where $$\mu^2\lambda k^n = 1,$$

and, consequently, $$\mu^2 k^{2t} = \frac{1}{\lambda k^{n-2t}};$$

$$-nkk'^2\frac{d}{dk}\log(\mu^2 k^{2t}) = nkk'^2\frac{d}{dk}\log(\lambda k^{n-2t}) = \frac{nkk'^2}{\lambda}\frac{d\lambda}{dk}+n(n-2t)k'^2$$
$$= M^{-2}\lambda'^2+n(n-2t)k'^2.$$

It thus appears that (2) is transformed into

$$(n-t+2)(n-t+1)\,a_{n-t+2}+2nkk'^2\frac{da_{n-t}}{dk}$$
$$-\left\{2a_2+(n-t)^2-(n^2-t^2)k^2\right\}a_{n-t}+(t+2)(t+1)\,k^2 a_{n-t-2}=0.$$

Now, if we write t for $n-t$, this is

$$(t+2)(t+1)\,a_{t+2}+2nkk'^2\frac{da_t}{dk}-\left\{2a_2+t^2-t(2n-t)k^2\right\}a_t$$
$$+(n-t+2)(n-t+1)\,k^2 a_{t-2}=0;$$

in other words (2) can be transformed into (1).

Further Note on Automorphic Functions. By W. BURNSIDE.

Received and read June 9th, 1892.

I propose in the present paper to continue the consideration of certain groups, and the automorphic functions connected with them, with which I dealt shortly in a previous paper, published in the current volume of the Society's *Proceedings*. In that paper it was shown that, for symmetrical fuchsian groups of the first class, automorphic functions taking every value twice only in the generating (or any) polygon can always be found, and therefore that the algebraic equation connecting two different automorphic functions is in this case an equation of the hyperelliptic class.

I shall here show how, when the group is given, to calculate the coefficients in the equation, incidentally expressing the two variables which the equation connects as uniform functions of a single parameter.

Herr H. Weber, in a paper published in the *Göttingen Nachrichten*, 1886, with the title "Ein Beitrag zu Poincaré's Theorie der Fuchs'schen Functionen," has already dealt with the problem of expressing the variables in a hyperelliptic equation as uniform functions of a parameter, and the mode of expression obtained in this paper is not essentially different from M. H. Weber's; still, though in this point possessing no novelty, the forms of the expressions here obtained are both infinite series and infinite products, while M. Weber's functions are all expressed in the form of infinite products, and moreover the group on which the functions depend is simpler than (in fact is a sub-group of) the group from which M. Weber starts.

The latter group is defined as follows:—

Take n circles with their centres lying on the real axis, and such that each one is outside all the others. Then n elliptic substitutions of period 2, with the n pairs of points where the circles meet the axis as their double points, are the generating substitutions of M. Weber's group. Any substitution of the group can therefore be represented by an even number of inversions at the system of $n+1$ circles formed by the n given circles and the axis of x. The symmetrical fuchsian group whose substitutions are the equivalents of an even number of inversions at the n circles is immediately seen to be a self-conjugate sub-group of the former, and it is on this group that the functions involved really depend.

After obtaining the equation connecting two independent automorphic functions, I go on to determine explicitly the relations between the integrals of the first species to which the equation gives rise, and the ϕ-functions of the corresponding group.

In the second paragraph, applying directly the definition which Prof. Klein gives in dealing with Abelian functions of a new element which he has introduced into the theory, I have shown how to express any automorphic function as the product of a number of factors which bear the same relation to it that a linear factor bears to a rational function.

Lastly, the relations between a symmetrical fuchsian group and its simplest sub-group (which is also symmetrical), as well as those between the functions connected with the two groups, are investigated; and, in the particular case when the hyperelliptic integrals reduce to elliptic by a quadratic substitution, the form of the generating polygon is determined.

1. *On the Algebraic Equation connecting Two Independent Automorphic Functions.*

I suppose for the present that the symmetrical fuchsian group is defined by $n+1$ circles $C_0, C_1, C_2 \dots C_n$, each outside all the others and with their centres on the axis of x. If C_p be taken also to represent an inversion with respect to the p^{th} circle, the n fundamental substitutions of the group may be written

$$S_p = C_p C_0 \qquad (p = 1, 2, \dots n).$$

If α, β are the double points of S_p, and α_i, β_i the homologues of α, β with respect to any substitution of the group, it was shown in my former paper that

$$\theta(z, J_p) = \sum_i \left(\frac{1}{z-\alpha_i} - \frac{1}{z-\beta_i}\right),$$

the summation extending to all *different* pairs of points α_i, β_i.

If α_i, β_i are the homologues of α, β for the substitution S_i, and α_j, β_j, for the substitution $C_0 S_i C_0$, which is clearly also a substitution of the group, then, since C_0 changes α into β, and S_i changes β into β_i, it follows that α_j and β_i are inverse points with respect to C_0, as also are β_j and α_i; hence, pairing the substitutions in this way,

$$\theta(z, J_p) = \frac{1}{z-\alpha} - \frac{1}{z-\beta} + \Sigma' \left[\frac{1}{z-\alpha_i} - \frac{1}{z-\beta_j} + \frac{1}{z-\alpha_j} - \frac{1}{z-\beta_i}\right].$$

Now, if P, Q are inverse points on a diameter MN of a circle, it may be easily verified that

$$\int_M^N \left(\frac{1}{z-P} - \frac{1}{z-Q}\right) dz = \pm \pi i,$$

where the integration is taken along the semicircle from M to N.

Hence, if a_0, b_0 be the points where C_0 meets the real axis, the above form of $\theta(z, J_p)$, taken in conjunction with the previous result that the integral is always a finite quantity, gives

$$\int_{a_0}^{b_0} \theta(z, J_p)\, dz = (2m+1)\pi i,$$

where m is an integer, the integration being conducted along the circle C_0; and therefore

$$\exp\left[\phi_p(b_0)-\phi_p(a_0)\right]=-1,$$

where S_p is any one of the fundamental substitutions.

If, instead of taking for the generating polygon the region external to $C_1, C_2, \dots C_n$, and to their inverses in C_0, there be taken the region external to $C_0, C_1, \dots C_{i-1}, C_{i+1} \dots C_n$ and to their inverses in C_i, the group remains the same, while C_i takes the place of C_0 in the previous investigation, and the substitutions

$$C_pC_i \quad (p=0, 1, \dots i-1, i+1, \dots n)$$

must be regarded as the fundamental substitutions. Expressed in terms of the former fundamental substitutions, these are

$$S_i^{-1},\ S_1S_i^{-1} \dots S_{i-1}S_i^{-1},\ S_{i+1}S_i^{-1} \dots S_nS_i^{-1}.$$

If, then, a_i, b_i are the points where C_i meets the real axis, and J is the homologue of infinity for any one of the n substitutions just written,

$$\int_{a_i}^{b_i}\theta(z, J)\,dz=(2m+1)\,\pi i,$$

m again being an integer. Now it has been proved before that

$$\theta(z, J_{p^{-1}q})=\theta(z, J_q)+\theta(z, J_{p^{-1}}).$$

Hence, if $q\neq i$,

$$\theta(z, J_q)=\theta(z, J_{i^{-1}q})-\theta(z, J_{i^{-1}}),$$

and

$$\int_{a_i}^{b_i}\theta(z, J_q)\,dz=2m'\pi i;$$

while

$$\int_{a_i}^{b_i}\theta(z, J_{i^{-1}})\,dz=-\int_{a_i}^{b_i}\theta(z, J_i)\,dz=(2m''+1)\,\pi i.$$

Hence, finally,

$$\exp\left[\phi_i(b_i)-\phi_i(a_i)\right]=-1,$$

$$\exp\left[\phi_q(b_i)-\phi_q(a_i)\right]=1, \qquad q\neq i.$$

It will be convenient to use the abbreviation z_i for $\dfrac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}$. With this notation it was shown in the previous paper that

$$\chi_{a,b}(z_p)-\chi_{a,b}(z)=\phi_p(b)-\phi_p(a),$$

and now, if for a, b in this formula a_0, b_0, a_1, b_1, &c. be successively written, there results

$$\exp\chi_{a_0,b_0}(z_p)=-\exp\chi_{a_0,b_0}(z),\quad p=1,2\ldots n,$$

and $$\exp\chi_{a_i,b_i}(z_p)=\exp\chi_{a_i,b_i}(z),\quad p=1,2\ldots i-1,i+1\ldots n,$$

$$\exp\chi_{a_i,b_i}(z_i)=-\exp\chi_{a_i,b_i}(z).$$

Hence $\exp 2\chi_{a_i,b_i}(z)$ is an automorphic function, and, since in the generating polygon it has a single double infinity at b_i and a single double zero at a_i, its square root $\exp\chi_{a_i,b_i}(z)$ is the analogue of a hyperelliptic function.

Also the above formulæ show that

$$\prod_0^n \exp\chi_{a_i,b_i}(z)$$

is unaltered by the fundamental substitutions, and hence this also is an automorphic function.

I return now to functions in the form of infinite series, and, to simplify the analysis as far as possible, I suppose the circle C_0 to be the axis of y, which, as I have shown, involves no real loss of generality.

Then, as shown on pp. 76, 77 of my former paper,

$$x_a=\psi_a(z)-\psi_{-a}(z)+\text{constant}$$

is an automorphic function which takes every value twice in the generating polygon. It will first be proved directly that between any two such functions a lineo-linear equation holds.

Replacing $\psi_a(z)$ and $\psi_{-a}(z)$ by the forms given on p. 68, and omitting a constant term, which if necessary can be replaced in the result,

$$x_a=\theta(a,z)-\theta(-a,z)$$

$$=\theta(a,z)+\theta(a,-z).$$

Hence

$$x_ax_b=\sum_i\sum_j(\gamma_ia+\delta_i)^{-2}(\gamma^jb+\delta_j)^{-2}\left[\frac{1}{a_i-z}+\frac{1}{a_i+z}\right]\left[\frac{1}{b_i-z}+\frac{1}{b_i+z}\right].$$

On resolving the product of the two brackets into partial fractions, the term depending on $\dfrac{1}{a_i - z}$ is

$$\sum_i \sum_j (\gamma_i a+\delta_i)^{-2} (\gamma_j b+\delta_j)^{-2} \frac{1}{a_i - z} \left[\frac{1}{b_j - a_i} + \frac{1}{b_j + a_i}\right]$$

$$= \sum_i \frac{(\gamma_i a+\delta_i)^{-2}}{a_i - z} \left[\theta(b, a_i) + \theta(b, -a_i)\right]$$

$$= \sum_i \frac{(\gamma_i a+\delta_i)^{-2}}{a_i - z} \left[\theta(b, a) + \theta(b, -a)\right]$$

$$= \theta(a, z) \left[\theta(b, a) + \theta(b, -a)\right].$$

Hence

$$x_a x_b = x_a \left[\theta(b, a) + \theta(b, -a)\right] + x_b \left[\theta(a, b) + \theta(a, -b)\right],$$

or

$$x_b = \frac{x_a \left[\theta(b, a) + \theta(b, -a)\right]}{x_a - \left[\theta(a, b) + \theta(a, -b)\right]}.$$

Now

$$x_a = 2\sum_i \frac{(\gamma_i a+\delta_i)^{-2} a_i}{a_i^2 - z^2},$$

so that x_a has a double zero at $z = \infty$, and, taking account of this, the above relation between x_a and x_b is what would have been obtained by assuming the one a linear function of the other from other considerations, and determining the constants from the zero and infinities.

The two infinities of x_a are in general distinct, since, if a is in the generating polygon, $-a$ is generally a different point also in the generating polygon. But, of the two circles C_p and C'_p, only one can be reckoned as belonging to the generating polygon, and hence, if a coincides with one of the points a_i or b_i, the two separate infinities of x_a coalesce into one double infinity. From this it follows that x_{b_i}/x_{a_i} can only differ from $\exp 2\chi_{a_i, b_i}(z)$ by a constant factor. Since C_0 is here taken for the axis of y, b_0 and a_0 are 0 and ∞.

Now, when a diminishes without limit,

$$\text{Lt. } x_a = \text{Lt.} \sum_i \left(\frac{(\gamma_i a+\delta_i)^{-2}}{\dfrac{a_i a+\beta_i}{\gamma_i a+\delta_i} - z} - \frac{(\gamma_i a-\delta_i)^{-2}}{\dfrac{a_i a-\beta_i}{\gamma_i a-\delta_i} - z} \right)$$

$$= \text{Lt.} \sum_i \left(\frac{1}{a - \dfrac{\delta_i z - \beta_i}{-\gamma_i z + a_i}} - \frac{1}{a + \dfrac{\delta_i}{\gamma_i}} + \frac{1}{a + \dfrac{\delta_i z - \beta_i}{-\gamma_i z + a_i}} - \frac{1}{a - \dfrac{\delta_i}{\gamma_i}} \right)$$

$$= 2a \sum_i \left[\left(\frac{\gamma_i}{a_i}\right)^2 - \left(\frac{\gamma_i z + \delta_i}{a_i z + \beta_i}\right)^2 \right];$$

and therefore $$x = \sum_i \left[\left(\frac{\gamma_i}{\alpha_i} \right)^2 - \left(\frac{\gamma_i z + \delta_i}{\alpha_i z + \beta_i} \right)^2 \right]$$

is an automorphic function taking every value twice, with a double infinity at the origin and a double zero at infinity; and if the value of the argument is expressed by writing $x(z)$, it follows that

$$\exp 2\chi_{\infty, 0}(z) = Cx,$$

and $$\exp 2\chi_{a_i, b_i}(z) = C' \frac{x - x(a_i)}{x - x(b_i)},$$

where C, C' are constants.

Finally, then, the two linearly-independent automorphic functions x and y, where

$$y = \exp \prod_0^n \chi_{a_i, b_i}(z),$$

are connected by the equation

$$y^2 = x \prod_1^n \frac{x - x(a_i)}{x - x(b_i)},$$

where the constants are directly calculable by using particular values of z in the series

$$\sum_i \left[\left(\frac{\gamma_i}{\alpha_i} \right)^2 - \left(\frac{\gamma_i z + \delta_i}{\alpha_i z + \beta_i} \right)^2 \right].$$

It is not without interest now to verify directly that the n independent hyperelliptic integrals of the first kind connected with the last equation are identical with n linearly independent ϕ-functions of the group.

The integrals may be written

$$\int \frac{dx}{[x - x(b_i)]\, y} \quad (i = 1.2 \ldots n),$$

and it is therefore to be shown that

$$\frac{1}{[x - x(b_i)]\, y} \frac{dx}{dz}$$

can be expressed in the form

$$\sum_{i=1}^{i=n} c_i \theta(z, J_i).$$

For this purpose it is necessary to determine the zeros and infinities of $\frac{dx}{dz}$.

Now, if a, b are any one of the pairs of points a_i, b_i, then

$$\frac{x-x(a)}{x-x(b)} = \exp 2\chi_{a,b}(z);$$

and therefore

$$\left[x(a)-x(b)\right]\frac{dx}{dz}$$
$$= \left[x-x(a)\right]\left[x-x(b)\right]\sum_i (\gamma_i z+\delta_i)^{-2}\left\{\frac{1}{z_i-a}-\frac{1}{z_i-b}\right\}.$$

When $z = a$, $x-x(a)$, considered as a function of z, has a double zero; and therefore $\frac{dx}{dz}$ has single zeros at the points a, b.

Also, since
$$\frac{dx}{dz} = \sum_i \frac{(\gamma_i z+\delta_i)^{-2}}{z_i^3},$$

it has a triple infinity at $z = 0$, and this is its only independent infinity; and, since x itself has a double zero at infinity, $\frac{dx}{dz}$ must have a triple zero there. This enumeration is complete, for it was shown, pp. 78, 79 of my former paper, that the number of zeros of any function of the form

$$\sum_i f(z_i)(\gamma_i z+\delta_i)^{-2}$$

in the generating polygon exceeded that of the infinities by $2n$, and while $\frac{dx}{dz}$ has a single triple infinity it has also been shown to have a triple zero and $2n$ simple zeros at the points a_1, b_1 ... a_n, b_n.

Considered as a function of z,

$$\frac{1}{\left[x-x(b_i)\right]y}$$

or
$$\frac{\left[x-x(b_1)\right]\ldots\left[x-x(b_{i-1})\right]\left[x-x(b_{i+2})\right]\ldots\left[x-x(b_n)\right]}{\sqrt{x\left[x-x(a_1)\right]\left[x-x(b_1)\right]\ldots\left[x-x(a_n)\right]\left[x-x(b_n)\right]}}$$

has simple infinities at a_1, a_2 ... a_n, b_i and ∞, simple zeros at b_1 ... b_{i-1}, b_{i+1} ... b_n, and finally a triple zero at 0.

Hence $\frac{1}{\left[x-x(b_i)\right]y}\frac{dx}{dz}$ has no infinity, and double zeros at the points $b_1 \dots b_{i-1}, b_{i+1} \dots b_n, \infty$, that is to say the equivalent of $2n$ simple zeros in the generating polygon; moreover, as already shown, it is of the form

$$\sum_i f(z_i)(\gamma_i z+\delta)^{-2}.$$

It therefore only remains to be shown that the constants C can be so chosen that

$$\sum_1^n C_p \theta(z, J_p),$$

which necessarily has a double zero at infinity, shall have $n-1$ other double zeros at the points $b_1, b_2 \dots b_{i-1}, b_{i+1} \dots b_n$.

This is immediately obvious; for, if $\theta(z, J)$ has a zero at z_0 in the generating polygon, it necessarily has a second at $-z_0$. Now if z_0 approaches one of the points b, a homologue of $-z_0$ will also approach b, and in the limit the two zeros will coincide and form a double zero. Hence, if the $n-1$ ratios of the constants are so determined that the expression in question vanishes at b_1, &c., it will necessarily have double zeros at these points.

Finally, then,

$$\frac{1}{\left[x-x(b_i)\right]y}\frac{dx}{dz} = \text{const.} \times \begin{vmatrix} \theta(z, J_1), & \theta(z, J_2), & \dots\dots & \theta(z, J_n) \\ \theta(b_1, J_1), & \dots\dots & \dots\dots & \theta(b_1, J_n) \\ \dots & \dots & \dots & \dots \\ \theta(b_{i-1}, J_1), & \dots\dots & \dots\dots & \theta(b_{i-1}, J_n) \\ \theta(b_{i+1}, J_1), & \dots\dots & \dots\dots & \theta(b_{i+1}, J_n) \\ \dots & \dots & \dots & \dots \\ \theta(b_n, J_1), & \dots\dots & \dots\dots & \theta(b_n, J_n) \end{vmatrix}$$

Corresponding relations can obviously be obtained between the integrals of the second and third species and the corresponding z-functions.

2. *On the Prime Factors of an Automorphic Function.*

In his recent memoirs on hyper-elliptic and Abelian functions, Prof. F. Klein has introduced, as a new element, an expression to which he gives the name "*Primform.*"

He defines it as follows. Let $w_1, w_2, \dots w_n$, be the n everywhere finite integrals connected with a given equation of deficiency n (or on a given $\overline{2n+1}$-ply connected Riemann's surface), and let $\phi_1, \phi_2, \dots \phi_n$ be n finite (*i.e.*, not infinitesimal) quantities such that

$$\frac{dw_1}{\phi_1} = \frac{dw_2}{\phi_2} = \dots = \frac{dw_n}{\phi_n}.$$

Then, if $d\omega$ is the common value of these fractions, and if $P_{x, y}^{\xi, \eta}$ is any integral of the third kind on the surface, the "*Primform*" $\Omega(x, y)$ is defined by the equation

$$\Omega(x, y) = \text{Lt.}_{dx=0, dy=0} \sqrt{d\omega_x d\omega_y \exp\left(-P_{x, y}^{x+dx, y+dy}\right)},$$

x, y denoting any two points on the Riemann's surface.

$\Omega(x, y)$ vanishes only when the point x coincides with the point y, and does not become infinite for any position of the points.

It is the exact analogue for the multiply-connected surface of the form

$$x_1 y_2 - x_2 y_1$$

for a simply-connected surface, where x_1/x_2 is any function of position on such a simply connected surface, which takes every value once only; and just as any uniform function on the simply-connected surface can be expressed in the form

$$\prod_{y, y'} \frac{x_1 y_2 - x_2 y_1}{x_1 y_2' - x_2 y_1'},$$

so any uniform function on the multiply-connected surface can be expressed in the form

$$\prod_{y, y} \frac{\Omega(x, y)}{\Omega(x, y')}.$$

In his memoir of the theory of Abelian functions (*Math. Ann.*, Vol. XXXVI.), Prof. Klein refers to a paper by Herr Schottky in *Crelle's Journal*, Vol. CI., where a precisely similar expression is introduced in connexion with the theory of automorphic functions. From the point of view taken by Herr Schottky, the close parallel that exists between the functions he deals with and the analogous functions of position on a Riemann's surface, are lost sight of.

I propose then, here, to take Prof. Klein's definition, and from that directly to construct and investigate some of the properties of the functions (and forms) on the z-plane that are equivalent to his *Primformen* and the derivable prime-functions on the surface.

It has been shown, p. 69 of my former paper, that the function

$$\chi_{a,b}(x, y) = \sum_i \log \frac{(z_i-a)(y_i-b)}{(z_i-b)(y_i-a)}$$

is the equivalent on the z-plane of the integral of the third kind on the Riemann's surface.

If the suffix refer to all substitutions except the identical one, this may be written

$$\exp \chi_{a,b}(z, y) = \frac{(z-a)(y-b)}{(z-b)(y-a)} \prod_i \frac{(z_i-a)(y_i-b)}{(z_i-b)(y_i-a)};$$

or, since $\chi_{a,b}(z, y) = -\chi_{b,a}(z, y)$,

$$(z-b)(y-a) \exp\left\{-\chi_{b,a}(z, y)\right\} = (z-a)(y-b) \prod_i \frac{(z_i-a)(y_i-b)}{(z_i-b)(y_i-a)}.$$

If now a approaches y, and b approaches z, the right-hand side remains finite, and also in the limit becomes a perfect square, for it may be at once verified that

$$\frac{(z_i-y)(y_i-z)}{(z_i-z)(y_i-y)} = \frac{(z_{-i}-y)(y_{-i}-z)}{(z_{-i}-z)(y_{-i}-y)}.$$

Hence the limiting form of the previous equation is

$$\mathrm{Lt}._{\substack{a=y \\ b=z}} \sqrt{(z-b)(a-y) \exp\left\{-\chi_{b,a}(z, y)\right\}} = (z-y) \prod_i \frac{(z_i-y)(y_i-z)}{(z_i-z)(y_i-y)},$$

where now, of each pair of inverse substitutions S and S^{-1}, only one is to be taken in the infinite product.

If now z_1/z_2 and y_1/y_2 be written for z and y, and the equation be multiplied all through by $y_2 z_2$, the left-hand side becomes

$$\mathrm{Lt}._{\substack{dz=0 \\ dy=0}} \sqrt{(z_1 dz_2 - z_2 dz_1)(y_2 dy_1 - y_1 dy_2) \exp\left\{-\chi_{z+dz, y+dy}(z, y)\right\}},$$

which agrees exactly with the expression used by Prof. Klein to define his *Primform*. The *Primform* on the z-plane may then be written

$$(zy) \prod_i \frac{(z_i y)(y_i z)}{(z_i z)(y_i y)},$$

where $(zy) \equiv z_1 y_2 - z_2 y_1,$

and the prime-function will be derived from this by dividing by y_1, z_1. It is to be remembered that in forming the infinite product, S and S^{-1} are not to be regarded as distinct substitutions. It is evidently immaterial for the present purposes whether the form or the function is dealt with, and a symbol $\Omega(z, y)$ will be taken for the function.

It is clear that $\Omega(z, y)$ has a single zero in each polygon, namely the homologue of y in that polygon; and that its only infinity is at infinity; also if z and y are interchanged the function changes sign.

To determine the relation between $\Omega(z_p, y)$ and $\Omega(z, y)$ it is necessary to return to the equation defining the function. This gives

$$\frac{\Omega(z_p, y)}{\Omega(z, y)} = \text{Lt.}_{\substack{a=y\\ b=z}} \sqrt{\frac{(z_p-b_p)(a-y)\exp\{-\chi_{b_p, a}(z_p, y)\}}{(z-b)(a-y)\exp\{-\chi_{b,a}(z, y)\}}}.$$

Now, from p. 69 of the former paper,

$$\chi_{b,a}(z_p, y)-\chi_{b,a}(z, y) = \phi_p(a)-\phi_p(b),$$

and
$$\chi_{b,a}(z, y) = \chi_{z,y}(b, a).$$

Hence
$$\exp\{\chi_{b,a}(z, y)-\chi_{b_p, a}(z_p, y)\}$$
$$= \exp\{\chi_{b,a}(z, y)-\chi_{b,a}(z_p, y)\}$$
$$\times\exp\{\chi_{z_p, y}(b, a)-\chi_{z_p, y}(b_p, a)\}$$
$$= \exp\{\phi_p(b)-\phi_p(a)+\phi_p(z_p)-\phi_p(y)\}.$$

Also,
$$\frac{z_p-b_p}{z-p} = \frac{1}{(\gamma_p z+\delta_p)(\gamma_p b+\delta_p)};$$

and therefore, writing $a = y$ and $b = z$,

$$\frac{\Omega(z_p, y)}{\Omega(z, y)} = (\gamma_p z+\delta_p)^{-1}\exp\tfrac{1}{2}\{\phi_p(z_p)+\phi_p(z)-2\phi_p(y)\}$$
$$= (\gamma_p z+\delta_p)^{-1}\exp\left\{\phi_p(z)-\phi_p(y)+\frac{a_{pp}}{2}\right\}.$$

If z_p be written for z in the product

$$\prod_1^m \frac{\Omega(z, y_r)}{\Omega(z, x_r)},$$

the function is reproduced multiplied by the factor

$$\exp\sum_1^m\left[\phi_p(x_r)-\phi_p(y_r)\right],$$

and if for n of the x's and y's suitably chosen homologues are written, this multiplier will become

$$\exp\left\{\sum_1^m\left[\phi_r(x_r)-\phi_p(y_r)\right]+\sum_1^n n_q a_{pq}\right\},$$

the quantities n_q being any arbitrarily assigned positive or negative integers; but when the x's and y's are the zeros and infinities of an automorphic function (p. 70 of the former paper) the integers n_q can be chosen so that the multipliers become

$$\exp\left[2m_p\pi i\right] \text{ or unity.}$$

Hence any automorphic function can be expressed as a product of prime-functions in the form

$$\prod_1^m \frac{\Omega(z,\,y_r)}{\Omega(z,\,x_r)}.$$

3. *On certain Sub-groups of the Symmetrical Group.*

If to the generating polygon of a symmetrical group be added the polygon into which it is transformed by one of the fundamental substitutions, say S_p, the new figure so formed is bounded by $4n-2$ circles, $2n-1$ of which are the inverses of the other $2n-1$ with respect to C_p, and which will therefore serve as the generating polygon of a new symmetric group. Moreover this new group is clearly a sub-group of the original one. Thus, with the notation of § 7 of my former paper, the fundamental substitutions of the new group are

$$\Sigma_1 = C_1'C_p,\quad \Sigma_2 = C_2'C_p,\ \ldots\ldots\ \Sigma_n = C_n'C_p,$$

$$\Sigma_{n+1} = C_1C_p,\ \ldots\ldots\ \Sigma_{2n+1} = C_nC_p,$$

Σ_{n+p} being the identical substitution.

Now the substitutions of the original group are

$$S_1 = C_1'C_0\,(\text{or } C_0C_1)\ \ldots\ldots\ S_n = C_n'C_0\,(\text{or } C_0C_n),$$

so that $$\Sigma_1 = S_1S_p\ \ldots\ldots\ \Sigma_p = S_p^2,\ \ldots\ldots\ \Sigma_n = S_nS_p,$$

$$\Sigma_{n+1} = S_1^{-1}S_p,\ \ldots\ldots\ \Sigma_{2n+1} = S_n^{-1}S_p.$$

It is also clear that the substitutions of the original group are all given without repetition by the forms Σ and ΣS_p, where Σ represents the totality of the substitutions of the original group.

Using now accents to distinguish functions formed with the substitutions of the sub-group, it follows that

$$\theta(z, z_0) = \sum_i \left[\frac{1}{z - S_i\infty} - \frac{1}{z - S_i z_0}\right]$$

$$= \Sigma_i \left[\frac{1}{z - \Sigma_i\infty} + \frac{1}{z - \Sigma_i S_p\infty} - \frac{1}{z - \Sigma_i z_0} - \frac{1}{z - \Sigma_i S_p z_0}\right]$$

$$= \theta'(z, z_0) + \theta'(z, S_p z_0),$$

and

$$\theta(z, J_q) = \theta(z, z_0) - \theta(z, S_q z_0)$$

$$= \theta'(z, z_0) - \theta'(z, S_p S_q z_0) + \theta'(z, S_p z_0) - \theta'(z, S_q z_0)$$

$$= \theta'(z, J'_q) - \theta'(z, J'_{n+q-1});$$

while

$$\theta(z, J_p) = \theta'(z, J'_p).$$

Hence for the sub-group n linearly independent $\theta'(z, J')$ functions are identical with the n θ-functions of the original group; and this will involve certain simple relations between the corresponding quasi-periods of the ϕ' functions.

If now the original group is also fuchsian, so also is the sub-group. Suppose then that

$$x' = f(z)$$

is an unchanged function for the sub-group, that takes every value twice in its generating polygon. Then

$$f(z) + f(S_p z) \quad \text{and} \quad f(z) f(S_p z)$$

are unchanged functions for the original group, which take every value twice in its generating polygon, and therefore, x being the already used function for the original group, x' is the root of a quadratic equation whose roots are linear in x; or x is a rational function of the second degree of x'.

These considerations may be applied to a particular case of some interest, as follows. Suppose that for a symmetric fuchsian group derived from two fundamental substitutions (and leading therefore to hyperelliptic integrals of the first order), the three circles C_0, C_1, C_2 are symmetrical with respect to a fourth circle C, which must therefore cut one of them, say C_1, at right angles. The group is then a sub-group of that formed by an even number of inversions at the circles C, C_0, and C_1, and the two fundamental substitutions of this group being one hyperbolic and the other elliptic (of period two), the corresponding algebraic equation is of deficiency 1, and the corresponding integrals elliptic. Hence the hyperelliptic integrals to which the group in question leads, are those which can be formed by a quadratic substitution from elliptic integrals. But these are of known form, and hence in this particular case a definite relation is obtained, without further calculation, between the nature of the group and the nature of the algebraic equation to which it leads. As a further verification, it may be very easily shown in this case, from the relation

$$\theta(z, J_p) = \theta'(z, J'_p),$$

proved on the last page, that

$$a_{12} = \tfrac{1}{2}a_{11},$$

which relation between the periods of the hyperelliptic integral involves the possibility of expressing it in terms of elliptic integrals by a quadratic substitution.

Note on Approximate Evolution. *By* H. W. Lloyd Tanner, M.A., F.R.A.S. Received June 6th, 1892. Read June 9th, 1892.

In a paper published in the *Proceedings* of the Society (Vol. xviii., pp. 171–178), Professor Hill has pointed out that the rule (Todhunter's *Algebra*, Art. 246) for contracting the process of finding the square and cube roots of a number, is incorrect in some cases. It is desirable to have a practical test for distinguishing the cases in which the rule is available from those in which it fails. Such a test is obtained by a slight modification of Todhunter's discussion (*loc. cit.*), which enables us also to state two limits between which the required root must lie.

Square Root.

It is convenient to take the decimal point in N, the number whose square root is to be found, so that the integral part of $\sqrt{N}$ may con-