n+(i+r-1)}$$

is an isomorphism² and $r - 1 \geq 1$. It follows that d_r is zero.

²The printed source gives $E_r^{p,n+(i+r-1)}$ as the target; the index $p+r$, required by the bidegree of d_r , is restored here.

Remark (1.8). One should take care that in general one cannot choose an isomorphism between X and $\sum_i H^i(X)[-i]$ which transforms u into the sum of the maps $H^i(u) : H^i(X) \rightarrow H^{i+2}(X)$. One easily obtains a counterexample by taking $H^i(X) = 0$ for $i \neq n$.

One can define an isomorphism, “more beautiful than the others”, between X and $\sum_i H^i(X)[-i]$, but it will not be used here.

Remark (1.9). Let $X \in D^b(\mathcal{A})$, let n and s be integers, and let u be an endomorphism of degree s (respectively $2s$) of X . Suppose that $H^i(X) = 0$ when i is not of the form ks ($0 \leq k \leq n$) (respectively $0 \leq k \leq 2n$), and that u^{n-2i} (respectively u^i) induces an isomorphism between $H^{is}(X)$ and $H^{(n-i)s}(X)$ (respectively between $H^{(n-i)s}(X)$ and $H^{(n+i)s}(X)$). Under these hypotheses, the proof of (1.5) still shows that X is isomorphic to $\sum_i H^i(X)[-i]$.

Remark (1.10). Let $(X_i)_{i \in \mathbb{Z}}$ be a family of objects of $D^b(\mathcal{A})$ and let u be a family of degree-2 homomorphisms $u_i : X_i \rightarrow X_{i+1}[2]$. If, for all i and j , u_i induces an isomorphism between $H^{n-j}(X_i)$ and $H^{n+j}(X_{i+j})$, it is still true that each X_i is isomorphic to the sum of its cohomology objects. To see this, it is enough to apply the preceding arguments formally to

$$u : \sum_i X_i \longrightarrow \sum_i X_i,$$

even if this sum does not exist in $D^b(\mathcal{A})$.

The same remark applies to the variant (1.9).

Theorem (1.11). Let $X \in D^b(\mathcal{A})$ and suppose that X admits degree-0 endomorphisms $\pi_i : X \rightarrow X$ such that $H^j(\pi_i) = \delta_{ij}$. Then X is isomorphic to $\sum_i H^i(X)[-i]$ in the category $D^b(\mathcal{A})$.

We again apply criterion (1.2) (i). The diagram

$$\begin{array}{ccc} E_r^{p,q} & \xrightarrow{d_r} & E_r^{p+r, q-r+1} \\ \downarrow \pi_q & & \downarrow \pi_q \\ E_r^{p,q} & \xrightarrow{d_r} & E_r^{p+r, q-r+1} \end{array}$$

is commutative; the first column is the identity and the second is zero, so that $d_r = 0$.

Corollary (1.12). Let $X \in D^b(\mathcal{A})$ be the sum of a finite family of objects $X_k \in D^b(\mathcal{A})$. If

$$X \simeq \sum_i H^i(X)[-i],$$

then, for every k ,

$$X_k \simeq \sum_i H^i(X_k)[-i].$$

Let j_k be the canonical injection of X_k into X , let pr_k be the canonical projection of X onto X_k , and let π_i be an endomorphism of X such that $H^j(\pi_i) = \delta_{ij}$. For every k , the $\text{pr}_k \pi_i j_k$ satisfy the hypothesis of (1.11).

Remark (1.13). One checks that, in order that the isomorphism between X and $\sum_i H^i(X)[-i]$ can be chosen so that the π_i are identified with the pr_i , it is necessary and sufficient that the π_i be pairwise orthogonal idempotents. There is then a unique isomorphism with this property and inducing the identity on cohomology.

2. Applications

Let $f : X \rightarrow Y$ be a continuous map between topological spaces, or even a morphism of topoi, let A be a sheaf of rings on X , and let $\mathcal{F}$ be a sheaf of A -modules. If $u \in H^2(X, A)$, then u defines a degree-2 endomorphism in $D^b(X)$, again denoted

$$u : \mathcal{F} \longrightarrow \mathcal{F}[2].$$

By functoriality, u defines a degree-2 endomorphism

$$Rf_*(u) : Rf_*\mathcal{F} \longrightarrow Rf_*\mathcal{F}[2].$$

We shall say that $(X, f, u, \mathcal{F}, n)$ satisfies the Lefschetz condition, or that $\mathcal{F}$ satisfies the Lefschetz condition relative to u , if, for every $i \geq 0$, the maps obtained by iterating i times the arrow $Rf_*(u)$,

$$Rf_*(u)^i : R^{n-i}f_*\mathcal{F} \longrightarrow R^{n+i}f_*\mathcal{F} \quad (i \geq 0),$$

are isomorphisms. Theorem (1.5) gives here:

Proposition (2.1). If $\mathcal{F}$ satisfies the Lefschetz condition relative to u , then, in $D^b(Y)$,

$$Rf_*\mathcal{F} \simeq \sum_i R^if_*\mathcal{F}[-i]$$

and in particular the Leray spectral sequence

$$H^p(Y, R^qf_*\mathcal{F}) \implies H^{p+q}(X, \mathcal{F})$$

degenerates.

If one had wished, one could have supposed that $\mathcal{F}$ was an (A, f^*B) -bimodule, where B is a sheaf of rings on Y , and stated (2.1) in $D^b(Y, B)$.

(2.2) The case of schemes, endowed with the étale topology, does not enter into the framework of (2.1). In what follows we shall suppose that the theory of $\mathbb{Z}_\ell$ -sheaves and $\mathbb{Q}_\ell$ -sheaves (not necessarily constructible) has been made, and that the derived category of these “sheaves” is reasonable. As far as the degeneration statements for spectral sequences are concerned, the reader who has no confidence in Jouanolou’s future work may verify, by returning to the proof of (1.5), that one can dispense with it. Let ℓ be a prime number. Throughout what follows, assume that ℓ is invertible on all schemes under consideration.

Recall that on every scheme X the $\mathbb{Z}_\ell$ -sheaf $\mathbb{Z}_\ell(1)$ is defined by the formula

$$\mathbb{Z}_\ell(1) = \varprojlim_n \mu_{\ell^n}.$$

This sheaf is an invertible $\mathbb{Z}_\ell$ -module, which permits one to define, for every $\mathbb{Z}_\ell$ -sheaf or $\mathbb{Q}_\ell$ -sheaf $\mathcal{F}$, the “sheaf” $\mathcal{F}(i)$ by the formula

$$\mathcal{F}(i) = \mathcal{F} \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(1)^{\otimes i} \quad (i \in \mathbb{Z}).$$

For every morphism $f : X \rightarrow Y$ of schemes and every pair of $\mathbb{Z}_\ell$ - or $\mathbb{Q}_\ell$ -sheaves $\mathcal{F}$ and $\mathcal{G}$ on X and Y respectively, one has

$$Rf_*(\mathcal{F})(i) = Rf_*(\mathcal{F}(i)) \quad \text{and} \quad f^*(\mathcal{G}(i)) = f^*(\mathcal{G})(i). \quad (2.3)$$

If $\mathcal{L}$ is an invertible sheaf on X , its first ℓ -adic Chern class lies in $H^2(X, \mathbb{Z}_\ell(1))$.

If $f : X \rightarrow Y$ is a morphism of schemes, if $\mathcal{F}$ is a $\mathbb{Z}_\ell$ -sheaf, respectively a $\mathbb{Q}_\ell$ -sheaf, on X , and if $u \in H^2(X, \mathbb{Z}_\ell(1))$, respectively $u \in H^2(X, \mathbb{Q}_\ell(1))$, then $\mathcal{F}$ will be said to satisfy the Lefschetz condition relative to u if, for every $i \geq 0$, the maps obtained by iterating i times the arrow $Rf_*(u)$,

$$Rf_*(u)^i : R^{n-i}f_*\mathcal{F} \longrightarrow R^{n+i}f_*\mathcal{F}(i) \quad (i \geq 0),$$

are isomorphisms. By (2.3), this implies that all the arrows

$$Rf_*(u)^i : R^{n-i}f_*\mathcal{F}(k) \longrightarrow R^{n+i}f_*\mathcal{F}(k+i)$$

are isomorphisms. Applying (1.10), one obtains:

Proposition (2.4). With the preceding notation, if $\mathcal{F}$ satisfies the Lefschetz condition relative to u , then, in $D^b(Y, \mathbb{Z}_\ell)$, respectively $D^b(Y, \mathbb{Q}_\ell)$, one has

$$Rf_*\mathcal{F} \simeq \sum_i R^i f_*\mathcal{F}[-i] \quad (2.5)$$

and in particular the Leray spectral sequence

$$H^p(Y, R^q f_*\mathcal{F}) \implies H^{p+q}(X, \mathcal{F})$$

degenerates.

If g is a morphism of schemes from Y to S , (2.4) also implies the degeneration of the Leray spectral sequence

$$R^p g_* R^q f_*\mathcal{F} \implies R^{p+q}(gf)_*\mathcal{F}.$$

(2.6.1) It remains to review a few interesting cases in which the Lefschetz condition is satisfied; the justifying details are given in (2.7). Let $f : X \rightarrow Y$, u , $\mathcal{F}$, and n be given. In the topological case, it is known from Godement [2], II, (4.7.1), that if f is proper and Y locally paracompact, the formation of higher direct images commutes with passage to fibers, so that the Lefschetz condition is checked fiber by fiber. In the case of schemes, the same reduction applies, the reference to Godement being replaced by SGA 4, XII, (5.1). Still in the case of schemes, if f is proper and smooth and $\mathcal{F}$ is twisted constant, that is, smooth in a newer terminology, then the $R^i f_*\mathcal{F}$ are still twisted constant (cf. SGA 4, XVI, (2.2)); hence, if Y is connected, it is even enough to verify the Lefschetz condition on one geometric fiber.

In all the examples given, f will be a proper smooth continuous map, respectively a proper smooth morphism of schemes, of relative topological dimension $2n$, respectively algebraic dimension n . A continuous map is called smooth if locally on the source it is isomorphic to the projection from $\mathbb{R}^n \times Y$ to Y .

(2.6.2) If f is a proper smooth morphism of Kähler varieties, the constant sheaf $\mathbb{C}$ satisfies the Lefschetz condition relative to the fundamental 2-class of X .

(2.6.3) If X is a complex relative scheme (see [3]), projective and smooth over the locally paracompact topological space Y , and endowed with a relatively ample invertible sheaf $\mathcal{O}(1)$, the constant sheaf $\mathbb{C}$ satisfies the Lefschetz condition relative to the first Chern class of $\mathcal{O}(1)$.

(2.6.4) Suppose that f is a projective smooth morphism of schemes, with Y connected, and that a relatively ample invertible sheaf $\mathcal{O}(1)$ on X is given. If one can lift to characteristic 0 a geometric fiber of f , and its polarization, then the constant $\mathbb{Q}_\ell$ -adic sheaf $\mathbb{Q}_\ell$ satisfies the Lefschetz condition relative to the ℓ -adic first Chern class of $\mathcal{O}(1)$.

It is known that one can lift to characteristic 0 in the preceding sense in the following cases:

- X is a Severi–Brauer variety over Y ;
- X is a nonsingular complete intersection of hypersurfaces in a Severi–Brauer variety over Y , for the polarization induced by the canonical polarization of the Severi–Brauer variety;
- X is a polarized abelian scheme over Y (Mumford, to appear).

It is conjectured that (2.6.4) remains true when no lifting to characteristic 0 is known; this is proved only for surfaces.

(2.6.5) If f is a projective smooth morphism of relative dimension 2 and if X is endowed with a relatively ample invertible sheaf $\mathcal{O}(1)$, then the constant sheaf $\mathbb{Q}_\ell$ satisfies the Lefschetz condition relative to the first Chern class of $\mathcal{O}(1)$.

(2.7) In cases (2.6.2) and (2.6.3), the given cohomology 2-class on X induces on the fibers of f the 2-class defined by the Kähler structure of the fibers. Thus, by (2.6.1), to prove (2.6.2) and (2.6.3) it suffices to prove (2.6.2) in the special case where Y is a point. The assertion then follows from Hodge theory and the Hodge–Lepage decomposition (Weil [10], IV, no. 6, corollary to theorem 5). The proof given in loc. cit. still applies if $\mathcal{F}$ is

a locally constant sheaf of finite-dimensional $\mathbb{C}$ -vector spaces, endowed with a locally constant positive definite Hermitian structure.

By (2.6.1), to verify (2.6.4) it suffices to treat the case where Y is the spectrum of a field of characteristic 0, and the Lefschetz principle allows one to suppose $Y = \text{Spec}(\mathbb{C})$.

It is then known (cf. SGA 4, XI, (4.4)) that, for every twisted constant $\mathbb{Q}_\ell$ -adic sheaf $\mathcal{F}$ on X , if $\mathcal{F}^{an}$ denotes the locally constant sheaf of $\mathbb{Q}_\ell$ -vector spaces on the topological space X^{an} deduced from $\mathcal{F}$, then the algebraic and transcendental cohomology $\mathbb{Q}_\ell$ -vector spaces of $\mathcal{F}$ and $\mathcal{F}^{an}$ are isomorphic:

$$H^i(X, \mathcal{F}) \xrightarrow{\sim} H^i(X^{an}, \mathcal{F}^{an}). \quad (2.7.2)$$

Choosing an embedding of $\mathbb{Q}_\ell$ into $\mathbb{C}$, one obtains isomorphisms

$$H^i(X, \mathcal{F}) \otimes_{\mathbb{Q}_\ell} \mathbb{C} \xrightarrow{\sim} H^i(X^{an}, \mathcal{F}^{an}) \otimes_{\mathbb{Q}_\ell} \mathbb{C}. \quad (2.7.3)$$

Over $\mathbb{C}$, the exponential map defines the so-called canonical isomorphism between $\mathbb{Z}/\ell^n$ and μ_{ℓ^n} , hence between $\mathbb{Z}_\ell$ and $\mathbb{Z}_\ell(1)$. The image, under the composite isomorphism

$$H^2(X, \mathbb{Z}_\ell(1)) \longrightarrow H^2(X, \mathbb{Z}_\ell) \longrightarrow H^2(X^{an}, \mathbb{Z}) \otimes \mathbb{Z}_\ell,$$

of the ℓ -adic first Chern class of $\mathcal{O}(1)$ coincides with the transcendental first Chern class of $\mathcal{O}(1)$. The isomorphism (2.7.3) therefore reduces the case of (2.6.4) to which we had come to (2.6.3).

For the proof of (2.6.5), see Kleiman [6], pp. 28 ff. The essential point is the following theorem (Weil [11], p. 127).

(2.8) Let S be a smooth projective surface over a field k , and let D be a smooth curve on S such that $\mathcal{O}(D)$ is ample. Then the composite map

$$\text{Pic}^0(S) \longrightarrow \text{Pic}^0(D) = \text{Alb}(D) \longrightarrow \text{Alb}(S)$$

is an isogeny.³

Remark (2.9). The degeneration theorem for the spectral sequence (2.2) deduced from (2.6.4) was conjectured by Grothendieck, from considerations of “weights”, when Y is projective and smooth over an algebraically closed field.

Remark (2.10). Serre pointed out to me that the degeneration theorems for spectral sequences deduced from (2.6.2) and (2.6.3) can also be proved by an easy extension of the method used by Blanchard ([1], II, 1), at least if Poincaré duality is available on the base. Conversely, the method followed here completes Theorems II (1.1) and II (1.2) of loc. cit.; the proof of the implication (ii) $\Rightarrow$ (i) is essentially Blanchard’s, who had to suppose the higher direct-image sheaves constant.

Theorem (2.11). Let $f : X \rightarrow Y$ be a proper smooth continuous map (2.6.1) of topological spaces of relative dimension $2n$, with Y locally paracompact, let k be a commutative field of characteristic zero, and let $u \in H^0(Y, R^2 f_* k)$. Suppose that:

- a) the sheaf $R^1 f_* k$ is constant, i.e. simple;
- b) the iterated cup products

$$u^i \wedge : R^{n-i} f_* k \longrightarrow R^{n+i} f_* k$$

are isomorphisms.

Then the following conditions are equivalent:

- (i) the class u is induced by an element of $H^2(X, k)$;
- (ii) the transgression

$$d_2 : H^0(Y, R^2 f_* k) \longrightarrow H^2(Y, f_* k)$$

is zero on u ;

³The printed source gives $\text{Pic}^0(X)$ and $\text{Alb}(X)$ here, although the surface introduced in the statement is S ; S is restored.

(iii) in $D^b(Y, k)$, one has

$$Rf_*k \simeq \bigoplus_i R^i f_*k[-i].$$

The implication (i) $\Rightarrow$ (iii) is contained in (2.1). Assertion (iii) implies the degeneration of the whole Leray spectral sequence

$$H^p(Y, R^q f_*k) \Longrightarrow H^{p+q}(X, k), \quad (2.12)$$

and a fortiori (ii). It remains to prove that (ii) implies (i).

Let $x \in Y$ and $X_x = f^{-1}(x)$. The induced class $u_x^n \in H^{2n}(X_x, k)$ is nonzero on every connected component of X_x , which is therefore orientable.⁴ Let Tr_u denote the composite map

$$H^{2n}(X_x, k) \xrightarrow{(u_x^n \wedge)^{-1}} H^0(X_x, k) = H_0(X_x, k) \xrightarrow{\text{Tr}} k.$$

Poincaré duality implies that the bilinear form

$$H^{n-i}(X_x, k) \times H^{n+i}(X_x, k) \longrightarrow k, \quad (x, y) \longmapsto \text{Tr}_u(xy)$$

is a perfect duality. This construction globalizes and gives

$$\text{Tr}_u : R^{2n} f_*k \longrightarrow k.$$

First prove that in (2.12), $d_2 u \in H^2(Y, R^1 f_*k)$ is zero. If $x \in H^0(Y, R^1 f_*k)$, then for degree reasons $xu^n = 0$, hence

$$d_2(xu^n) = d_2 x u^n - n x u^{n-1} d_2 u = 0.$$

By b), the map

$$u^{n-1} \wedge : H^0(Y, R^1 f_*k) \longrightarrow H^0(Y, R^{2n-1} f_*k)$$

is an isomorphism. Thus, if (ii) holds, for every $y \in H^0(Y, R^{2n-1} f_*k)$ one has

$$y d_2 u = 0,$$

and in particular

$$\text{Tr}_u(y d_2 u) = 0. \quad (2.13)$$

Let V be a k -vector space such that $R^1 f_*k$ is isomorphic to the constant sheaf V . Then $d_2 u \in H^2(Y, V)$ and, by (2.13), for every linear form w on V , the image of $d_2 u$ under w in $H^2(Y, k)$ is zero. Thus $d_2 u = 0$.

If $d_2 u = 0$, the proof of (1.5) shows that all differentials d_2 of (2.12) are zero, so that $E_2 = E_3$. For degree reasons, $u^{n+1} = 0$, hence

$$d_3 u^{n+1} = (n+1)u^n d_3 u = 0.$$

Since $d_3 u \in H^3(Y, R^0 f_*k)$, b) implies that $d_3 u = 0$. For degree reasons $d_r u = 0$ for $r > 3$, and u therefore comes from an element of $H^2(X, k)$.

(2.14) In the scheme-theoretic setting, the preceding result remains valid, but it is usable only in the “geometric” case, more precisely when $\mathbb{Q}_\ell$ is isomorphic to $\mathbb{Q}_\ell(1)$ on Y .

(2.15) Corollary (1.12) allows some of the preceding statements to be extended to certain nonconstant sheaves.

Proposition (2.16). Let g and f be two composable continuous maps, respectively two morphisms of schemes,

$$X \xrightarrow{g} Y \xrightarrow{f} Z,$$

and let $\mathcal{F}$ be a sheaf, respectively a $\mathbb{Z}_\ell$ -sheaf or $\mathbb{Q}_\ell$ -sheaf, on X . Suppose that

- a) $Rg_*\mathcal{F} \in D^b(Y)$ and $Rg_*\mathcal{F} \simeq \bigoplus_i R^i g_*\mathcal{F}[-i]$;
- b) $R(fg)_*\mathcal{F} \in D^b(Z)$ and $R(fg)_*\mathcal{F} \simeq \bigoplus_i R^i (fg)_*\mathcal{F}[-i]$.

⁴After introducing x , the printed source switches to X_s and u_s^n ; the subscript is regularized to x here.

Then, for every k , one has

$$Rf_*(R^k g_* \mathcal{F}) \simeq \bigoplus_i R^i f_*(R^k g_* \mathcal{F})[-i].$$

By b), the complex

$$R(fg)_* \mathcal{F} = Rf_* Rg_* \mathcal{F} \simeq \bigoplus_k Rf_*(R^k g_* \mathcal{F}[-k])$$

is a sum of its cohomology objects; by (1.12), the complexes $Rf_* R^k g_* \mathcal{F}$, which up to a shift are direct factors thanks to a), have the same property.

Proposition (2.17). Let $\mathcal{E}$ be a locally free vector bundle on a scheme S , let $\mathcal{F}$ be a twisted constant ℓ -adic sheaf on $\mathbb{P}(\mathcal{E})$, and let f be the projection from $\mathbb{P}(\mathcal{E})$ to S . In the derived category, one has

$$Rf_* \mathcal{F} \simeq \bigoplus_i R^i f_* \mathcal{F}[-i].$$

Projective space is simply connected and has torsion-free cohomology (SGA 1, XI, 1.1 and SGA 4, XVI, 2.2). Hence

$$\mathcal{F} = f^* f_* \mathcal{F}, \quad Rf_* \mathcal{F} = Rf_* \mathbb{Z}_\ell \overset{\mathrm{L}}{\otimes} f_* \mathcal{F},$$

which reduces the proof to the case $\mathcal{F} = \mathbb{Z}_\ell$. Let u be the Chern class of the invertible sheaf $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{E})$. Then $\mathbb{Z}_\ell$ is known to satisfy the Lefschetz condition relative to u , and (2.2) concludes the proof.

(2.18) From (2.17), with the help of (2.16), one obtains the same result for flag bundles of all kinds defined by $\mathcal{E}$. In the topological setting, the preceding applies directly to vector bundles over $\mathbb{C}$, and extends to vector bundles over $\mathbb{H}$ by (1.9). For real vector bundles, one must restrict to the case where $\mathcal{F}$ comes from a sheaf of 2-torsion on the base.

Remark (2.19). Let X be an abelian scheme of relative dimension g over a base Y . For every integer n , multiplication by n , denoted $[n]$, defines an endomorphism

$$[n]^* : Rf_* \mathbb{Q}_\ell \longrightarrow Rf_* \mathbb{Q}_\ell.$$

Taking rational linear combinations, with denominators bounded in terms of g , of these endomorphisms, one constructs endomorphisms

$$\pi_i : Rf_* \mathbb{Q}_\ell \longrightarrow Rf_* \mathbb{Q}_\ell$$

such that $H^j(\pi_i) = \delta_{ij}$. Applying (1.11), one concludes, without any projectivity hypothesis, that

$$Rf_* \mathbb{Q}_\ell \simeq \bigoplus_i R^i f_* \mathbb{Q}_\ell[-i].$$

The idea used here is due to Lieberman.

3. The infinitesimal form of the semicontinuity theorem

I recall in this paragraph some results which one manages to find in EGA III, § 7. The formulation (3.4) was pointed out to me by L. Illusie.

(3.0) Throughout this paragraph, A denotes a fixed noetherian local ring, $\mathfrak{m}$ its maximal ideal, and k its residue field. Let $D_{\mathrm{coh}}^-(A)$ denote the subcategory of the derived category of A -modules formed by bounded-above complexes of finite-type A -modules.

Proposition (3.1). Let M be a finite-type module over A .

- (i) For every homomorphism, not necessarily local, from A to an Artinian ring B , and every finite-type B -module N , one has

$$\lg_B(M \otimes_A N) \leq \lg_B(N) \cdot \dim_k(M \otimes_A k). \quad (3.1.1)$$

- (ii) The following conditions on M are equivalent:

- a) M is a free module;
- b) for all B and N as in (i), one has

$$\lg_B(M \otimes_A N) = \lg_B(N) \cdot \dim_k(M \otimes_A k); \quad (3.1.2)$$

- c) for every $n \in \mathbb{N}$, condition (3.1.2) is satisfied for $B = N = A/\mathfrak{m}^n$;
- d) for every $\mathfrak{p} \in \text{Ass}(A)$, $M_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$, and condition (3.1.2) is satisfied for $B = N = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$.

If A has no embedded components, these conditions are also equivalent to:

- e) for every maximal point $\mathfrak{p}$ of $\text{Spec}(A)$ one has

$$\lg_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \lg(A_{\mathfrak{p}}) \cdot \dim_k(M \otimes_A k). \quad (3.1.2')$$

The proof, an application of Nakayama's lemma, is left to the reader.

(3.2) For every complex K and every integer n , let $\tau_{\geq n}(K)$ denote the complex defined by

$$\tau_{\geq n}(K)^i = \begin{cases} 0, & i < n, \\ \text{coker}(d^{n-1}), & i = n, \\ K^i, & i > n, \end{cases} \quad (3.2.1)$$

so that

$$H^i(\tau_{\geq n}(K)) = \begin{cases} 0, & i < n, \\ H^i(K), & i \geq n. \end{cases} \quad (3.2.2)$$

Let K be an object of $D_{\text{coh}}^-(A)$ (cf. (3.0)). Replacing K , if necessary, by an isomorphic complex in the derived category, one may suppose the components of K to be free of finite type.

For all B and N as in (3.1) (i), one has at the level of complexes

$$\tau_{\geq n}(K \otimes_A^L N) \simeq \tau_{\geq n}(K) \otimes_A N.$$

Taking the Euler–Poincaré characteristics of the two sides gives

$$\sum_{i \geq 0} (-1)^i \lg_B(H^{n+i}(K \otimes_A^L N)) = \lg_B(\text{coker}(d^{n-1}) \otimes_A N) + \sum_{i > 0} (-1)^i \lg_B(K^{n+i} \otimes_A N).$$

Subtract from this identity the analogous identity for $B = N = k$, multiplied by $\lg_B(N)$. By (3.1) (ii), $a \Leftrightarrow b$, and since the K^i are free, one obtains

$$\begin{aligned} \sum_{i \geq 0} (-1)^i \lg_B(H^{n+i}(K \otimes_A^L N)) - \lg_B(N) \sum_{i \geq 0} (-1)^i \dim_k(H^{n+i}(K \otimes_A^L k)) \\ = \lg_B(\text{coker}(d^{n-1}) \otimes_A N) - \lg_B(N) \dim_k(\text{coker}(d^{n-1}) \otimes_A k), \end{aligned} \quad (3.2.3)$$

which allows one to apply (3.1) to the first member of (3.2.3).

Theorem (3.3) (exchange and semicontinuity theorems). Let $K \in \text{Ob } D_{\text{coh}}^-(A)$ (cf. (3.0)) and $n \in \mathbb{Z}$.

(i) For every homomorphism from A to an Artinian ring B and every finite-type B -module N , one has

$$\sum_{i \geq 0} (-1)^i \lg_B(H^{n+i}(K \otimes_A^L N)) \leq \lg_B(N) \sum_{i \geq 0} (-1)^i \dim_k(H^{n+i}(K \otimes_A^L k)). \quad (3.3.1)$$

(ii) The following conditions on K and n are equivalent:

- a) The cohomological functor in the A -module N , $H^*(K \otimes_A^L N)$, satisfies the exchange property in degree $*$ = n , i.e. the following equivalent conditions on K and n are satisfied:
 - a.1) the functor $H^n(K \otimes_A^L N)$ is left exact;
 - a.2) there is a module Q such that this functor is isomorphic to $\text{Hom}_A(Q, N)$;
 - a.3) the functor $H^{n-1}(K \otimes_A^L N)$ is right exact;
 - a.4) there is a module P such that this functor is isomorphic to $P \otimes_A N$;
 - a.5) the map $H^{n-1}(K) \rightarrow H^{n-1}(K \otimes_A^L k)$ is surjective;
 - a.6) there exists a bounded-above complex K' , quasi-isomorphic to K and with flat components, respectively free components of finite type, such that $d'^{n-1} = 0$.
- b) For all B and N as in (i), one has

$$\sum_{i \geq 0} (-1)^i \lg_B(H^{n+i}(K \otimes_A^L N)) = \lg_B(N) \sum_{i \geq 0} (-1)^i \dim_k(H^{n+i}(K \otimes_A^L k)). \quad (3.3.2)$$

If A has no embedded components, these conditions are also equivalent to:

- c) for every maximal point $\mathfrak{p}$ of $\text{Spec}(A)$ one has

$$\sum_{i \geq 0} (-1)^i \lg_{A_{\mathfrak{p}}}(H^{n+i}(K_{\mathfrak{p}})) = \lg(A_{\mathfrak{p}}) \sum_{i \geq 0} (-1)^i \dim_k(H^{n+i}(K \otimes_A^L k)). \quad (3.3.2')$$

Remark (3.3.3). One could have lengthened the list by adding the conditions analogous to (3.1) (ii) c) and d).

Assertion (i) follows from (3.2.3) and (3.1) (i). By EGA III, (7.4.2), and its proof, conditions a.1), a.3), and a.6) are equivalent and, if K is represented by a bounded-above complex of finite-type free modules, they also mean that $\text{coker}(d^{n-1})$ is free; taking (3.1) (ii) $a \Leftrightarrow b \Leftrightarrow e$ and (3.2.3) into account proves the equivalence of a.1), a.3), a.6), b), and c). Trivially, a.6) $\Rightarrow$ a.2) $\Rightarrow$ a.1) and a.6) $\Rightarrow$ a.4) $\Rightarrow$ a.3). The equivalence of condition a.5) with the others is contained in EGA III, (7.5.2).

Corollary (3.4). Let A , K , and n be as in (3.3), and let $k \in \mathbb{N}$.

(i) For all B and N as in (3.3) (i), one has

$$\sum_{0 \leq i \leq 2k} (-1)^i \lg_B(H^{n+i}(K \otimes_A^L N)) \leq \lg_B(N) \sum_{0 \leq i \leq 2k} (-1)^i \dim_k(H^{n+i}(K \otimes_A^L k)). \quad (3.4.1)$$

(ii) The following conditions on K , n , and k are equivalent:

- a) The cohomological functor in N , $H^*(K \otimes_A^L N)$, satisfies the exchange property in degrees $*$ = n and $*$ = $n + 2k + 1$.
- a' The complex K is quasi-isomorphic to a bounded-above complex K' with flat components, respectively free components of finite type, such that $d'^{n-1} = d'^{n+2k} = 0$.
- b) For all B and N as in (i), equality holds in (3.4.1).

If A has no embedded components, these conditions are also equivalent to:

- c) for every maximal point $\mathfrak{p}$ of $\text{Spec}(A)$, equality holds in (3.4.1) for $B = N = A_{\mathfrak{p}}$.

(3.4.2) For $k = 0$, these are properties only of the single hypertor $H^n(K \otimes_A^L N)$, considered as a functor in the A -module N . They can also be expressed by saying that $H^n(K)$ is free and that, for all N , the canonical map from $H^n(K) \otimes_A N$ to $H^n(K \otimes_A^L N)$ is an isomorphism.

(3.4.3) On the other hand, if K is a perfect complex, one deduces from (3.4), for $n \ll 0$, a result analogous to (3.3), with the summation over $i \geq 0$ in (3.3.1, 2, 2') replaced by a summation over $i \leq 0$.

Assertion (i) and the equivalence of a), b), and c) follow from (3.3) and from the identity

$$\sum_{0 \leq i \leq 2k} (-1)^i x_i = \sum_{0 \leq i} (-1)^i x_i + \sum_{0 \leq i} (-1)^i x_{i+2k+1}.$$

To prove a'), one successively applies (3.3) (ii) $a \Leftrightarrow a.6$) to (K, n) and to $(\tau_{\geq n}(K), n + 2k + 1)$.

(3.5) When A is Artinian, (3.4) gives, for $k = 0$, the inequality

$$\lg_A H^n(K) \leq \lg(A) \dim_k (H^n(K \otimes_A^L k)), \quad (3.5.1)$$

and, if equality holds in (3.5.1), then the exchange condition is satisfied in degrees n and $n + 1$, and $H^n(K)$ is a free A -module.

4. The Gysin morphism in Hodge cohomology

This paragraph is written from notes of A. Grothendieck.

(4.0) Let $f : X \rightarrow Y$ be a proper morphism between schemes X and Y which are smooth and purely of relative dimensions n and m over a base S . We suppose that S is noetherian and has a dualizing complex, in order to refer to [4] for the definition of $Rf^!$; this restriction is probably unnecessary. Finally, put $d = n - m$.

(4.1) Every relative differential form on Y defines, by inverse image, a relative differential form on X , hence a morphism

$$f^* \Omega_{Y/S}^p = Lf^* \Omega_{Y/S}^p \longrightarrow \Omega_{X/S}^p. \quad (4.1.1)$$

By the duality theory for smooth morphisms and the transitivity of $Rf^!$, one has canonically

$$Rf^! \Omega_{Y/S}^m = \Omega_{X/S}^n[d]. \quad (4.1.2)$$

Apply to $\Omega_{Y/S}^p$ and $\Omega_{Y/S}^m$ the induction formula

$$Rf^! \operatorname{RHom}(K, L) = \operatorname{RHom}(Lf^* K, Rf^! L),$$

which is obtained readily from the definition of $Rf^!$ by biduality (Hartshorne [4], chap. VII, § 3). Since

$$\operatorname{RHom}(\Omega_{Y/S}^p, \Omega_{Y/S}^m) \simeq \Omega_{Y/S}^{m-p},$$

one obtains

$$Rf^! \Omega_{Y/S}^{m-p} = \operatorname{RHom}(Lf^* \Omega_{Y/S}^p, \Omega_{X/S}^n)[d].$$

The arrow (4.1.1) therefore defines, by transposition, an arrow⁵

$$\Omega_{X/S}^{n-p} = \operatorname{RHom}(\Omega_{X/S}^p, \Omega_{X/S}^n) \longrightarrow Rf^! \Omega_{Y/S}^{m-p}[d]. \quad (4.1.3)$$

By duality theory, giving an arrow (4.1.3) amounts to giving an arrow

$$Rf_* \Omega_{X/S}^{n-p} \longrightarrow \Omega_{Y/S}^{m-p}[-d]. \quad (4.1.4)$$

Definition (4.2). Let X, Y, f, S, n, m , and d be as in (4.0). The arrow (4.1.4)

$$\operatorname{Tr}_f : Rf_* \Omega_{X/S}^p \longrightarrow \Omega_{Y/S}^{p-d}[-d]$$

is called the Gysin morphism, as are the induced maps

$$\operatorname{Tr}_f : H^q(X, \Omega_{X/S}^p) \longrightarrow H^{q-d}(Y, \Omega_{Y/S}^{p-d}).$$

⁵In the first member of (4.1.3), the printed source switches once from p to q ; the index p used in (4.1.1), (4.1.4), and (4.2) is retained here.

Proposition (4.3). Let $f : X \rightarrow Y$ be a proper birational morphism of smooth schemes over a field k . The maps

$$f^* : H^q(Y, \Omega_Y^p) \longrightarrow H^q(X, \Omega_X^p)$$

are injective.

One reduces to the case where X and Y are purely of dimension n . The composite morphism

$$\mathrm{Tr}_f \circ f^* : \Omega_Y^p \longrightarrow \mathrm{R}f_* f^* \Omega_Y^p \longrightarrow \mathrm{R}f_* \Omega_X^p \longrightarrow \Omega_Y^p$$

is the identity, because it coincides with the identity on a dense open subset. Applying the functor $H^q(Y, \cdot)$, one finds that the composite morphism

$$\mathrm{Tr}_f \circ f^* : H^q(Y, \Omega_Y^p) \longrightarrow H^q(X, \Omega_X^p) \longrightarrow H^q(Y, \Omega_Y^p)$$

is the identity, which proves (4.3).

5. Hodge and De Rham cohomology

(5.1) Let $f : X \rightarrow S$ be a smooth morphism of schemes. By definition, the relative De Rham cohomology sheaves of X over S are given by

$$H_{\mathrm{DR}}^*(X/S) = \mathrm{R}^* f_*(\Omega_{X/S}^\bullet),$$

where $\Omega_{X/S}^\bullet$, the relative De Rham complex, is an S -linear differential complex. There is a spectral sequence

$$E_1^{pq} = \mathrm{R}^q f_* \Omega_{X/S}^p \implies H_{\mathrm{DR}}^{p+q}(X/S). \quad (5.1.1)$$

(5.2) When $S = \mathrm{Spec}(\mathbb{C})$ and X is proper and smooth, GAGA ([7]) shows that the cohomology of each coherent sheaf Ω_X^p , and hence the hypercohomology of $\Omega_X^\bullet$, are the same in the algebraic sense (Zariski topology) and in the transcendental sense (usual topology of X^{an}); in particular, since $\Omega_{X^{\mathrm{an}}}^\bullet$ is a resolution of the constant sheaf $\mathbb{C}$, one has

$$H_{\mathrm{DR}}^n(X) = H^n(X^{\mathrm{an}}, \mathbb{C}),$$

and complex conjugation on the constant sheaf $\mathbb{C}$ induces a complex conjugation, of transcendental nature, on $H_{\mathrm{DR}}^n(X)$. The spectral sequence (5.1.1) is here

$$E_1^{pq} = H^q(X, \Omega_X^p) \implies H_{\mathrm{DR}}^{p+q}(X); \quad (5.2.1)$$

let $F^p(H^n(X))$ denote the decreasing filtration which it defines on its abutment. By Weil [10], chap. IV, no. 4, th. 2, the spectral sequence (5.2.1) can be computed as the spectral sequence of the double complex $H^0(X^{\mathrm{an}}, \Omega^{p,q})$, filtered by the first degree, where $\Omega^{p,q}$ denotes the sheaf of C^∞ differential forms bihomogeneous of bidegree (p, q) (Weil [10], chap. II, no. 1).

Suppose now that X is projective, so that X^{an} is a Kähler variety. The harmonic forms then form a double subcomplex (Weil [10], chap. II, no. 6, cor. 1 to th. 2) of the preceding double complex, on which d' and d'' vanish. The inclusion of this subcomplex induces an isomorphism on the E_1 terms of the spectral sequences defined by these double complexes, filtered by the first degree, as one sees by considering the operator H (harmonic component), a retraction of the double complex $H^0(X^{\mathrm{an}}, \Omega^{p,q})$ onto the subcomplex of harmonic forms and satisfying

$$1 - H = d'(2\delta'G) + (2\delta'G)d'$$

(Weil [10], chap. IV, no. 1: (I) and lemma 3; no. 3: cor. 2 and chap. II, no. 5, th. 2 (IX)).

If $H^{p,q}(X)$ denotes the image in $H^{p+q}(X^{\mathrm{an}}, \mathbb{C})$ of the space of harmonic forms of type (p, q) , one therefore has

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X), \quad (5.2.2)$$

$$F^p(H^n(X)) = \bigoplus_{i \geq p} H^{i, n-i}(X), \quad (5.2.3)$$

$$\overline{H^{p,q}(X)} = H^{q,p}(X) \quad (\text{complex conjugation}), \quad (5.2.4)$$

and the spectral sequence (5.2.1) degenerates.

Returning to the general case where X is only assumed proper and smooth over $\mathbb{C}$, put

$$H^{p,q}(X) = F^p(H^{p+q}(X)) \cap \overline{F^q(H^{p+q}(X))}, \quad (5.2.5)$$

so that

$$\overline{H^{p,q}(X)} = H^{q,p}(X). \quad (5.2.6)$$

By (5.2.3), this notation is compatible with the one used when X is projective. Put also

$$h^{p,q} = \dim H^q(X, \Omega_X^p). \quad (5.2.7)$$

Proposition (5.3). Let X be a proper smooth scheme over $\mathbb{C}$:

- (i) the spectral sequence (5.2.1) degenerates;
- (ii) one has

$$F^p(H^n(X)) = \bigoplus_{i \geq p} H^{i,n-i}(X), \quad (5.3.1)$$

and in particular

$$H^n(X) = \bigoplus_{p+q=n} H^{p,q}(X) \quad (\text{direct sum}), \quad (5.3.2)$$

$$h^{p,q} = h^{q,p} = \dim H^{p,q}(X). \quad (5.3.3)$$

One reduces to the case where X is connected. Let N be its dimension. By Chow's lemma and resolution of singularities (Hironaka [5]), there exists a projective birational morphism $g : X' \rightarrow X$, with X' projective and smooth over $\mathbb{C}$. The spectral sequence

$$E_1^{pq} = H^q(X, \Omega_X^p) \implies H_{\text{DR}}^{p+q}(X) \quad (5.3.4)$$

maps, by inverse image, into the analogous spectral sequence

$$E_1'^{pq} = H^q(X', \Omega_{X'}^p) \implies H_{\text{DR}}^{p+q}(X'), \quad (5.3.5)$$

which degenerates by (5.2). By (4.3), the morphism $g^* : E_1 \rightarrow E_1'$ is injective. One concludes, by induction on $r \geq 1$, that $d_r = 0$, that $E_r = E_{r+1}$, and that g^* injects E_{r+1} into E_{r+1}' :

$$\begin{array}{ccc} E_r^{pq} & \xrightarrow{d_r} & E_r^{p+r, q-r+1} \\ g^* \downarrow & & g^* \downarrow \\ E_r'^{pq} & \xrightarrow{0} & E_r'^{p+r, q-r+1}. \end{array}$$

Since the spectral sequences (5.3.4) and (5.3.5) degenerate, and since g^* is injective on their initial term, it is still injective on their abutment. From formulas (5.2.3), (5.2.4), applied to X' , it follows that

$$F^p(H^n(X')) \cap \overline{F^{n-p+1}(H^n(X'))} = \{0\};$$

hence one obtains the analogous formula

$$F^p(H^n(X)) \cap \overline{F^{n-p+1}(H^n(X))} = \{0\}, \quad (5.3.6)$$

because g^* sends $F^p(H^n(X))$ to $F^p(H^n(X'))$ and commutes with complex conjugation. In particular,

$$\sum_{i \geq p} h^{i,n-i} + \sum_{i \geq n-p+1} h^{i,n-i} \leq \dim H^n(X) \leq \sum_i h^{i,n-i},$$

which may be written

$$\sum_{i \geq p} h^{i,n-i} \leq \sum_{i \leq n-p} h^{i,n-i}. \quad (5.3.7)$$

By Serre duality, $h^{i,j} = h^{N-i,N-j}$, inequality (5.3.7), for $N - n$, implies the opposite inequality, so that

$$\dim F^p(H^n(X)) + \dim \overline{F^{n-p+1}(H^n(X))} = \dim H^n(X).$$

By (5.3.6), therefore,

$$H^n(X) = F^p(H^n(X)) \oplus \overline{F^{n-p+1}(H^n(X))}. \quad (5.3.8)$$

The decomposition (5.3.8) induces on $F^{p-1}(H^n(X))$, which contains one of the factors, a decomposition

$$F^{p-1}(H^n(X)) = F^p(H^n(X)) \oplus H^{p-1,n-p+1}(X),$$

and formula (5.3.1) follows by descending induction on p ; (5.3.2) and (5.3.3) follow at once from (5.3.1) and (5.2.6), completing the proof of (5.3).

Corollary (5.4). Under the hypotheses of (5.3), one has:

- (i) A complex cohomology class a on X lies in $H^{p,q}(X)$ if and only if it can be represented by a closed form α of type (p, q) .
- (ii) Every cohomology class can be represented by a form α satisfying $d'\alpha = d''\alpha = 0$.
- (iii) If the form α satisfies $d'\alpha = d''\alpha = 0$, the following conditions are equivalent:
 - a) $(\exists \beta) \alpha = d\beta$;
 - b) $(\exists \beta) \alpha = d'\beta$;
 - c) $(\exists \beta) \alpha = d''\beta$.⁶

By definition, $a \in H^{p,q}(X)$ if and only if in the class of a there exist closed forms α_1 and α_2 whose components of type (p', q') are zero for $p' < p$ (respectively $q' < q$). Then there exists β such that $\alpha_1 - \alpha_2 = d\beta$. Let β_1 (respectively β_2) be the sum of the components of β of type (p', q') with $p' \geq p$ (respectively $q' \geq q$); one has $\beta = \beta_1 + \beta_2$ and

$$\alpha = \alpha_1 - d\beta_1 = \alpha_2 + d\beta_2$$

is a closed form of type (p, q) in the class of a . This proves (i). It suffices to prove (ii) for a bihomogeneous, hence represented by a closed bihomogeneous form α by (i). Such a form automatically satisfies $d'\alpha = d''\alpha = 0$.

If α satisfies $d'\alpha = d''\alpha = 0$, its bihomogeneous components are closed, and each of the conditions a), b), c) is equivalent to the conjunction of the analogous conditions for the various components of α , trivially for b) and c), and by (5.3) (ii) and (5.4) (i) for a). To prove (iii), suppose therefore that α is closed of type (p, q) . The cohomology class a defined by α lies in $H^{p,q}(X)$, hence in $F^p(H^{p+q}(X))$, and is zero if and only if it already lies in $F^{p+1}(H^{p+q}(X))$, i.e. if its image in $H^q(X, \Omega_X^p)$ is zero. This proves a) $\Leftrightarrow$ c), and the equivalence a) $\Leftrightarrow$ b) follows by conjugation.

Theorem (5.5). Let S be a scheme of characteristic 0 and let $f : X \rightarrow S$ be a proper smooth morphism. Then:

- (i) the sheaves $R^q f_* \Omega_{X/S}^p$ are locally free of finite type and their formation is compatible with arbitrary base change;
- (ii) the spectral sequence (5.1.1)

$$E_1^{pq} = R^q f_* \Omega_{X/S}^p \Longrightarrow H_{\text{DR}}^{p+q}(X/S)$$

degenerates;

- (iii) at every point of S , the sheaves $R^q f_* \Omega_{X/S}^p$ and $R^p f_* \Omega_{X/S}^q$ have the same rank (Hodge symmetry).

The second assertion of (i) follows from the first and from EGA III, (7.8.5), applied to each of the $\Omega_{X/S}^p$; it implies that it suffices to verify (iii) for S the spectrum of a field.

Standard arguments allow one to reduce successively to the case where S is affine, since the question is local on S ; noetherian, by passage to an inductive limit of rings; the spectrum of a noetherian local ring, since the question is local; the spectrum of a complete noetherian local ring, by faithful flatness of completion; and finally the spectrum of an Artinian local ring. To make this last reduction, write the complete noetherian local ring R as the projective limit of its Artinian quotients $R/\mathfrak{m}^n$. By the comparison theorem EGA III, (4.1.5), the E_1 term

⁶(Added in proofs.) One can show that these conditions are also equivalent to the existence of a form β such that $\alpha = d'd''\beta$.

of the spectral sequence (5.1.1) is the projective limit of the E_1 terms of the analogous spectral sequences over the $R/\mathfrak{m}^n$, so that if these degenerate, then (5.1.1) also degenerates. If assertion (i) is true over the $R/\mathfrak{m}^n$, these E_1 terms form a projective system of free modules, obtained from one another by reduction modulo $\mathfrak{m}^n$, so that their projective limit is a free module.

Suppose therefore that $S = \operatorname{Spec}(A)$ with A an Artinian local ring with residue field k of characteristic 0. This ring admits a field of representatives (EGA, 0_{IV}, (19.8.6), (i), for $W = k$, $u = \text{identity}$) and can therefore be regarded as a local k -algebra of finite dimension. The Lefschetz principle permits one to suppose $k = \mathbb{C}$, in which case (iii) follows from (i) and (5.3.3).

To complete the proof of (5.5), it remains to prove the first assertion of (i), and (ii), under the additional hypothesis:

$$S = \operatorname{Spec}(A) \quad \text{for a finite-dimensional local } \mathbb{C}\text{-algebra } A \text{ with maximal ideal } \mathfrak{m}. \quad (5.5.1)$$

The spectral sequence (5.1.1) becomes

$$E_1^{pq} = H^q(X, \Omega_{X/S}^p) \implies R^{p+q}\Gamma(\Omega_{X/S}^\bullet). \quad (5.5.2)$$

Lemma (5.5.3). Under the hypotheses (5.5) and (5.5.1), the complex $\Omega_{X/S}^{\text{an}, \bullet}$ is a resolution of the constant sheaf A on the topological space X^{an} .

Since the complex $\Omega_{X/S}^{\text{an}, \bullet}$ is A -linear, its $\mathfrak{m}$ -adic filtration is a filtration by subcomplexes. For this filtration,

$$\operatorname{Gr} \Omega_{X/S}^{\text{an}, \bullet} = \operatorname{Gr} A \otimes_{\mathbb{C}} \operatorname{Gr}^0 \Omega_{X/S}^{\text{an}, \bullet}.$$

The complex $\operatorname{Gr}^0 \Omega_{X/S}^{\text{an}, \bullet}$ is a resolution of the constant sheaf $\mathbb{C}$, since it is just the De Rham complex of X_{red} . Thus the complex $\operatorname{Gr} \Omega_{X/S}^{\text{an}, \bullet}$ is a resolution of $\operatorname{Gr} A$, and the assertion follows.

The space X is compact, so by GAGA (cf. (5.2)) the algebraic and transcendental hypercohomologies of $\Omega_{X/S}^\bullet$ coincide, and by (5.5.3) they also coincide with the cohomology of X^{an} with values in A . Hence

$$\lg_A R^n \Gamma(\Omega_{X/S}^\bullet) = \lg(A) \dim_{\mathbb{C}} R^n \Gamma(\Omega_{X_{\text{red}}}^\bullet). \quad (5.5.4)$$

By (3.5.1) and EGA III, (6.10.5), one knows that

$$\lg_A H^q(X, \Omega_{X/S}^p) \leq \lg(A) \dim_{\mathbb{C}} H^q(X_{\text{red}}, \Omega_{X_{\text{red}}}^p), \quad (5.5.5)$$

with equality possible only if $H^q(X, \Omega_{X/S}^p)$ is A -free.

From (5.5.2) one obtains

$$\sum_{p+q=n} \lg_A H^q(X, \Omega_{X/S}^p) \geq \lg_A R^n \Gamma(\Omega_{X/S}^\bullet), \quad (5.5.6)$$

equality being possible for all n only if the spectral sequence (5.5.2) degenerates. Chaining these inequalities, one finds

$$\lg(A) \sum_{p+q=n} \dim_{\mathbb{C}} H^q(X_{\text{red}}, \Omega_{X_{\text{red}}}^p) \geq \lg(A) R^n \Gamma(\Omega_{X_{\text{red}}}^\bullet), \quad (5.5.7)$$

equality being possible only if the conclusions of (5.4) are satisfied. This equality follows from (5.3) (i) applied to X_{red} .

6. Application to coherent cohomology

Theorem (6.1). Let S be a scheme of characteristic 0, and let $f : X \rightarrow S$ be a projective smooth morphism of relative dimension n . For every p , for example $p = 0$, one has in $D^b(S, \mathcal{O}_S)$

$$Rf_* \Omega_{X/S}^p \simeq \bigoplus_q R^q f_* \Omega_{X/S}^p[-q].$$

Let u denote the first Chern class of a relatively ample invertible sheaf on X , with values in $H^1(X, \Omega_{X/S}^1)$. By cup product, the class u defines homomorphisms in $D^b(S)$,

$$u \wedge : Rf_* \Omega_{X/S}^p \longrightarrow Rf_* \Omega_{X/S}^{p+1}[1],$$

and a degree-two endomorphism

$$u \wedge : \bigoplus_p \mathrm{R}f_* \Omega_{X/S}^p[-p] \longrightarrow \left(\bigoplus_p \mathrm{R}f_* \Omega_{X/S}^p[-p] \right) [2]. \quad (6.1.1)$$

Lemma (6.2). The endomorphism (6.1.1) satisfies the hypotheses of (1.5).

By (5.4) (i) and the Lefschetz principle, it suffices to prove (6.2) when $S = \mathrm{Spec}(\mathbb{C})$. The assertion then follows from the Lefschetz theorem (see (2.6.2) or (2.6.3)) and from the Hodge decomposition.

By (1.5), the complex $\bigoplus_p \mathrm{R}f_* \Omega_{X/S}^p[-p]$ is isomorphic, in the derived category, to the sum of its cohomology objects, and the same is true for each of the $\mathrm{R}f_* \Omega_{X/S}^p$, which, up to a shift, are direct factors of it (1.12).

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