

APPENDIX: COHOMOLOGY WITH PROPER SUPPORT AND CONSTRUCTION OF THE FUNCTOR $f^!$.

by P. DELIGNE¹

Verdier has shown that, in the topological case, the formalism of Poincaré duality reduces to local problems upstairs (see [1]). To transpose his construction to the schematic setting, one needs a theory of cohomology “with proper support” for coherent sheaves.

Unless explicitly stated otherwise, all preschemes considered are noetherian and all presheaves are quasi-coherent.

NO. 1. FORMALITIES OF PRO-OBJECTS.

Proposition 1. *Let C be a U -category² in which finite inductive limits exist. Let h be a functor $C^\circ \rightarrow (\text{ens})$. The following conditions are equivalent:*

- (i) *h is an inductive limit, over a small³ filtered ordered set, of representable functors.*
- (ii) *h is an inductive limit, over a small filtered category, of representable functors.*
- (iii) *h transforms finite projective limits into finite inductive limits, and there is a small set of objects such that every element of an $h(X)$ factors through one of them.*

The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are trivial, and (iii) $\Rightarrow$ (ii) is standard: h is the limit of the h_F over the category of pairs (F, α) , where $\alpha \in h(F)$ and F lies in the subcategory of C stable under finite inductive limits generated by the given small set. It remains to prove that (ii) $\Rightarrow$ (i). If $\mathcal{L}$ and $\mathcal{M}$ are two filtered categories, a functor $F : \mathcal{L} \rightarrow \mathcal{M}$ is called cofinal if $\varinjlim_{\mathcal{L}} h_{F(L)}$ is the final functor. For every $G : \mathcal{M} \rightarrow \mathcal{N}$, one then has $\varinjlim_{\mathcal{M}} G = \varinjlim_{\mathcal{L}} GF$. Thus it remains to prove:

Lemma 1. *For every small filtered category $\mathcal{L}$, there exists a cofinal functor from a small filtered ordered set to $\mathcal{L}$.*

¹This is an expanded version of a letter from P. Deligne to R. Hartshorne (letter of March 3, 1966). The footnotes were added by the copyist.

² U denotes a fixed universe throughout.

³“small” means “ $\in U$ ”.

The first projection $\mathcal{L} \times \mathbb{N} \rightarrow \mathcal{L}$ is a cofinal functor, where $\mathbb{N}$ has its natural order. This permits one to reduce to the case where $\text{Ob } \mathcal{L}$, preordered by the relation $\text{Hom}(X, Y) \neq \emptyset$, has no greatest element. Order by inclusion the set E of finite subcategories of $\mathcal{L}$ having a single final object, where finite means having a finite set of arrows. Define a functor from E to $\mathcal{L}$ by sending each of these categories to its final object. Under the hypotheses made, E is filtered and this functor is cofinal.

The functors satisfying the equivalent conditions of Proposition 1 are the Ind-objects of C . They form a full subcategory $\text{Ind } C$ of $\underline{\text{Hom}}(C^\circ, (\text{Ens}))$, stable under inductive limits. The assignment $C \mapsto \text{Ind } C$ is a functor $(\text{cat}) \rightarrow (\text{cat})$. If filtered inductive limits exist in C , then for every $h \in \text{Ind } C$ the functor $X \mapsto \text{Hom}(h, h_X)$ is corepresentable; hence there is a functor, denoted $\varinjlim$, from $\text{Ind } C$ to C . In particular, one always has a functor $\text{Ind } \text{Ind } C \rightarrow \text{Ind } C$, and every functor $C \rightarrow \text{Ind } D$ extends to a functor $\text{Ind } C \rightarrow \text{Ind } D$. In $\text{Ind } C$, filtered inductive limits are exact and will be denoted “ $\varinjlim$ ”; in particular, identifying C with a full subcategory of $\text{Ind } C$, if $(X_i)_{i \in I}$ is a filtered inductive system in C , then

$$\varinjlim_{i \in I} X_i = \varinjlim_{i \in I} \text{“} \varinjlim \text{”} X_i.$$

A functor on $\text{Ind } C$ is representable if and only if it transforms finite projective limits into finite inductive limits, filtered inductive limits into filtered inductive limits, and its restriction to C satisfies the smallness condition of Proposition 1(iii). Indeed, Proposition 1 then shows that its restriction to C is an Ind-object, to which the functor is everywhere equal because of the condition on limits.

The preceding statements, except Proposition 1(iii) and the preceding assertion, remain true if one uses everywhere limits of sequences, or countable limits, which is the same thing.

By duality one defines the category $\text{pro } C$ of pro-objects, namely the full subcategory of $\underline{\text{Hom}}(C, (\text{Ens}))^\circ$.

Proposition 2. *Let X be a quasi-compact, quasi-separated prescheme, not necessarily noetherian. The category of quasi-coherent sheaves on X is equivalent to the category of Ind-objects of the category of quasi-coherent sheaves of finite presentation on X .*

The map is “ $\varinjlim$ ” $\mathcal{F}_i \mapsto \varinjlim \mathcal{F}_i$. If $\mathcal{F}$ is of finite presentation, then for every filtered inductive system $(\mathcal{G}_i)_{i \in I}$,

$$\text{Hom}_X(\mathcal{F}, \varinjlim \mathcal{G}_i) = \varinjlim \text{Hom}_X(\mathcal{F}, \mathcal{G}_i).$$

By a finite gluing argument, which commutes with $\varinjlim$, one reduces to the affine case. Thus the preceding functor is fully faithful. Its image is stable under filtered inductive limits and finite sums. It remains only to prove that for every $x \in X$, every $\mathcal{F}$ on X , and every $s \in \mathcal{F}_x$, there exists a sheaf $\mathcal{G}$ of finite presentation and a map from $\mathcal{G}$ to $\mathcal{F}$ whose image contains s .

The sheaf $\mathcal{F}$ will be a limit of images of sheaves of finite presentation, and each of these will be a quotient of a sheaf of finite presentation by a limit of sub-sheaves of finite type.

On a quasi-compact neighbourhood U of x , let $f : \mathcal{C} \rightarrow \mathcal{F}$ be a map with $\mathcal{C}$ of finite presentation and with s in its image. It is enough to extend $\mathcal{C}$ and f to all of X ; proceeding step by step, it is enough to know how to extend them to $U \cup V$ when V is affine. This amounts to carrying out an extension from $U \cap V$ to V .

Lemma 2. *Let $\mathcal{C}$ be a sheaf of finite presentation on a quasi-compact open U of an affine scheme X , let $\mathcal{F}$ be a sheaf on X , and let f be a map from $\mathcal{C}$ to $\mathcal{F}|_U$. It is possible to extend $\mathcal{C}$ and f to $\bar{\mathcal{C}}$ and $\bar{f}$, defined on X , with $\bar{\mathcal{C}}$ of finite presentation.*

Let ι be the inclusion of U in X , and let $\mathcal{C}_1$ be the fiber product of $\iota_*\mathcal{C}$ and $\mathcal{F}$ over $\iota_*\iota^*\mathcal{F}$. Then $\mathcal{C}_1$ extends $\mathcal{C}$ and maps to $\mathcal{F}$. If one writes $\mathcal{C}_1$ as a limit of its sub-sheaves of finite type, then, since U is quasi-compact and $\mathcal{C}$ is of finite presentation, one of these already extends $\mathcal{C}$. Writing the latter as a quotient of $\mathcal{O}^n$ by a limit of sub-sheaves of finite type of $\mathcal{O}^n$, one sees in the same way that $\mathcal{C}_1$ may be replaced by a sheaf of finite presentation.

Corollary 1. *A contravariant additive functor $F : (q \text{ coh}) \rightarrow \text{Ab}$ is representable if and only if it is left exact and transforms filtered inductive limits into filtered inductive limits.*

Corollary 2. *Every sheaf of finite presentation defined on a quasi-compact open U of X extends to a sheaf of finite presentation on X .*

Let $\mathcal{A}$ be an abelian category. We shall have to work essentially in $\text{pro } D^b\mathcal{A}$, the full subcategory of $\text{pro } D(\mathcal{A})$ consisting of the “ $\varprojlim$ ” K_i for which the cohomology of K_i is uniformly bounded as i varies. The functor $\mathbb{H}^p$ extends to a functor $\text{pro } D(\mathcal{A}) \rightarrow \text{pro } \mathcal{A}$.

Proposition 3. *The functors $\mathbb{H}^p$ form a conservative⁴ family of functors on $\text{pro } D^b\mathcal{A}$.*

⁴That is, if u is an arrow such that all $\mathbb{H}^p(u)$ are invertible, then u is invertible.

Let $f : K \rightarrow L$ be an arrow of $\text{pro } D^b \mathcal{A}$ such that the maps $\mathbb{H}^p(K) \rightarrow \mathbb{H}^p(L)$ are isomorphisms in $\text{pro } \mathcal{A}$. We must check that, for every M in $D^b(\mathcal{A})$,

$$\text{Hom}(K, M) \longleftarrow \text{Hom}(L, M)$$

is an isomorphism. Write $K = \varprojlim K_\alpha$. There is a convergent spectral sequence

$$\text{Ext}^p(\mathbb{H}^q K_\alpha, M) \Longrightarrow \text{Ext}^{p+q}(K_\alpha, M),$$

whence, on passing to the limit, another convergent spectral sequence

$$\varinjlim \text{Ext}^p(\mathbb{H}^q K_\alpha, M) \Longrightarrow \varinjlim \text{Ext}^{p+q}(K_\alpha, M),$$

since q is uniformly bounded. Using the fact that every functor, here Ext , passes to pro-objects, and using the functor $\varinjlim : \text{Ind Ab} \rightarrow \text{Ab}$, we may rewrite this as

$$\varinjlim \text{Ext}^p(\mathbb{H}^q K, M) \Longrightarrow \varinjlim \text{Ext}^{p+q}(K, M) = \text{Hom}(K[-p-q], M).$$

This spectral sequence remains valid for $M \in D(\mathcal{A})$. The proposition follows immediately from the existence of these spectral sequences.

NO. 2. FUNDAMENTAL LEMMAS.

Recall that all preschemes considered are noetherian.

Proposition 4 (Duality theorem for an open immersion). *Let U be an open subset of X , let j be the inclusion, let $\mathcal{J}$ be an ideal defining the closed complement, let $\mathcal{F}$ be a coherent sheaf on U , and let $\mathcal{G}$ be quasi-coherent on X . Let $\overline{\mathcal{F}}$ be a coherent extension of $\mathcal{F}$. Then*

$$(5) \quad \text{Hom}_U(\mathcal{F}, j^* \mathcal{G}) = \text{Hom}_X(\varprojlim \mathcal{J}^n \overline{\mathcal{F}}, \mathcal{G}).$$

The map $\varinjlim \text{Hom}_X(\mathcal{J}^n \overline{\mathcal{F}}, \mathcal{G}) \rightarrow \text{Hom}_U(\mathcal{F}, \mathcal{G})$ is injective: if the image under f of $\mathcal{J}^n \overline{\mathcal{F}}$ in $\mathcal{G}$ has its support in the complement of U , then it is annihilated by a power of $\mathcal{J}$, say $\mathcal{J}^k$, and the image of $\mathcal{J}^{n+k} \overline{\mathcal{F}}$ is zero.

Now assume that X is affine, and let $f \in \text{Hom}_U(\mathcal{F}, \mathcal{G})$. For k sufficiently large, every section s of $\mathcal{J}^k \overline{\mathcal{F}}$ over X has an image over U which extends to a section of $\mathcal{G}$ over X . Replacing $\overline{\mathcal{F}}$ by $\mathcal{J}^k \overline{\mathcal{F}}$, we may

⁵This formula is of course well known; cf., for example, P. Gabriel's thesis.

suppose that we have a diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & R & \longrightarrow & \mathcal{O}^n & \longrightarrow & \overline{\mathcal{F}} \longrightarrow 0 \\
 & & \downarrow & & \parallel & & \downarrow \\
 0 & \longrightarrow & R_1 & \longrightarrow & \mathcal{O}^n & \longrightarrow & \mathcal{G}
 \end{array}$$

in which the dotted arrows are defined on U .

For ℓ sufficiently large, by Krull's theorem and by the fact that $R/(R \cap R_1)$ is concentrated outside U , one has $R \cap \mathcal{J}^\ell \cdot \mathcal{O}^n \subset R_1$, which permits f to be extended to a map $\overline{f} : \mathcal{J}^\ell \overline{\mathcal{F}} \rightarrow \mathcal{G}$.

In the general case, let U_i be a finite affine covering of X , and let U_{ijk} be finite affine coverings of $U_{ij} = U_i \cap U_j$. Since $\varinjlim$ commutes with finite products, one obtains

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathrm{Hom}_U(\mathcal{F}, j^* \mathcal{G}) & \longrightarrow & \prod_i \mathrm{Hom}_{U_i \cap U}(\mathcal{F}, j^* \mathcal{G}) & \Longrightarrow & \prod_{ijk} \mathrm{Hom}_{U_{ijk} \cap U}(\mathcal{F}, j^* \mathcal{G}) \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \varinjlim \mathrm{Hom}_X(\mathcal{J}^n \overline{\mathcal{F}}, j^* \mathcal{G}) & \longrightarrow & \prod_i \varinjlim \mathrm{Hom}_{U_i}(\mathcal{J}^n \overline{\mathcal{F}}, \mathcal{G}) & \Longrightarrow & \prod_{ijk} \varinjlim \mathrm{Hom}_{U_{ijk}}(\mathcal{J}^n \overline{\mathcal{F}}, \mathcal{G})
 \end{array}$$

which completes the proof.

Proposition 4 gives an explicit formula for the functor $(\mathrm{coh} \text{ on } U) \rightarrow \mathrm{pro}(\mathrm{coh} \text{ on } X)$ left adjoint to j^* . We shall denote this functor by $j_!$ (extension by zero)⁶; it follows from Krull's theorem that it is exact. This also follows from the fact that, since X is noetherian, an injective object of the category of quasi-coherent sheaves on X , restricted to U , remains injective as a quasi-coherent sheaf on U .

Proposition 5 (Independence of the compactification).

$$\begin{array}{ccc}
 U & \hookrightarrow & X \\
 \parallel & & \downarrow f \\
 U & \hookrightarrow & Y
 \end{array}$$

Let $f : X \rightarrow Y$ be a proper morphism inducing an isomorphism between the open subset U of Y and $f^{-1}(U)$. Let $\mathcal{J}$ be a sheaf of ideals defining $Y \setminus U$, and put $\mathcal{I} = f^* \mathcal{J}$. If $\mathcal{F}$ is coherent on X , then

- (i) for $k > 0$, the projective system $\mathbb{R}^k f_* \mathcal{I}^n \mathcal{F}$ is essentially zero, that is, it defines the zero pro-object;
- (ii) for $k = 0$, for all sufficiently large n , $f_* \mathcal{I}^{n+1} \mathcal{F} = \mathcal{J} \cdot f_* \mathcal{I}^n \mathcal{F}$.

⁶This terminology and notation plainly conflict with the generally accepted ones, for example in Godement's book and in SGA 1962. The reader may prefer to read $\mathfrak{J}_!$ instead of $j_!$.

One has

$$\bigoplus_{n=0}^{\infty} \mathbb{R}^k f_* \mathcal{I}^n \mathcal{F} \quad \text{is a finite type } \bigoplus_{n=0}^{\infty} \mathcal{J}^n \text{-module (EGA III).}$$

Thus, for n sufficiently large, the map $\mathcal{J} \otimes \mathbb{R}^k f_* \mathcal{I}^n \mathcal{F} \rightarrow \mathbb{R}^k f_* \mathcal{I}^{n+1} \mathcal{F}$ is surjective; (ii) follows. If $k > 0$, $\mathbb{R}^k f_*$ is zero on U , so there exists N such that $\mathcal{J}^N \mathbb{R}^k f_* \mathcal{I}^n \mathcal{F}$ is zero for every n . The diagram

$$\begin{array}{ccccc} \mathcal{J}^p \otimes \mathbb{R}^k f_* \mathcal{I}^n \mathcal{F} & \longrightarrow & \mathbb{R}^k f_* \mathcal{I}^{n+p} \mathcal{F} & \longrightarrow & 0 \quad (\text{if } n \geq n_0) \\ \downarrow & & \downarrow & & \\ \mathcal{J}^p \cdot \mathbb{R}^k f_* \mathcal{I}^n \mathcal{F} & \longrightarrow & \mathbb{R}^k f_* \mathcal{I}^n \mathcal{F} & & \end{array}$$

proves that for N sufficiently large and $k > 0$ the map

$$\mathbb{R}^k f_* \mathcal{I}^{n+N} \mathcal{F} \rightarrow \mathbb{R}^k f_* \mathcal{I}^n \mathcal{F}$$

is zero, which is (i).

NO. 3. DEFINITION OF $\mathbb{R}f_!$.

A morphism f , respectively a composable pair of morphisms, respectively a triple, will be called compactifiable if one can respectively find commutative diagrams

$$\begin{array}{ccc} \begin{array}{ccc} X \hookrightarrow \bar{X} \\ f \downarrow \swarrow \\ S \end{array} & \begin{array}{ccc} X \hookrightarrow \bar{X} \hookrightarrow \overline{\bar{X}} \\ g \downarrow \swarrow & \searrow \swarrow \\ Y \hookrightarrow \bar{Y} \\ f \downarrow \swarrow \\ S \end{array} & \begin{array}{cccc} X \hookrightarrow \bar{X} \hookrightarrow \overline{\bar{X}} \hookrightarrow \overline{\overline{\bar{X}}} \\ h \downarrow \swarrow & \searrow \swarrow & \searrow \swarrow \\ Y \hookrightarrow \bar{Y} \hookrightarrow \overline{\bar{Y}} \\ g \downarrow \swarrow & \searrow \swarrow & \searrow \swarrow \\ Z \hookrightarrow \bar{Z} \\ f \downarrow \swarrow \\ S \end{array} \end{array}$$

in which the horizontal arrows are open immersions and the oblique arrows are proper. Every compactifiable morphism is separated and of finite type, and Nagata asserts in [2] that his proof gives the converse for integral noetherian schemes, but the hypotheses he imposes on the base are not clear.⁷

We propose, for every compactifiable f , to define

$$\mathbb{R}f_! : \text{pro } D_{\text{coh}}^b(X) \rightarrow \text{pro } D_{\text{coh}}^b(S).$$

Recall that the category $D_{\text{coh}}^b(S)$, the bounded derived category of coherent sheaves on S , is a full subcategory of the derived category of

⁷Mumford is said to have checked, via Nagata's proof, that every separated morphism of finite type of noetherian schemes is compactifiable.

all sheaves, not necessarily quasi-coherent, on S ; its essential image consists of complexes with bounded coherent cohomology.

If f is proper, one has $\mathbb{R}f_* : D_{\text{coh}}^b(X) \rightarrow D_{\text{coh}}^b(S)$ of finite dimension, and it induces $\mathbb{R}f_! : \text{pro } D_{\text{coh}}^b(X) \rightarrow \text{pro } D_{\text{coh}}^b(S)$. If f is an open immersion, $f_! : (\text{coh on } X) \rightarrow \text{pro}(\text{coh on } S)$ induces

$$D^b(\text{coh on } X) \rightarrow D^b \text{pro}(\text{coh on } S),$$

which maps to $\text{pro } D_{\text{coh}}^b(S)$, and this map extends to

$$\mathbb{R}f_! : \text{pro } D_{\text{coh}}^b(X) \rightarrow \text{pro } D_{\text{coh}}^b(S).$$

If a coherent complex K on X is extended to $\overline{K}$ on S , its image is “ $\varprojlim$ ” $\mathcal{J}^n \overline{K}$, where $\mathcal{J}$ defines $S \setminus X$.

For a composite $f = gh$, with g proper and h an open immersion, set $\mathbb{R}f_! = \mathbb{R}g_! \mathbb{R}h_!$. One must verify:

Independence of the compactification.

Consider a commutative diagram

$$\begin{array}{ccc} & & X' \\ & \nearrow j' & \downarrow g \\ X & & \\ & \searrow j'' & \downarrow \\ & & X'' \\ & \nwarrow \bar{f} & \\ & S & \end{array}$$

where j' and j'' are open immersions, and $\bar{f}$ and g are proper maps. We have $j'X = g^{-1}j''X \cap j'X$. We must prove that

$$\mathbb{R}\bar{f}_* \mathbb{R}j'_! = \mathbb{R}(\bar{f}g)_* \mathbb{R}j'_!,$$

or equivalently, since $\mathbb{R}(\bar{f}g)_* = \mathbb{R}\bar{f}_* \mathbb{R}g_*$, that

$$\mathbb{R}j''_! = \mathbb{R}g_* \mathbb{R}j'_!.$$

One reduces to the case where X is dense in X' , so as to be in the situation of Proposition 5. Let K be a bounded coherent complex on X , extended to $\overline{K}$ on X' . Then $g_* \overline{K}$ extends K on X'' , and one has a map

$$(1) \quad g_* \mathcal{J}'^n \overline{K} \rightarrow \mathbb{R}g_* \mathcal{J}'^n \overline{K},$$

where $\mathcal{J}'$ denotes a sheaf of ideals defining $X' \setminus X$. Proposition 5(ii) shows that $j'_! K = “\varprojlim” g_* \mathcal{J}'^n \overline{K}$. Passing to the “limit”, one obtains

$$“\varprojlim” \mathbb{R}^p g_* \mathcal{J}'^n \overline{K}^q \implies “\varprojlim” \mathbb{R}^{p+q} g_* \mathcal{J}'^n \overline{K}.$$

Proposition 5(i) shows that this spectral sequence degenerates, and Proposition 3 implies that the arrow deduced from (1) by passing to the “limit” is an isomorphism. This gives the desired formula.

It remains to verify the following.

(a) For every diagram of the form

$$\begin{array}{ccccc}
 & & & X' & \\
 & & \nearrow j' & \downarrow h & \\
 X & \xrightarrow{\quad} & X'' & \downarrow g & \\
 & \searrow j'' & & & \\
 & & \searrow j''' & \bar{X} & \\
 \downarrow f & & & \nwarrow \bar{f} & \\
 S & & & &
 \end{array}$$

one has a compatibility

$$\begin{aligned}
 \mathbb{R}\bar{f}_* \mathbb{R}j'_! &= \mathbb{R}\bar{f}_* \mathbb{R}g_* \mathbb{R}j''_! \\
 &= \mathbb{R}\bar{f}_* \mathbb{R}g_* \mathbb{R}h_* \mathbb{R}j'_! \\
 &= \mathbb{R}\bar{f}_* \mathbb{R}(gh)_* \mathbb{R}j'_! \\
 &= \mathbb{R}f_* \mathbb{R}j'''_!.
 \end{aligned}$$

Since two compactifications can always be dominated by a third, for example $X' \times_S X''$, this is enough to prove independence of the choices.

(b) For every compactifiable pair, an identification

$$\mathbb{R}(fg)_! = \mathbb{R}f_! \mathbb{R}g_!.$$

(c) For every compactifiable triple, a compatibility

$$\begin{array}{ccc}
 \mathbb{R}(fgh)_! & = & \mathbb{R}(fg)_! \mathbb{R}h_! \\
 \parallel & & \parallel \\
 \mathbb{R}f_! \mathbb{R}(gh)_! & = & \mathbb{R}f_! \mathbb{R}g_! \mathbb{R}h_!
 \end{array}$$

One would then have proved:

Theorem 1. *For $f : X \rightarrow Y$ compactifiable, $K \in \text{pro } D_{\text{coh}}^b(X)$ and $L \in \text{pro } D_{\text{coh}}^b(Y)$, put*

$$\text{Hom}_f(K, L) = \text{Hom}_Y(\mathbb{R}f_! K, L).$$

Modulo compactifiability questions, this makes the categories

$$\text{pro } D_{\text{coh}}^b(X)$$

into a cofibrant category over noetherian preschemes, with the arrows compactifiable.

NO. 4. DEFINITION OF $\mathbb{R}f^!$.

It follows from a general theorem of Verdier [1] that the functor $\mathbb{R}f_* : D(X) \rightarrow D(S)$ has a right adjoint $D^+(S) \rightarrow D^+(X)$. It deserves a name, say $\mathbb{R}f^!$, only when f is proper, and its definition is not directly suited for computation.

First, one must make explicit a procedure for computing $\mathbb{R}f_*$ at the level of complexes, of the form

$$\mathbb{R}f_*K = f_*C^*(K),$$

where C^* is a finite acyclic resolution depending functorially and exactly on the object to which it is applied, and compatible with filtered inductive limits. Then, for every injective I on S , the functor $\text{Hom}(f_*C^p\mathcal{F}, I)$ is exact in the quasi-coherent sheaf $\mathcal{F}$ on X and transforms $\varinjlim$ into $\varprojlim$; hence it is representable by $f_p^!I$, an injective quasi-coherent sheaf. The $f_p^!I$ form a complex, which defines $\mathbb{R}f^! : D^+S \rightarrow D^+X$, right adjoint to $\mathbb{R}f_* : D(X) \rightarrow D(S)$.

Here is how to define such a resolution.

1) If S is separated: take a finite covering of X by opens affine over S , for instance affine opens, and put $\mathbb{R}f_*K = f_*\check{C}(\mathcal{U}, K)$, the alternating Čech complex.

2) Otherwise: the canonical flasque resolutions have two defects here.

(a) $f_*C^*\mathcal{F}$ is no longer quasi-coherent. This causes no problem as long as, as here, quasi-coherent injectives are injective as sheaves.

(b) C^* does not commute with filtered inductive limits. This defect is easy to correct: write a quasi-coherent sheaf as $\mathcal{F} = \varinjlim \mathcal{F}_i$, with the $\mathcal{F}_i$ of finite presentation, and put $C^*\mathcal{F} = \varinjlim C^*\mathcal{F}_i$. This does not depend on the choices.

For an open immersion, take $\mathbb{R}f^!K = f^*K$.

For a composite morphism $f = gh$, with g proper and h an open immersion, take $\mathbb{R}f^! = \mathbb{R}g^!\mathbb{R}h^!$. Combining Proposition 4 with the preceding construction gives:

Theorem 2 (Duality). *Let $f : X \rightarrow S$ be compactifiable, $f = gh$. The functors*

$$\mathbb{R}f_! : \text{pro } D_{\text{coh}}^b(X) \rightarrow \text{pro } D_{\text{coh}}^b(S) \quad \text{and} \quad \mathbb{R}f^! : D^+(S) \rightarrow D^+(X)$$

are adjoint to one another.

One should verify the independence of the compactification and a formalism of compatibilities and identifications analogous to the case of $\mathbb{R}f_!$. Particular cases of the formula sheet follow from the adjunction formula, by uniqueness of a genuine adjoint.

In the general case one may use the following proposition.

Proposition 6. *Let $f : X \rightarrow S$ be compactifiable and let $L \in D_{\text{qc}}^+(S)$. The fiber at $x \in X$ of $\mathcal{H}^p \mathbb{R}f^! L$ is given by*

$$(\mathcal{H}^p \mathbb{R}f^! L)_x = \varinjlim_{x \in U} \text{Hom}_S^p(\mathbb{R}f_!(U, \mathcal{O}_U), L).$$

One may assume that X is proper and that L is given as a complex of injective quasi-coherent sheaves. Let U be an affine open of X , and let $\mathcal{J}$ be an ideal defining $X \setminus U$. Then

$$\begin{aligned} (\mathcal{H}^p \mathbb{R}f^! L)(U) &= \mathbb{H}^p(\mathbb{R}f^! L(U)) = \mathbb{H}^p \text{Hom}_U(\mathcal{O}, \mathbb{R}f^! L) \\ &= \mathbb{H}^p \varinjlim \text{Hom}_X(\mathcal{J}^n, \mathbb{R}f^! L) = \varinjlim \text{Hom}_S^p(\mathbb{R}f_* \mathcal{J}^n, L) \\ &= \text{Hom}_S^p(\mathbb{R}f_!(U, \mathcal{O}_U), L), \end{aligned}$$

and the proposition follows.

We shall assume for $\mathbb{R}f^!$ a variance formalism analogous to that of Theorem 1 for $\mathbb{R}f_!$.

If one wants to make the definition of $\mathbb{R}f^!$ more explicit in the proper case, one can observe that if a quasi-coherent sheaf $\mathcal{F}$ represents a functor F , then for every open U and every $\mathcal{J}$ defining $X \setminus U$, one has

$$\mathcal{F}(U) = \varinjlim \text{Hom}(\mathcal{J}^n, \mathcal{F}) = \varinjlim F(\mathcal{J}^n).$$

NO. 5. COMPUTATION OF $\mathbb{R}f^!$.

For a finite morphism, $\mathbb{R}f^!$ is what one wants; to see this, it is enough to take the identity as resolving functor in the computation of $\mathbb{R}f_*$. For projective space, the uniqueness of an adjoint leaves no choice; one uses the fact that the adjunction formula is available for $\mathbb{R}f_! = \mathbb{R}f_* : D(X) \rightarrow D(S)$. We shall return to the arrow. For an open immersion, $\mathbb{R}f^! = \mathbb{R}f^*$.

Thus for every quasi-projective open U of X over S , one can compute the restriction of $\mathbb{R}f^!$ to U . In particular, if one can check locally that $\mathbb{R}^q f^! K = 0$ for $q \neq p$, then one knows $\mathbb{R}f^! K$, since an object of the derived category with only one non-zero cohomology sheaf is determined by that sheaf. Recall an important case where this condition is satisfied, with $K = \mathcal{O}$.

Proposition 7. *Let $f : X \rightarrow Y$ be compactifiable. Suppose that locally on X and on Y one can find a diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow g & \swarrow h \\ & S & \end{array}$$

where g and h are locally complete intersections of relative dimensions d' and d'' , and let $d = d' - d''$.

Then $\mathbb{R}f^!\mathcal{O}$ is reduced to a single cohomology sheaf, situated in degree $-d$, and that sheaf is invertible.

Indeed, $\mathbb{R}g^!\mathcal{O} = \mathbb{R}f^!\mathbb{R}h^!\mathcal{O}$. We know, by local calculations in projective space, that $\mathbb{R}g^!\mathcal{O}$ and $\mathbb{R}h^!\mathcal{O}$ are of the indicated type, situated in degrees $-d'$ and $-d''$. Since everything is local, one may suppose that $\mathbb{R}^{-d''}h^!\mathcal{O}$ is isomorphic to $\mathcal{O}$, and the assertion follows.

To pass from this to more general complexes, one would first have to prove formulas of the form

$$\mathbb{R}f^!(K \overset{\mathbb{L}}{\otimes} L) = \mathbb{R}f^!(K) \overset{\mathbb{L}}{\otimes} \mathbb{L}f^*L.$$

Only the definition of the arrow is problematic; its being an isomorphism is a local question. Of course boundedness restrictions are needed. I have checked none of the following in detail.

First method for defining the arrow.

Assume that K and L lie in $D_{\text{coh}}^b(S)$, not merely in the quasi-coherent setting, and also that

- (a) $\mathbb{R}f^!K$ is bounded, it being automatically coherent, which is a local problem;
- (b) $K \overset{\mathbb{L}}{\otimes} L$ is bounded;
- (c) $\mathbb{R}f^!K \overset{\mathbb{L}}{\otimes} \mathbb{L}f^*L$ is bounded.

To define

$$\mathbb{R}f^!K \overset{\mathbb{L}}{\otimes} \mathbb{L}f^*L \rightarrow \mathbb{R}f^!(K \overset{\mathbb{L}}{\otimes} L),$$

it is enough by duality to define

$$\mathbb{R}f_!(\mathbb{R}f^!K \overset{\mathbb{L}}{\otimes} \mathbb{L}f^*L) \rightarrow K \overset{\mathbb{L}}{\otimes} L.$$

By Kunneth,

$$\mathbb{R}f_!(\mathbb{R}f^!K \overset{\mathbb{L}}{\otimes} \mathbb{L}f^*L) = \mathbb{R}f_!\mathbb{R}f^!K \overset{\mathbb{L}}{\otimes} L,$$

and the required arrow is induced by the adjunction arrow $\mathbb{R}f_!\mathbb{R}f^!K \rightarrow K$.

Second method.

Assume that f is compactifiable, flat, and locally a relative complete intersection. Then $\mathbb{R}f^!\mathcal{O} = \omega[d]$, where ω is invertible. Let $\mathbb{R}f_!$ denote a procedure for computing $\mathbb{R}f_!$ at the level of complexes, namely such a procedure for a compactification $\bar{f} : \bar{X} \rightarrow S$ of f , assumed to be of

the type described in No. 4 and such that $C^p\mathcal{F}$ is zero for $p > d$, which can be arranged by truncation. Then there is an arrow

$$\mathbb{R}^d \bar{f}_* \mathcal{F}[-d] \rightarrow \mathbb{R} \bar{f}_* \mathcal{F} \quad \text{for every } \mathcal{F} \text{ on } \bar{X},$$

since the last cohomology sheaf maps to its complex. From this one obtains morphisms of complexes

$$\text{Hom}_S(\mathbb{R} \bar{f}_* \mathcal{F}, I) \rightarrow \text{Hom}_S(\mathbb{R}^d \bar{f}_* \mathcal{F}[-d], I) \quad \text{for every complex } I \text{ on } S.$$

Let I be injective. The functor $\mathbb{R}^d \bar{f}_* \mathcal{F}[-d]$ is right exact in $\mathcal{F}$, so the functor in $\mathcal{F}$ on the right of the arrow is representable. Define I_1 by

$$\text{Hom}_S(\mathbb{R}^d \bar{f}_* \mathcal{F}, I) = \text{Hom}_{\bar{X}}(\mathcal{F}, I_1) \quad \text{and} \quad \mathbb{R} f_1^! I = I_1[d].$$

The preceding construction defines

$$\mathbb{R} f^! I \rightarrow \mathbb{R} f_1^! I,$$

and, for a complex of injectives, $\mathbb{R} f_1^!$ is defined term by term.

It remains to evaluate $\mathbb{R} f_1^! I$ on affine opens U of X . It is given by $\varinjlim \text{Hom}(\mathbb{R}^d f_! \mathcal{J}^n, I)$. Since this problem is local upstairs, it is easy to show by projective methods that the dual of the ind-object

$$“\varinjlim” \mathbb{R}^d f_! \mathcal{J}^n = \mathbb{R}^d f_!(U, \mathcal{O})$$

is $f_*(U, \omega)$, represented as the limit of its submodules of sections having a pole of order at least n outside U , with $n \rightarrow \infty$.

This gives

$$\mathbb{R} f_1^! I = \omega \otimes f^* I[d],$$

and the general formula

$$\mathbb{R} f^! K = f^* K \otimes \omega[d].$$

BIBLIOGRAPHY

- [1] J. L. Verdier, Seminaire Bourbaki, November 1965, expose 300.
- [2] M. Nagata, Imbedding of an abstract variety in a complete variety. *Journal of Math. of Kyoto University*, vol. 2, no. 1, 1962.