On Cremonian Congruences. By Dr. T. Archer Hirst, F.R.S.

[*Read May* 13*th*, 1880.]*

Definition.

1. The well-known point-to-point correspondence between two planes, which was first investigated by Cremona in two Memoirs, published in the Transactions of the Institute of Bologna for 1863 and 1865,† is now usually referred to as a Cremonian Correspondence. The two-fold system of right-line rays, each of which passes through a pair of corresponding points of two planes thus related, but arbitrarily situated in space, may therefore be appropriately termed a *Cremonian Congruence.*

2. Although far from being always general ones of their order and class, Cremonian Congruences, from their very mode of generation, cannot fail to secure the interest of Geometers. This interest is enhanced, moreover, by the fact that they include many well-known and important types, whose direct investigation, by purely geometrical methods, has hitherto received comparatively little attention.

The present paper is occupied mainly with the study of Cremonian Congruences, whose class, or whose order, does not exceed the second; the general properties of Cremonian Congruences will be treated in it only so far as this main object of the paper may appear to demand.

General Properties.

3. Let α and β be the two planes between which a Cremonian Correspondence of the n^{th} degree is supposed to exist. An arbitrary plane π will cut each of them in a line to which a curve of the order n corresponds, point by point, in the other. Every such plane π, therefore, will in general contain n pairs of corresponding points, and consequently n rays of the Cremonian Congruence;—in other words, *this congruence is of the class n.*

4. Should the plane π pass through the intersection of the two generating planes α and β, without coinciding with either of them, all the congruence-rays in it will, in general, be coincident with $\alpha\beta$; hence *the intersection of the planes α and β is a n-ple ray of the Cremonian Congruence.*

* The publication of this paper has been unavoidably delayed for more than three years; the form in which it now appears is, substantially, that which it had when communicated, in abstract, to the Society.

† Both memoirs were afterwards republished in the *Bulletin des Sciences Mathématiques et Astronomiques* (t. v., p. 206, 1873), to which, under the designation *Bulletin*, reference will be hereafter made.

5. When the plane π, through $\overline{\alpha\beta}$, coincides with either of the generating planes, it obviously contains the singly infinite number of congruence-rays connecting the several points of $\overline{\alpha\beta}$ with the points in that plane which respectively correspond to them. *The generating planes α and β, therefore, are singular planes of the congruence under consideration.*

6. The several pairs of corresponding points of α and β are projected from an arbitrary point P in space by two pencils between which a Cremonian correspondence of the n^{th} degree likewise exists. The $n+2$ self-corresponding rays which these concentric pencils possess* are evidently congruence-rays, and, moreover, the only ones passing through P; in other words, *the Cremonian Congruence generated by the planes α and β is of the order $n+2$.*

7. Of the $n+2$ congruence-rays which pass through an arbitrary point of α, or β, however, all but one necessarily lie in that plane. The excepted one connects the point in question with its correspondent in β, or α; the remaining $n+1$ connect points of β, or α, situated on $\overline{\alpha\beta}$, with their correspondents in α or β. Hence *the congruence-rays in each of the two singular planes α and β envelope a congruence-curve of the class $n+1$, to which*, by Art. 4, *$\overline{\alpha\beta}$ is a n-ple tangent.* These congruence-curves will be denoted by a_{n+1} and b_{n+1} respectively; being unicursal, they are, in general, of the order $2n$.

8. The correspondents, above alluded to, of the several points of $\overline{\alpha\beta}$ lie, by hypothesis, on a curve $\bar{a}^n$ in α, and $\bar{b}^n$ in β. These curves, being unicursal and of the order n, are in general of the class $2(n-1)$. They cut $\overline{\alpha\beta}$ in the n pairs of corresponding points which this line contains; each, in fact, cuts it in the n points at which it is touched by the congruence-curve situated in the plane of the other, as will be evident on remembering that every tangent of the congruence-curve in α, or β, joins a point on $\overline{\alpha\beta}$ to its correspondent on $\bar{a}^n$, or $\bar{b}^n$, (Art. 7). The curves $\bar{a}^n$ and a_{n+1}, moreover, (and in like manner $\bar{b}^n$ and b_{n+1}) have contact with each other at every point where the former is touched by the tangent of the latter which connects that point with its correspondent on $\overline{\alpha\beta}$. It may be readily shown, by Chasles' *Principle of Correspondence*, that the number of such points of contact is in general $2n-1$.

9. The fixed (principal) points through which every curve, of the order n, corresponding to a right line passes, once or oftener, are *singular points* of the Cremonian Congruence. Each, in fact, is the

* *Bulletin*, Art. 35.

correspondent, not of one point solely, but of a singly infinite number of points, all of which are situate on a (principal) line or unicursal curve whose order is equal to the multiplicity of the principal point in question, and whose own multiple points are themselves coincident with principal points.* Through every principal point of either of the two planes α or β, therefore, a singly infinite number of congruence-rays pass, all of which are situated on a *congruence-cone*, whose trace on β or α is the principal curve corresponding to its vertex.

The multiple generators of each such cone are obviously congruence-rays of the same order of multiplicity. If A_i, for instance, be a principal point of α, of the order i, whose corresponding principal curve b^i passes k times through a principal point B_j of β, then, as Cremona has shown,† the principal curve a^j, corresponding to the latter point, will pass k times through A_i. Hence it follows that, of the n congruence-rays situated in any plane passing through both these points, k, at least, must coincide with A_iB_j itself.

The singularity would be still more marked if either of the points A_i or B_j, or both, were situated on the intersection $\overline{\alpha\beta}$; but I do not pause to consider it.

10. The class of the focal surface of a Cremonian Congruence is readily determined from the circumstance that every tangent plane thereof (since it contains, by definition, two infinitely close congruence-rays) must, in general, intersect α and β in lines which touch their corresponding curves.

Now, a plane π, by turning around an arbitrary line p, generates two pencils of rays $[p\alpha]$ and $[p\beta]$, in α and β, which are in perspective with each other, as well as projective with the pencils of curves, of order n, corresponding to their several rays. The two projective pencils in each plane, say in α, generate, by the intersection of their associated elements, a curve a^{n+1} of the order $n+1$, which passes once through $(p\alpha)$ and the correspondent of $(p\beta)$, and i times through every principal point A_i. The class of the generated curve, therefore, is in general

$$n(n+1)-\Sigma i(i-1)\,\alpha_i = 4n-2 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where α_i denotes the number of principal points in α of the order i, and it is to be remembered that

$$\Sigma i^2\alpha_i = n^2-1, \quad \text{and} \quad \Sigma i\,\alpha_i = 3(n-1),$$

if the summation be extended from $i=1$ to $i=n-1$.‡ By a well-known theorem, the tangent to the curve a^{n+1} at the point $(p\alpha)$ is the ray of the pencil $[p\alpha]$ associated with the curve, of the projective

* *Bulletin*, Art. 5. † *Bulletin*, Art. 12. ‡ *Bulletin*, Arts. 4 and 8.

pencil, which passes through $(p\alpha)$*; but this ray and curve do not, in general, touch each other at $(p\alpha)$; whereas every ray of the pencil $[p\alpha]$ which touches α^{n+1} at a point not coincident with $(p\alpha)$, likewise touches its associated curve at that point. Hence it follows that the plane through p and any such ray possesses the property, above referred to, of intersecting α and β in lines which touch their corresponding curves, and therefore of touching the focal surface. According to the equation (1), the number of such planes passing through p, in other words *the class of the focal surface, is* $4(n-1)$.

11. If p were an ordinary congruence-ray,—that is to say, if it intersected α and β in a pair of corresponding points, neither being a principal point,—then the curves associated with the several rays of the pencil $[p\alpha]$ would pass through its centre, and the generated curve α^{n+1} would there acquire an additional double point, the tangents at which would touch the curves respectively associated with them.† Although the class of α^{n+1} would be diminished by 2, therefore, every plane of the pencil $[p]$ by which it is touched would become a tangent plane of the focal surface, and moreover, with each of the planes touching it at its double point, two consecutive tangent planes of that surface would coincide; in short, as is well known, every congruence-ray p is a double tangent of the focal surface; the two planes in question are its planes of contact—*focal planes*, according to Kummer.‡

12. The only tangent planes to the focal surface which pass through an arbitrary line p situated in either of the generating planes α and β, and which do not coincide with that plane, are those which touch the curve corresponding to p. Their number is, in general, $2(n-1)$ (Art. 8); whence, and from the fact that the focal surface is of the class $4(n-1)$ (Art. 10), we infer that $2(n-1)$ *of the tangent planes to the focal surface which pass through an arbitrary line of either of the two generating planes coincide with that plane.*

13. As multiple tangent planes, α and β touch the focal surface along curves of the class $2(n-1)$. *These curves of contact are precisely those which correspond to the intersection* $\overline{\alpha\beta}$ (Art. 8).

In proof of this, since the latter curves are of the same class as the former, it will suffice to show that every tangent of the one likewise touches the other. Now, if p touch the curve $\overline{\alpha}^n$, for instance, the curve, in β, corresponding to p will touch $\overline{\alpha\beta}$; so that, of the tangent planes to the focal surface which pass through p, one more than the

* Cremona's *Curve Piane*, Arts. 51*a*.
† Cremona's *Curve Piane*, Art. 52.
‡ *Allgemeine Theorie der geradlinigen Strahlensysteme* (*Journal für die reine und angewandte Mathematik*, Band 57, p. 202).

ordinary number, $2(n-1)$, will coincide with α: in other words, p is a tangent to the curve of contact between α and the focal surface.

14. It follows at once, from Art. 7, that the congruence-curves a_{n+1} and b_{n+1} in α and β also lie on the focal surface. But the tangent plane to the latter which passes through an arbitrary tangent p of one of these curves, say a_{n+1}, is in general inclined to the plane α which contains it. For, since every such line p joins a point on $\overline{\alpha\beta}$ to its correspondent in α, the curve in β which corresponds to it will necessarily *cut* $\overline{\alpha\beta}$ at the same point as p itself does, and the plane through p which touches that curve at $(p, \overline{\alpha\beta})$ is clearly the tangent plane in question.

15. It may further be observed that, since every point of α situated on the focal surface must lie either on the curve of contact $\overline{a}''$ or on the congruence-curve a_{n+1}, the section of this surface made by the plane α must consist of the former curve, counted twice, and of the latter one, which is in general of the order $2n$, counted once; in other words, *the order of the focal surface of a Cremonian Congruence is, in general,* $4n$, —an inference which is in accordance with the well-known rule, that the difference between the order and the class of any congruence is half the difference between the order and the class of its focal surface.*

16. The above results become modified, in various ways, when special positions are given to the two planes between which a general Cremonian correspondence of the degree n has been established. Of these modifications, a few of the more noteworthy shall now be briefly considered.

17. If two corresponding points coincide in C, on the intersection $\overline{\alpha\beta}$, one of the congruence-rays proceeding from every point in space will obviously pass through it; so that the congruence $(n+2, n)$ will break up into the congruence $(1, 0)$, consisting of all rays through C, and a residual congruence $(n+1, n)$ of which C is a singular point. This self-corresponding point C, in fact, becomes the centre of a pencil of congruence-rays situated in a singular plane γ; for, of the n congruence-rays situated in an arbitrary plane passing through C, it is clear that one, and only one, will always pass through this point.

18. Since the existence of a self-corresponding point C diminishes, by one, the order of the Cremonian Congruence, the class of each of the congruence-curves in α and β must suffer a like reduction (Art. 7). The intersection $\overline{\alpha\beta}$ is a $(n-1)$-ple tangent of these congruence-

* This theorem is attributed by Sophus Lie (*Göttinger Nachrichten*, 1870, No. 4) to Felix Klein; see also Schubert's *Abzählende Geometrie*, p. 64.

curves a_{n} and b_{n}, and both obviously touch the singular plane γ. The curves $\overline{a}^{n}$ and $\overline{b}^{n}$, corresponding to $\overline{\alpha\beta}$, both pass through C; their remaining intersections with $\overline{\alpha\beta}$ being, as before (Art. 8), the points of contact, with $\overline{\alpha\beta}$, of the congruence-curves b_{n} and a_{n}, respectively.

19. The processes of Arts. 10 and 12 being unaltered by the hypothetical presence of a self-corresponding point C, the class $4(n-1)$ of the focal surface, and the multiplicity $2(n-1)$ of the tangent planes α and β of this surface, remain unchanged. By virtue of the general relation referred to in Art. 15, however, the order of the focal surface is reduced, by two, to $4n-2$.

This surface, moreover, acquires a node (double point) at C. For, if a be any line in α, passing through C, and b the tangent, at C, to the curve in β which corresponds to a, *two* of the n congruence-rays in the plane (ab) will coincide with the intersection of the latter and the singular plane γ; in other words, (ab) and γ are the two focal planes of the congruence-ray γ, (ab), C being the point of contact of the former with the focal surface. But a and b are obviously corresponding rays of two projective pencils in the planes α and β respectively, so that (ab) envelopes a quadric cone—the cone of contact, in fact, at the node C, of the focal surface. γ is clearly a tangent plane of this cone, and the latter likewise touches α and β along the tangents, at C, to the curves $\overline{a}^{n}$ and $\overline{b}^{n}$ which correspond to $\overline{\alpha\beta}$ (Art. 8).

20. The correspondence being still perfectly general, the planes α and β may always be placed so that two self-corresponding points, C and D, will present themselves. The residual congruence (n, n), then generated, is of the order as well as of the class n, and the congruence-curves in α and β are each of the class $n-1$, $\overline{\alpha\beta}$ being a common $(n-2)$-ple tangent thereof. The self-corresponding points C and D are the centres of congruence pencils in fixed singular planes γ and δ, respectively. The order of the focal surface is reduced to equality with its class, which is $4(n-1)$, as before. Of this surface, C and D are double points, and γ and δ double tangent planes; whilst the multiplicity of α and β, as tangent planes thereof, is $2(n-1)$, as before.

21. A noteworthy special case, where two self-corresponding points C and D present themselves, may occur if $\overline{\alpha\beta}$ becomes a self-corresponding line; that is to say, a line upon which the correspondent of each of its own points is situated. This can only happen, of course, when there are principal points, of each of the planes α and β, situated on $\overline{\alpha\beta}$, whose joint-multiplicities amount to $n-1$. Each of the singular planes γ and δ, containing congruence-pencils whose centres are C and D, will in such a case pass through the line $\overline{\alpha\beta}$, and the quadric cones which touch the focal surface at its double points C and D

(Art. 19) will each degenerate to a line-pair, one element of which will be $\overline{\alpha\beta}$ itself.

22. The case, however, with which we shall be most concerned in the sequel is still more special than the one briefly indicated in the last Article. It is that in which more than two, and therefore all points of $\overline{\alpha\beta}$ are self-corresponding ones. Every line incident with $\overline{\alpha\beta}$ may then be regarded as a line passing through two corresponding points, and, if the linear complex consisting of all such lines be detached from the aggregate possessing the property in question, a congruence of the class $n-1$ will remain.

The order of this residual congruence will, of course, be greater by unity than the number of rays, passing through an arbitrary point of β (say), which join two corresponding, but not coincident, points situated in that plane. Now, under the present hypothesis, the congruence-rays in β must consist exclusively of rays belonging to the pencils whose centres are the several principal points of α situated on $\alpha\beta$, and each such ray will have to be counted as often as it intersects the principal curve corresponding to the centre from which it proceeds, at a point not coincident with that centre. But, since every point of $\overline{\alpha\beta}$ is supposed to be self-correspondent, every such principal curve passes once, and in general only once, through its corresponding principal point; so that if $\overline{A}_i$ be any one of the latter, every ray of the pencil $(\overline{A}_i, \beta)$ will be a $(i-1)$-ple congruence-ray, and the total number of such rays passing through an arbitrary point of β will be $\Sigma\,(i-1)\,\bar{a}_i$, where $\bar{a}_i$ denotes the number of principal points $\overline{A}_i$, of the order i, situated on $\overline{\alpha\beta}$, and the summation is to be extended from $i = 1$ to $i = n-1$. From this, and the fact that the joint multiplicities of all points $\overline{A}_i$ is, by the hypothesis of Art. 21, $\quad \Sigma i\bar{a}_i = n-1,$

we at once deduce, for the required order of our congruence, the expression* $\quad n-\Sigma\bar{a}_i.$

But another expression for this same order is, obviously, $n-\Sigma\bar{\beta}_i$, where $\bar{\beta}_i$ is the number of principal points of the order i, belonging to the plane β, which are situated on $\overline{\alpha\beta}$. Hence we infer that *when every point of the intersection of the planes α and β is self-correspondent, the number,*

$$\Sigma\bar{a}_i = \Sigma\bar{\beta}_i = m \quad\text{..........} (1),$$

of principal points situated on their intersection $\overline{\alpha\beta}$ is the same for both planes, and the order of the residual congruence which is generated is less, by this number m, than the degree n of the Cremonian correspondence existing between the two planes. Its class, as already stated, is $n-1$.

* This expression may be readily verified by the more general method indicated in Art. 6.

23. By the method adopted in Art. 10, it can readily be shown that the focal surface of the congruence $(n-m, n-1)$, under consideration, is of the class $4n-2m-6$, and of the order $4(n-m-1)$. With each of the generating planes α and β coincide $2(n-m-1)$ of the tangent planes which can be drawn to the surface through an arbitrary line in that plane.

Congruences of the First Class.

24. Proceeding to the applications of the foregoing general results, I observe that congruences of the first class are generated by the method of Art. 3 when $n=1$; that is to say, when the correspondence between the planes α and β is homographic.* Such a congruence is, in general, of the third order (Art. 6). Its focal surface, which is of the class 0 (Art. 10), that is to say developable, and of the order 4 (Art. 15), touches the planes α and β along the right lines $\bar{a}$, $\bar{b}$, which correspond to $\overline{\alpha\beta}$ (Art. 13), and cuts them in the congruence-conics, a_2 and b_2, which are enveloped by the congruence-rays joining the points of $\overline{\alpha\beta}$ to their correspondents on $\bar{a}$ and $\bar{b}$, respectively (Art. 15).

Every tangent plane of this focal surface is a singular plane of the congruence; for the latter is only of the first class, whereas the plane in question contains, by hypothesis, two, and actually an infinite number of congruence-rays. In virtue of the last-mentioned property, the plane under consideration necessarily intersects α and β in a pair of corresponding lines. Conversely, every plane containing two corresponding lines of α and β must be a singular one; the congruence-curve in it being the conic which touches these lines at their intersections with $\bar{a}$ and $\bar{b}$. The two lines, in fact, are tangents, respectively, to a_2 and b_2, and the plane which contains them, since it touches two conics having a common tangent $\overline{\alpha\beta}$, envelopes, by a well-known theorem, a developable of the third class. Every two such planes obviously intersect in a congruence-ray; the aggregate of all such intersections, in fact, is precisely the congruence (3, 1) under investigation. When the two planes are consecutive ones, the congruence-ray in which they intersect, and which joins corresponding points of the conics a_2 and b_2, becomes a generator of the developable focal surface, and consequently a tangent to the skew cubic which

* In 1832, at the end of his *Systematische Entwickelung der Abhängigkeit geometrischer Gestalten*, Steiner proposed this question: "To what law are the lines subjected which join the corresponding points of two projective (homographic) planes, or what curved surface is touched by them?" (Steiner's *Gesammelte Werke*, Bd. 1, p. 453.) The second part of the *Abhängigkeit*, however, in which this and similar problems were to have been treated, was never published, and the question, or rather its correlative, appears to have been first solved by Reye, in 1870, in the *Journal für die reine und angewandte Mathematik*, Bd. 74, p. 1.

forms its edge of regression. The three congruence-rays which proceed from any point of this skew cubic coincide obviously with the tangent to the curve at that point.

25. When the homographic planes α and β have a self-corresponding point C, the congruence of the first class which they generate is only of the second order (Art. 17). The point C, however, becomes the centre of a pencil of congruence-rays situated in a plane γ, which passes through the points A and B towards which all the congruence-rays in the planes α and β, respectively, now converge (Art. 18). The focal surface is developable, as before; but it is now of the second order, and has a node at C (Art. 19). It not only touches α and β along the lines $\bar{a}$ and $\bar{b}$, through C, which correspond to $\overline{\alpha\beta}$, but by Art. 14 also cuts these planes in the points A and B. These results are most readily explained by again regarding the focal surface as the envelope of planes, each containing a pair of corresponding lines. There are now, in fact, two distinct series of such planes, since there are two different modes of grouping the several rays of the congruence. The planes of one of these series all pass through C, and envelope a quadric cone which touches α and β in the manner above described; the planes of the other series form a pencil whose axis intersects α and β in A and B. The congruence, consequently, has a *focal line*, as well as a *focal cone*. Each plane through the former contains a congruence-conic which lies on the latter; whilst each tangent plane of the latter contains a congruence-pencil whose centre is on the former. The conic in the plane (ABC), which is common to both series, degenerates to a point-pair (C, D), where D is the point in which the focal line and focal cone touch each other. If the two latter be regarded as constituting a developable of the third class, the congruence (2, 1) under consideration will not be, as before, the aggregate of the intersections of *all* its pairs of planes, but only of those which belong to different series; in other words, the congruence (2, 1) consists of all right lines which touch the focal cone, and are at the same time incident with the focal line.

26. When, finally, the homographic planes α and β have two self-corresponding points C and D, and therefore a self-corresponding line $\overline{\alpha\beta}$, they generate a congruence of the first order, as well as of the first class (Art. 20). The focal surface is now of the order, as well as of the class 0, and the points C and D upon it are of such a kind that at each there are an infinite number of tangent planes, all intersecting in the same line (Art. 21). From this, and the fact that every such tangent plane is a singular plane of the congruence (Art. 24), we may at once infer the existence of two focal lines c and d; the former passing through C, and the latter through D. Each, in fact, is the

axis of a pencil of planes, whereof each constituent contains a pencil of congruence-rays, since it intersects α and β in a pair of corresponding lines having a self-corresponding point. The congruence (1, 1) under consideration is thus resolved, in two different ways, into a series of pencils whose planes pass through one focal line, while their centres lie on the other. It is, in short, the aggregate of all the intersections of planes belonging to different pencils, or, if we please, of all right lines which are incident with *both* the focal lines c and d.*

27. A congruence of the first order and first class is also generated, by the method described in Art. 22, when $n = 2$; that is to say, when the planes α and β are in quadric correspondence. Two, associated, principal points thereof are necessarily situated on $\overline{\alpha\beta}$; but if A_1' and A_1'' be the two principal points of α which are not situated thereon, and if B_1' and B_1'' be the principal points of β respectively associated with them, it is clear that every plane through $\overline{A_1'B_1'}$, as well as every plane through $\overline{A_1''B_1''}$, will intersect α and β in a pair of corresponding lines having a self-corresponding point, and will therefore contain a pencil of congruence-rays. The axes $\overline{A_1'B_1'}$ and $\overline{A_1''B_1''}$ of the two pencils of planes are, obviously, the focal lines of the congruence (1, 1) under consideration.†

Congruences of the First Order.

28. Cremonian Congruences of the first order are more numerous than those of the first class. They can all be generated by the method described in Art. 22. To this end, it is essential that each of the planes α and β, between which a correspondence of the degree n exists, should possess $n-1$ principal points situated on their intersection $\overline{\alpha\beta}$, and, since the sum of their multiplicities must also be equal to $n-1$, that they should all be principal single points. This condition being satisfied, and the planes α and β appropriately placed, the focal surface of the generated congruence $(1, n-1)$ will be of the class $2(n-2)$ and of the order 0 (Art. 23); that is to say, it will degenerate to a focal curve of the class $2(n-2)$.

29. The above condition can be satisfied (see Art. 30), for all values of

* The above generation of the congruence (1, 1) was given, in 1868, by Reye, in a paper published in the *Journal für die reine und angewandte Mathematik*, Bd. 69, p. 365.

† The theorem converse to the one on which this mode of generation is based, viz., that *the several rays of a congruence* (1, 1) *determine a quadric correspondence between two arbitrary planes*, is in reality Steiner's,—at least, in the case where the focal lines are real. He founded upon it, in fact, the well-known mode of transformation to which he gave the name of *skew* (schief) *projection* (*Gesammelte Werke*, Bd. 1, p. 407). Reye also gives the theorem in the paper quoted in the preceding note.

$n > 1$,* by a correspondence of the type, first studied by De Jonquières,† which is characterised by having, in each plane, a principal point of the order $n-1$ of multiplicity, and consequently $2(n-1)$ principal single points. When $(n-1)$ of the latter are situated on the intersection of both planes, the curves in α and β, of the order n, corresponding to $\overline{\alpha\beta}$ will each break up into this line itself, and the $n-1$ principal lines which correspond, respectively, to the above-mentioned principal points. They connect the latter points, in fact, with the principal multiple point, and each contains one of the principal single points of its own plane which are not situated on $\overline{\alpha\beta}$.

Since every plane π, passing through the principal multiple points A_{n-1} and B_{n-1}, cuts the planes α and β in a pair of corresponding lines, which intersect in a self-corresponding point, it necessarily contains a pencil (P, π) of congruence-rays. *The congruence* $(1, n-1)$ *under consideration*, being the aggregate of all such pencils, *has, consequently*, $f \equiv \overline{A_{n-1}B_{n-1}}$ *for a focal line, and the locus of P for a focal curve.* The only points in which the latter cuts either of the planes α and β being the principal points, not situated on $\overline{\alpha\beta}$, it is clearly a curve f^{n-1} of the order $n-1$; moreover, since each plane π contains but one point P, this focal curve must be incident at $n-2$ points with the focal line f. Each of the $n-2$ planes π in which this incidence occurs touches, in fact, at A_{n-1} and B_{n-1} a pair of associated branches of the principal curves corresponding, respectively, to B_{n-1} and A_{n-1}.

30. The congruence $(1, n-1)$ above generated is identical with the one described by Kummer in his *Algebraische Strahlensysteme*,‡ and it may readily be shown, conversely, that every congruence which has a focal line and a focal curve, of the order $n-1$, incident at $n-2$ points with that focal line, will determine a correspondence of the n^{th} degree, and of the Jonquièrian type (Art. 29), between two arbitrary planes α and β. Every point of the intersection $\overline{\alpha\beta}$, moreover, will be self-correspondent, and $n-1$ principal single points of each plane will necessarily be situated thereon. The disposition of principal points predicated in Art. 29 is thus realised.

31. It will be observed that when $n=2$, in Art. 29, the congruence $(1, 1)$ generated in Art. 27 is reproduced, and that the value $n = 3$ yields a congruence which is precisely the correlative of that described in Art. 25. To generate the congruence $(1, 3)$ correlative to that in Art. 24, however, by the method of Art. 22, a quartic correspondence

* When $n=1$, the planes α and β, as is well known, are in perspective with one another; in other words, they generate a congruence $(1, 0)$.

† *Comptes Rendus de l'Académie des Sciences*, tome 49, 1859; and *Nouvelles Annales de Mathématique, deuxième serie*, t. 3, 1864.

‡ *Abhandlungen der K. Acad. der Wissenschaften zu Berlin*, 1866, § 2, V.

between α and β of a different type from that employed in Art. 29 is requisite. In each plane, in fact, there must be three principal double, and three principal single points;* the latter being all situated on $\overline{\alpha\beta}$.

The principal double points A_2', A_2'', A_2''' of α, and B_2', B_2'', B_2''' of β being arbitrarily placed, let the principal single points A_1', A_1'', A_1''' of α, and B_1', B_1'', B_1''' of β correspond, respectively, to the principal lines $\overline{B_2''B_2'''}$, $\overline{B_2'''B_2'}$, $\overline{B_2'B_2''}$, $\overline{A_2''A_2'''}$, $\overline{A_2'''A_2'}$, $\overline{A_2'A_2''}$, and be situated at the intersections of the latter with $\overline{\alpha\beta}$. Granting for a moment (see Art. 34) the possibility of such a disposition, it is manifest, from Art. 28, that the generated congruence will be of the first order and third class, and that its focal surface will degenerate to a curve of the fourth class. Unlike the congruence (1, 3) generated in Art. 29, however, there is here no focal line, but simply a focal curve f^3 of the third order.

32. To show this, and at the same time to elucidate the relation which exists between the two congruences (1, 3) just referred to, we may proceed as follows:—Let A and B be any two corresponding points of the quartic correspondence in Art. 31. To each conic of the pencil $(A_2'A_2''A_2'''A)$ will correspond, point to point, a conic of the pencil $(B_2'B_2''B_2'''B)$,† and since the two conics will necessarily intersect in two self-corresponding points, the congruence-lines joining their remaining pairs of corresponding points will, as is well known, form a quadric regulus.‡ The aggregate of all such *reguli* is clearly the congruence under consideration; and, since one congruence-ray only passes through an arbitrary point in space, these reguli, or, to speak more strictly, the quadric surfaces on which they lie, form a pencil the base-curve of which breaks up into the congruence-ray $\overline{AB}$, and a cubic which is incident at two points, not only with that ray, but with every other belonging to the congruence. This is the focal cubic f^3; it obviously passes through, and is determined by, the six principal double points of the correspondence.

33. The quadric reguli conjugate, respectively, to those of the pencil considered in the last Article—that is to say, situated with the latter on the several quadric surfaces of the pencil—manifestly form a second congruence of the same order and class as the first one. This

---

* *Bulletin*, p. 216, Art. 20.

† The complete curve, of the eighth order, corresponding to the first of those conics breaks up into the second, and the three principal conics corresponding, solely, to the principal double points A_2', A_2'', A_2''', respectively.

‡ I here adopt a convenient term suggested to me by Prof. Cayley. In the absence of self-corresponding points, the regulus would be of the fourth degree, as Cremona has shown in his *Preliminari*, § 64, p. 43.

new congruence, however, is of the type considered in Art. 29, since each of its rays is clearly incident, at one point solely, not only with the line $\overline{AB}$, but also with the cubic f^3. The number of congruences of this latter kind, which can be similarly associated with a given one of the first kind, is doubly infinite,—seeing that every chord of f^3 may play the part above assigned to $\overline{AB}$.

34. It only remains to be shown, conversely, that a congruence (1, 3) whose rays are chords of any given skew cubic f^3 necessarily determines a correspondence, between two arbitrary planes α and β, of the precise kind predicated in Art. 31.

In the first place, every point of $\overline{\alpha\beta}$ is rendered self-correspondent by the congruence-ray which passes through it. Moreover, the rays of the congruence which are incident with an arbitrary line of either plane, say a in α, generate, as is well known, a quartic scroll upon which f^3 is a double curve, and of which the three congruence-rays situated in α are generators. Every such scroll, therefore, intersects β in a quartic curve, corresponding to a, which possesses three double points, at the points in which f^3 pierces β, and passes necessarily through the intersections of $\overline{\alpha\beta}$ with the above three generators situated in α. The former represent the principal double, the latter the principal single, points of the quartic correspondence employed in Art. 31.

35. All congruences of the first order belong to one of the two types considered in Arts. 29 and 31,* and the method of resolution into quadric reguli employed in Art. 32 may be applied to each of the first three of the series of congruences considered in Art. 29, but to no others. The reguli conjugate to those into which any one of these three congruences may be resolved, form, in the aggregate, a second congruence of the same order and class as the original one. In the case of $n = 4$, these two associated congruences, as we have seen in Art. 33, are of different types; in the two other cases, however, they are of the same type.

When $n = 3$, for example, in Art. 29, every conic of the pencil $(A_2A_1A_1'A)$, in α, corresponds to a conic of the pencil $(B_2B_1''B_1'''B)$, in β; where A_2, B_2 are the principal double points of the correspondence, A_1, A_1', B_1'', B_1''' the principal single points not situate on $\overline{\alpha\beta}$, and A, B an arbitrary pair of corresponding points. The congruence-rays joining the corresponding points of any two such conics generate, as before, a quadric regulus, and all such reguli, a pencil in which every ray of the congruence (1, 2) under consideration is included. The base of this pencil breaks up into the line $\overline{AB}$ common to all its

* Kummer's *Algebraischen Strahlensysteme*, § 2, IV., and § 2, V.

reguli, the focal line $\overline{A_2B_2}$ of the congruence, and its focal conic which, as we know, is incident with each of the above two lines, and likewise passes through A_1, A_1', B_1'', B_1''' (Art. 29). The congruence (1, 2) formed by the reguli conjugate to those just considered has obviously the same focal conic; but $\overline{AB}$ and $\overline{A_2B_2}$ change places, the former being the focal line, and the latter an ordinary congruence-ray.

Again, when $n = 2$, every conic in α, of the pencil $(A_1A_1'AA')$ corresponds to a conic in β of the pencil $(B_1B_1'BB')$, where A_1, B_1 and A_1', B_1' are the two pairs of associated principal points not situated on $\overline{\alpha\beta}$, and A, B and A', B' are any two pairs of corresponding points. The congruence-rays passing through corresponding points of corresponding conics in these two pencils form a quadric regulus, and all such reguli a pencil in which every ray of the given congruence (1, 1) is included. The base-curve of the pencil obviously breaks up into the focal lines $\overline{A_1B_1}$, and $\overline{A_1'B_1'}$ of that congruence, and into the congruence-rays $\overline{AB}$ and $\overline{A'B'}$ common to every regulus of the pencil. The congruence (1, 1) formed by all the reguli conjugate to the above, is precisely of the same type as the original one; $\overline{A_1B_1}$ and $\overline{A_1'B_1'}$, however, are now ordinary congruence-rays, whilst $\overline{AB}$ and $\overline{A'B'}$ are focal lines.

By giving suitable positions to any two of the four points A, A', B, B' which do not correspond to each other, all the ∞^4 possible modes of resolution into quadric reguli are obtainable.

Resolution of a Congruence into Quadric Reguli.

36. Before proceeding to the detailed consideration of Cremonian Congruences of the second class, or second order, it will be convenient to give a brief, but more general statement of the properties resulting from the resolution of a congruence (m, n) into a one-fold system of quadric reguli.

(**a**) Of the quadric surfaces upon which such reguli are situated, m will pass through an arbitrary point, and n will touch an arbitrary plane; whence it follows that the reguli conjugate to the original ones will form a second, *associated*, congruence (m, n) of the same order and class as the first.

(**b**) Two successive quadric surfaces of the system will intersect in a quartic curve, from every point of which will proceed, in general, two ultimately coincident rays of each of the associated congruences.* The latter, consequently, will, in general, have the envelope of the given system of quadric surfaces for a common focal surface.

* Exceptions to this rule have already presented themselves in the last Article. I do not consider them further.

(**c**) A point P through which every quadric of the system passes must be a singular point of both the associated congruences (m, n). The sum of the orders of the two congruence-cones,* having P for a common vertex, is in all cases n, and between their generators a $(1, 1)$ correspondence exists, corresponding generators being defined to be those which lie on the same quadric surface. The plane of two such generators envelopes the cone of contact with the common focal surface at its node (multiple point) P. The class of this cone cannot exceed n.†

Upon each of the n quadrics, in fact, which touch an arbitrary plane π, passing through P, there will be a line of each of the conjugate reguli situated entirely in π. But only one of these will pass through P, seeing that π, being arbitrary, will not touch at P any one of the n quadrics under consideration. Hence it is the sum of the orders of the two congruence-cones at P which is equal to n.

(**d**) Correlatively, a plane π which is touched by every quadric of the system must be a singular plane of each of the two associated congruences (m, n). The sum of the classes of the two congruence-curves in this plane π is in all cases m, and between their tangents a $(1, 1)$ correspondence exists, provided corresponding tangents be those which are generators of the same quadric. The locus of the intersection of two such tangents is the curve of contact of the common focal surface with its multiple tangent plane π. Its order cannot exceed n.†

(**e**) If the system of quadrics contain a cone, its generators will belong to both systems of conjugate reguli, and therefore be common rays of the two associated congruences (m, n).

(**f**) In like manner, if a quadric surface of the system degenerate to a conic, all its tangents will be common rays of the two associated congruences.

The existence of these degenerate forms (**e**) and (**f**) will, of course, cause a diminution to take place in the class of the cone of contact referred to in (**c**), as well as in the order of the curve of contact to which allusion is made in (**d**).

(**g**) If any quadric surface of the system degenerate to a pair of planes γ, δ, together with a pair of points C, D situated on their intersection (point-and-plane pair), these planes and points will obviously be singular ones of both the associated congruences. One of the latter, in fact, will contain the pair of congruence-pencils (C, γ), (D, δ), and the other the pair (C, δ), (D, γ).

---

* Each may degenerate to a plane pencil.

† A diminution is caused by the presence of coincident corresponding generators, as explained below under (**e**), (**f**), and (**g**).

Congruences of the Second Class.

37. Congruences of the second class, and of any order not exceeding the fourth, may be generated in the manner described in Arts. 3 to 21, provided a quadric correspondence exists between the planes α and β.

The congruence (4, 2) presents itself whenever this correspondence is unconditioned. The intersection $\overline{\alpha\beta}$ is a double ray thereof, as well as a double tangent of each of the congruence-curves a_3, b_3, of the third class, which α and β contain (Art. 7). The tangents of these curves connect the several points of $\overline{\alpha\beta}$ with their correspondents on the conics $\overline{a}^2$ and $\overline{b}^2$, and the points in which the former is cut by the latter, respectively, are likewise the points in which $\overline{\alpha\beta}$ is touched by b_3 and a_3. Moreover, $\overline{a}^2$ and a_3, as well as $\overline{b}^2$ and b_3, have triple contact with each other (Art. 8).*

The focal surface F_4^8 of the congruence is of the order 8 and class 4 (Arts. 10 and 15); it touches the planes α and β at the several points of the conics $\overline{a}^2$ and $\overline{b}^2$ (Art. 13), and cuts them in the congruence-curves a_3 and b_3 (Art. 14).

The conic $\overline{a}^2$ passes through the three principal points A_1', A_1'', A_1''', and $\overline{b}^2$ through their respective associates B_1', B_1'', B_1'''. All six are singular points of the congruence; centres, in fact, of congruence-pencils situated in the planes which connect them with their corresponding principal lines. These six singular planes will be thus denoted:

$$\alpha_1 \equiv (A_1'\overline{B_1''B_1'''}), \quad \beta_1 \equiv (B_1'\overline{A_1''A_1'''}),$$
$$\alpha_2 \equiv (A_1''\overline{B_1'''B_1'}), \quad \beta_2 \equiv (B_1''\overline{A_1'''A_1'}),$$
$$\alpha_3 \equiv (A_1'''\overline{B_1'B_1''}), \quad \beta_3 \equiv (B_1'''\overline{A_1'A_1''}).$$

To a right line in α passing through any principal point corresponds, point by point, a right line passing through the associated principal point in β, and two, but only two, such lines will be incident with their corresponding ones. The congruence-lines which connect the corresponding points of two such incident lines envelope a congruence-conic, touching α and β at points on the conics $\overline{a}^2$ and $\overline{b}^2$. Let the six singular planes thus obtained be

$$\gamma_1 \text{ and } \delta_1, \quad \gamma_2 \text{ and } \delta_2, \quad \text{and} \quad \gamma_3 \text{ and } \delta_3,$$

passing, respectively, through

$$A_1' \text{ and } B_1', \quad A_1'' \text{ and } B_1'', \quad \text{and} \quad A_1''' \text{ and } B_1''';$$

and let the congruence-conics they respectively contain be designated by (γ_1') and (δ_1'), (γ_2') and (δ_2'), and (γ_3') and (δ_3').

38. The congruence (4, 2) above generated, possessing six singular points and fourteen singular planes, is precisely the correlative of the

* Schröter, *Journal für die reine und angewandte Mathematik*, Bd. 64, p. 38.

one described by Kummer in § 9 of his *Algebraische Strahlensysteme.* That it is a perfectly general one, of its order and class, follows at once from the circumstance that any such general congruence must necessarily determine a quadric correspondence between those two of its fourteen singular planes which are distinguished from the rest by containing congruence-curves of the third class. The truth of this statement may be proved in a manner similar to that employed in Art. 34.

39. The congruence (4, 2) under consideration contains three systems of quadric reguli; they proceed from the three pairs of pencils of corresponding lines referred to in Art. 37. Each system includes four degenerate reguli. Each of two of these consists, as we have seen, of the tangents to a conic. Each of the remaining two is represented by a pencil-pair, as shown below; its directrices are principal lines.

Corresponding Directrices.		Pencil-pair.	System.
$\overline{A_1'A_1''}$	$\overline{B_1'B_1'''}$	$(A_1''\alpha_2)$, $(B_1'''\beta_3)$	(i.)
$\overline{A_1'A_1'''}$	$\overline{B_1'B_1''}$	$(A_1'''\alpha_3)$, $(B_1''\beta_2)$	
$\overline{A_1''A_1'''}$	$\overline{B_1''B_1'}$	$(A_1'''\alpha_3)$, $(B_1'\beta_1)$	(ii.)
$\overline{A_1''A_1'}$	$\overline{B_1''B_1'''}$	$(A_1'\alpha_1)$, $(B_1'''\beta_3)$	
$\overline{A_1'''A_1'}$	$\overline{B_1'''B_1''}$	$(A_1'\alpha_1)$, $(B_1''\beta_2)$	(iii.)
$\overline{A_1'''A_1''}$	$\overline{B_1'''B_1'}$	$(A_1''\alpha_2)$, $(B_1'\beta_1)$	

No two reguli of the same system have any generator in common; but two reguli belonging to different systems have always one, and in general only one, common generator.* Should the latter have more than one, however, they will have an infinite number of common generators. The above pencil-pairs present instances of this.

The quadric surfaces which contain the several reguli of one and the same system, all pass through two fixed points and touch eight fixed planes. These points are the centres of the pencils whence the several reguli of the system proceed. Two of the eight planes are obviously α and β; two others connect the centres above referred to with their corresponding principal lines; the remaining four are the planes which contain congruence-conics (Art. 37), but do not pass through either of the last-mentioned centres.

The quadrics of each of the three systems envelope the focal surface

* See end of Art. 45.

of the congruence under consideration (Art. 36**b**), and the reguli upon them which are conjugate to those of which this congruence is the aggregate form another congruence of the same order and class as the original one (Art. 36**a**). The original congruence, and the three thus derived from it, form the four congruences (4, 2) which, as is well known, have a common focal surface, F_4^8, of the eighth order and fourth class.

In the following Table the original congruence is denoted by **C**, the three associated ones, proceeding respectively from the pairs of projective pencils (A_1'), (B_1'); (A_1''), (B_1''); (A_1'''), (B_1'''), by $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$. To each a column is appropriated in which the nature of the congruence-curve is indicated which lies in each of the fourteen singular planes. To each of the latter a separate line is devoted. A congruence-pencil is denoted by the letter indicative of its centre; congruence-conics, by the symbols already used in Art. 37, and other similar ones; and congruence-curves of the third class by the number (3), simply. The theorems of Art. 36 suffice, in all cases, for the verification of these indications.

	C	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_3$
α	(3)	A_1'	A_1''	A_1'''
β	(3)	B_1'	B_1''	B_1'''
α_1	A_1'	(3)	B_1'''	B_1''
β_1	B_1'	(3)	A_1'''	A_1''
α_2	A_1''	B_1'''	(3)	B_1'
β_2	B_1''	A_1'''	(3)	A_1'
α_3	A_1'''	B_1''	B_1'	(3)
β_3	B_1'''	A_1''	A_1'	(3)
γ_1	(γ_1')	(γ_1')	(γ_1'')	(γ_1'')
δ_1	(δ_1')	(δ_1')	(δ_1'')	(δ_1'')
γ_2	(γ_2')	(γ_2'')	(γ_2')	(γ_2'')
δ_2	(δ_2')	(δ_2'')	(δ_2')	(δ_2'')
γ_3	(γ_3')	(γ_3'')	(γ_3'')	(γ_3')
δ_3	(δ_3')	(δ_3'')	(δ_3'')	(δ_3')

40. The three congruences $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$ determine quadric correspondences between the pairs of planes α_1, β_1; α_2, β_2; α_3, β_3, respectively, and from each congruence the other two, as well as **C** itself, may be derived in exactly the same manner as they were themselves derived from **C**. Hence it readily follows that the rays of any one of the possible six *pairs* of congruences may be so grouped as to form conju-

gate reguli on a system of quadric surfaces, all passing through two points, touching eight planes, and enveloping the common focal surface F_4^8. Although the properties of these systems are discernible in the Table of the last Article, they may be thus enunciated :—

Every two of the four associated congruences (4, 2) have the tangents of two conics for common rays. These conics are situated in the planes which pass through the two singular points that are common to every quadric surface of the system originating from the two congruences in question.* Both conics, as well as every quadric surface of the system to which (as degenerate forms) they belong, touch the remaining four planes containing congruence-conics. The four other planes which are touched by every quadric of the system under consideration are those which contain the congruence-curves, of the third class, belonging to the two congruences whence the system proceeds. The four singular planes not yet referred to are those of the two plane-and-point pairs included in the systems of quadrics. The mode in which they are paired is obvious from the points they contain.

41. The fact that the quadrics of each of the above six systems envelope the common focal surface of the four congruences (4, 2), enables us readily to determine the singularities of that surface.

Each of the six singular points, for instance, is a double point thereof; the quadric cone of contact thereat is touched by each of the six singular planes which pass through the point in question. Through this point, in fact, every quadric surface of each of two systems passes,† and in each of these systems the corresponding generators, through the point, are corresponding rays of two plane pencils (Art. 36**c**). The cone of contact above referred to is obviously the envelope of the plane of two such rays.

Each of the fourteen singular planes, again, is a double tangent-plane of the focal surface, touching the latter along a conic.

Each of the first eight singular planes in the Table of Art. 39, for example, is touched by every quadric of each of three different systems; and of the corresponding generators in it (relative to any one of these systems) one is always a tangent to the third class congruence-curve situated in the plane, and the other a ray of a pencil. Seeing that two self-corresponding generators exist, viz.: those belonging to the two conics included in any one of the three systems under consideration (Art. 39), the locus of the intersection of corresponding generators is a conic (Art. 36**d**). This is the conic of contact between the plane under consideration and

* The same two planes also contain the two conics whose tangents are rays common to the remaining two of the four associated congruences.

† That there are *two* systems, follows readily from the preceding note.

the focal surface. It passes, obviously, through the three singular points situated in the plane.

Each of the last six singular planes in the Table of Art. 39, on the other hand, is touched by every quadric of each of four systems, and (relative to any one of these systems) the pairs of corresponding generators in the plane are tangents of the two different congruence-conics which it contains. Here, as before, two self-corresponding generators have to be taken into account; so that the locus of the intersection of the remaining pairs of corresponding generators is again a conic; viz., the conic of contact between the plane under consideration and the focal surface. This conic passes through the two singular points situated in its plane, because each of the latter belongs to one of the two plane-and-point pairs included in any one of the four systems of quadrics above alluded to.

42. The correspondence between α and β being still quadric, as in Art. 37, we will now consider the effect of the presence of one self-corresponding point.

It follows from Arts. 17 to 19, where the general effect of such a point was described, that the congruence (3, 2) now generated, although still of the second class, is only of the third order; that the intersection $\overline{\alpha\beta}$, instead of being a double, is now only a single congruence-ray; that the congruence-curves in α and β which touch it, instead of being of the third, are only of the second class; and further that, with the self-conjugate point C, an additional, fifteenth, singular plane γ is introduced containing a congruence-pencil whose centre is C.

The focal surface F_4^6 moreover, although still of the fourth class, is now only of the sixth order; it cuts the planes α and β, in fact, in the congruence-conics they contain, and touches them at all points of the conics, $\bar{a}^2$ and $\bar{b}^2$, corresponding to $\overline{\alpha\beta}$. The last-mentioned conics, it may be observed, pass through C; each has double contact with the congruence-conic in its own plane, and cuts $\overline{\alpha\beta}$ a second time in the point at which this line is touched by the congruence-conic in the other plane (Art. 8).

43. The above, however, are far from being the only modifications, in preceding results, due to the introduction of the self-corresponding point C. Through the latter, for example, three of the six planes pass which formerly contained congruence-conics; and these three, say γ_1, γ_2, γ_3, now contain congruence-pencils whose respective centres C_1, C_2, C_3 constitute, with C, four additional singular points of the congruence, thus raising the total number from six to ten. Again, since each of the three last-mentioned planes obviously intersects γ in a congruence-line, the centres C_1, C_2, C_3 must all lie, with C, in γ. For a similar reason, the pairs of planes α_1, β_1; α_2, β_2; and α_3, β_3, containing

congruence-pencils whose respective centres are A_1', B_1'; A_1'', B_1''; and A_1''', B_1''' (Art. 37), must pass, in pairs, through C_1, C_2, and C_3, respectively. Finally, the planes δ_1, δ_2, δ_3, each of which obviously intersects, in congruence-rays, the two planes of the group γ_1, γ_2, γ_3 whose suffixes are unlike its own, must pass, respectively, through the pairs of points C_2, C_3; C_3, C_1; C_1, C_2. In short, four of the ten singular points of the congruence **C**, under consideration, must lie in each of its fifteen singular planes. The Table of Art. 49 shows, at once, what points lie in each of these planes.

44. The congruence **C** determines a quadric correspondence, not only between the planes α and β, by which it was generated, but likewise between any two of the five planes α, β, δ_1, δ_2, δ_3 containing congruence-conics, and these two have always a self-corresponding point. The proof of the first of these statements is similar to that given in Art. 34; the correctness of the second follows immediately from the fact that, in the congruence **C**, the point common to any two of the five planes in question is always the centre of a congruence-pencil in a third plane. It may further be observed that associated principal points of the quadric correspondence between the two planes under consideration always lie in one and the same column of the Table just referred to.

45. The congruence **C** contains five systems of quadric reguli. Three of these proceed, as in Art. 39, from the pencils of corresponding lines, in α and β, whose centres are the associated principal points A_1', B_1'; A_1'', B_1''; and A_1''', B_1'''. Of the remaining two, one proceeds from the pencils $C(\beta)$ and (A_1', A_1'', A_1''', C), whose elements (lines and conics) correspond to each other; the other, in like manner, from the pencils $C(\alpha)$ and (B_1', B_1'', B_1''', C). It is well known, in fact, that if the points of a conic and of a right line (or of another conic), not in the same plane with the former, correspond projectively, the lines connecting their corresponding points form a quadric regulus, provided one point (or two points) be self-correspondent. From this it readily follows, too, that the congruence **C** contains no other quadric reguli than those included in the above five systems.

Each of these systems contains, as before, four degenerate reguli, only one of which, however, now consists of the tangents to a conic, the other three being pencil-pairs. The latter, together with the corresponding elements of the generating pencils which serve as their directrices, are given in the following Table :—

Corresponding Directrices.		Pencil-pair.	System.
$\overline{A_1'A_1''}$	$\overline{B_1'B_1'''}$	$(A_1''\alpha_2)$, $(B_1'''\beta_3)$	
$\overline{A_1'A_1'''}$	$\overline{B_1'B_1''}$	$(A_1'''\alpha_3)$, $(B_1''\beta_2)$	(i.)
$\overline{A_1'C}$	$\overline{B_1'C}$	$(C\gamma)$, $(C_1\gamma_1)$	
$\overline{A_1''A_1'''}$	$\overline{B_1''B_1'}$	$(A_1'''\alpha_3)$, $(B_1'\beta_1)$	
$\overline{A_1''A'}$	$\overline{B_1''B_1'''}$	$(A_1'\alpha_1)$, $(B_1'''\beta_3)$	(ii.)
$\overline{A_1''C}$	$\overline{B_1''C}$	$(C\gamma)$, $(C_2\gamma_2)$	
$\overline{A_1'''A_1'}$	$\overline{B_1'''B_1''}$	$(A_1'\alpha_1)$, $(B_1''\beta_2)$	
$\overline{A_1'''A_1''}$	$\overline{B_1'''B_1'}$	$(A_1''\alpha_2)$, $(B'\beta_1)$	(iii.)
$\overline{A_1'''C}$	$\overline{B_1'''C}$	$(C\gamma)$, $(C_3\gamma_3)$	
$(\overline{CA_1'}, \overline{A_1''A_1'''})$	$\overline{CB_1'}$	$(C_1\gamma_1)$, $(B_1'\beta_1)$	
$(\overline{CA_1''}, \overline{A_1'''A_1'})$	$\overline{CB_1''}$	$(C_2\gamma_2)$, $(B_1''\beta_2)$	(iv.)
$(\overline{CA_1'''}, \overline{A_1'A_1''})$	$\overline{CB_1'''}$	$(C_3\gamma_3)$, $(B_1'''\beta_3)$	
$\overline{CA_1'}$	$(\overline{CB_1'}, \overline{B_1''B_1'''})$	$(C_1\gamma_1)$, $(A_1'\alpha_1)$	
$\overline{CA_1''}$	$(\overline{CB_1''}, \overline{B_1'''B_1'})$	$(C_2\gamma_2)$, $(A_1''\alpha_2)$	(v.)
$\overline{CA_1'''}$	$(\overline{CB_1'''}, \overline{B_1'B_1''})$	$(C_3\gamma_3)$, $(A_1'''\alpha_3)$	

The relation (Art. 39) also subsists, in virtue of which no two reguli of the same system have any generator in common, whilst two reguli belonging to different systems have, in general, one, and only one, common generator. This relation, in fact, is due to the circumstance that to an intersection, not at a principal point, of any two directrices situated in one of the generating planes, necessarily corresponds an intersection of their corresponding directrices in the other plane.

46. The reguli conjugate to those of any one of the above five systems form, in the aggregate, another congruence (3, 2) having the same singular points and planes as **C**, and the same focal surface $F_4'^{6}$; the latter being, in fact, the envelope of the quadric surfaces upon which the several pairs of conjugate reguli of the system are situated.

The five congruences thus associated with **C**, are indicated by the symbols $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, $\mathbf{C}_4$, $\mathbf{C}_5$ in the Table of Art. 49, where a column is reserved wherein to record the singularities of each; all these singularities, it may be observed, are readily deducible from the theorems of Art. 36. Each of these five congruences determines, as **C** does, a quadric correspondence between any two of its five singular planes containing congruence-conics. The associated principal points thereof, as well as the self-corresponding point, are found from the Table in the manner already explained in Art. 44. From any one of these congruences, moreover, all the others, as well as **C**, may be derived in precisely the same manner as they were themselves derived from **C** in Art. 45; whence it follows that the rays of any two of the six associated congruences group themselves into conjugate reguli, situated on a system of quadric surfaces, enveloping their common focal surface F_4^6, passing through four fixed singular points, and touching eight fixed singular planes. Each system includes four degenerate quadrics; viz., one doubled plane (bounded by a conic) and three plane-and-point pairs.

47. Any two of the six associated congruences being given, the doubled plane of the system of quadrics proceeding from them is at once recognised in the Table of Art. 49, as that which contains their common congruence-conic. The four singular points situated in this plane are those through which every quadric of the system passes. The three point-and-plane pairs of the system are recognised with equal facility. The *points* of each pair, in fact, appear twice in the columns appropriated to the two congruences; viz., once on each of the lines appropriated to the *planes* of the pair. The eight singular planes, none of which forms a constituent part of any one of the above four degenerate quadrics, are those which are touched by every quadric of the system to which the latter belong.

48. The singularities of the common focal surface F_4^6 of the six associated congruences, regarded as the envelope of the quadric surfaces belonging to any one of the fifteen systems above referred to, may be investigated by aid of the theorems of Art. 36 in the manner described in Art. 41. The results at which we thus arrive are well known, viz.:—

First,—Each of the ten singular points of the configuration is a double point of the focal surface, the quadric cone of contact at which touches the six singular planes which pass through that point.

Secondly,—Each of the fifteen singular planes of the configuration is a double plane of the focal surface, the curve of contact therewith being a conic passing through the four singular points situated in that plane, and having double contact with the congruence-conic it contains. It is in the latter conic that the focal surface cuts the plane in question.

49. The following Table, to which reference has already been frequently made, is arranged like the one at the end of Art. 39. Since

no ambiguity could arise from so doing, however, the congruence-conics of the several planes are here indicated by the symbol (2), simply.

	C	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_3$	$\mathbf{C}_4$	$\mathbf{C}_5$
α	(2)	A_1'	A_1''	A_1'''	(2)	C
β	(2)	B_1'	B_1''	B_1'''	C	(2)
α_1	A_1'	(2)	B_1'''	B_1''	(2)	C_1
β_1	B_1	(2)	A_1'''	A_1''	C_1	(2)
α_2	A_1''	B_1'''	(2)	B_1'	(2)	C_2
β_2	B_1''	A_1'''	(2)	A_1'	C_2	(2)
α_3	A_1'''	B_1''	B_1'	(2)	(2)	C_3
β_3	B_1'''	A_1''	A_1'	(2)	C_3	(2)
γ_1	C_1	C	(2)	(2)	B_1'	A_1'
δ_1	(2)	(2)	C_3	C_2	A_1'	B_1'
γ_2	C_2	(2)	C	(2)	B_1''	A_1''
δ_2	(2)	C_3	(2)	C_1	A_1''	B_1''
γ_3	C_3	(2)	(2)	C	B_1'''	A_1'''
δ_3	(2)	C_2	C_1	(2)	A_1'''	B_1'''
γ	C	C_1	C_2	C_3	(2)	(2)

50. I pass, finally, to the case where two self-corresponding points of the quadric correspondence between α and β present themselves. Since they, together with five pairs of arbitrarily chosen corresponding points, suffice to determine the correspondence,* the possibility of their co-existence is, of course, manifest. Moreover, from what has been already explained in Art. 42, it follows, at once, that the congruence generated in such a case is not only of the second class, but also of the second order; that the intersection $\overline{\alpha\beta}$ is no longer a ray thereof; that in place of congruence-curves we have congruence-pencils in α and β; and that, with the new self-corresponding point D, a sixteenth singular plane δ is introduced, containing a congruence-pencil of which that point is the centre.

The focal surface F_4^4, moreover, is now of the fourth order, as well as of the fourth class; it does not cut either of the planes α or β, but simply touches them at the several points of the conics $\overline{a}^2$ and $\overline{b}^2$, passing through C and D, which correspond to $\overline{\alpha\beta}$.

* On the Correlation of Two Planes, Art. 47. (*Proceedings of the London Mathematical Society*, Vol. v., 1874.)

51. Since the line $\overline{\alpha\beta}$ intersects each of its corresponding conics in self-corresponding points, the congruence-rays joining the several points of the former with their corresponding ones in α or β, all converge, as is well known, to a fixed point on the conic $\overline{a}^2$ or $\overline{b}^2$. The points A_1 and B_1, thus determined, are new singular points of the generated congruence (2, 2); they are centres, in fact, of congruence-pencils in α and β, respectively.* In addition to them, there are, of course, the three singular points D_1, D_2, D_3 introduced, with the self-corresponding point D, in precisely the same manner as C_1, C_2, C_3 were introduced, with C, in Art. 43.

On the whole, therefore, the congruence **C**, now under consideration, contains sixteen singular points as well as sixteen singular planes; each of the former being the centre of a congruence-pencil situated in one of the latter, as shown in the first two columns of either of the Tables of Art. 59.

52. It has already been shown in (Arts. 42 and 43) how it may be proved that each of the first fifteen singular planes, given in the Table of Art. 49, passes through four singular points, and it is manifest that the sixteenth δ has precisely the same properties as γ. It is, moreover, obvious from the preceding Article that α and β each contain two singular points more than they formerly did. In short, it may be proved, by reasoning precisely similar to that employed in Art. 43, that every singular plane now contains six singular points. These are all indicated on the several lines of Table I. of Art. 59. From the different arrangement of the latter, given in Table II. of Art. 59, the six singular planes which pass through each singular point are rendered manifest, and a further inspection of the two Tables likewise reveals the well-known property of the configuration, in virtue of which each of the $16 \times 15 = 240$ intersections of two singular planes passes through two singular points, and *vice versâ* every line passing through two singular points lies in two singular planes.

53. Any one of the sixteen singular planes of the congruence **C** under consideration being selected, there are, as we have just seen, five others which pass through the centre of the congruence-pencil it contains, or say *its centre*, and ten which do not do so. We may say, indeed, that there are $\frac{16 \times 5}{2} = 40$ *pairs* of singular planes which pass through one

* The conics of each of the pencils $(A_1A_1'A_1''A_1''')$ and $(B_1B_1'B_1''B_1''')$ cut $\overline{\alpha\beta}$ in pairs of points in involution. In each pencil there are three line-pairs, one element of each of which, being a principal line of the correspondence, cuts $\overline{\alpha\beta}$ in a point corresponding to a principal point, and, therefore, in line with it and the centre of the congruence-pencil in its plane. Hence it follows that the line-pairs of the two pencils determine the same points on $\overline{\alpha\beta}$, and that the two involutions are identical.

another's centres, and $\frac{16 \times 10}{2} = 80$ which do not possess this property. Between the planes of each of these 80 pairs, the rays of the congruence **C** determine a quadric correspondence precisely similar to that, between α and β, by which it was generated in Art. 50. Between the planes of each of the above 40 pairs, however, the congruence **C** determines a cubic correspondence disposed in the manner described in Art. 22.

54. In proof of these statements, I observe that the rays of **C** which are incident with an arbitrary line form a quartic scroll passing through all the sixteen singular points of the configuration, and upon which that line, and another not incident therewith, are double ones. When the first line lies in one of the planes of the pair under consideration, the scroll breaks up into the congruence-pencil in that plane, and a cubic scroll (passing through the ten singular points not situated in the plane) on which the first line is single, while the second one is still double, and passes through the centre of the detached pencil.

Now, in the first of the two cases with which we are at present concerned, the cubic scroll intersects the second plane of our pair—one of the 80 of Art. 53—in a congruence-ray, and in a conic which passes through three singular points, and corresponds, point by point, to the arbitrary line in the first plane. In short, the correspondence between the two planes is in this case a quadric one.

In the second case, the intersection of the pair of planes—one of the 40 of Art. 53—is a congruence-ray, and through every point thereof a second ray passes which converts that point into a self-corresponding one. The double line, moreover, of the cubic scroll is incident with the intersection of the two planes at the centre of the first, so that the scroll cuts the second plane in a cubic of which the above centre is a double point, and which passes through the four singular points of the plane not situated on the intersection.

The method of recognising the associated principal points of the quadric, or cubic, correspondence between any pair of planes will be explained in Art. 60.

55. Returning to the quadric correspondence between α and β by means of which the congruence **C** was generated (Art. 50), I observe that the latter still contains the five systems of quadric reguli proceeding from the pairs of pencils described in Art. 45. With each of these five systems, however, another is now *associated* which proceeds, in like manner, from the corresponding elements of two pencils whose base-points are the principal, and the self-corresponding points of α and β which are not included amongst the base-points of the pencils whence the former system of reguli proceeds.

Thus, with the system proceeding from any one of the following five pairs of pencils on the left, is associated the system which proceeds from the pair of pencils indicated on the same line on the right of the page:—

α	β	System.	α	β
(A_1')	(B_1')	(i.)	$(CDA_1''A_1''')$	$(CDB_1''B_1''')$
(A_1'')	(B_1'')	(ii.)	$(CDA_1'''A')$	$(CDB_1'''B_1')$
(A_1''')	(B_1''')	(iii.)	$(CDA_1'A_1'')$	$(CDB_1'B_1'')$
$(CA_1'A_1''A_1''')$	(C)	(iv.)	(D)	$(DB_1'B_1''B_1''')$
(C)	$(CB_1'B_1''B_1''')$	(v.)	$(DA_1'A_1''A_1''')$	(D)

The four degenerate reguli included in each of the systems above referred to are all pencil-pairs. Arranged in associated systems, they are readily seen to be those exhibited in the Table on the next page. The corresponding elements of the pencils whence, as directrices, each such degenerate regulus proceeds, are placed, in juxta-position with the latter, in the extreme right and left-hand columns.

56. All the sixteen congruence-pencils of **C**, it will be observed, are represented by the degenerate reguli included in each of the five pairs of associated systems, and four such pencils are common to each system and every other not associated therewith.

Moreover, since the co-planar directrices of any two reguli belonging to associated systems intersect in two points, the reguli in question have necessarily two generators in common (Art. 45). From this it follows, as a particular case, that the quadric surfaces on which the several reguli of any system are situated all pass through the eight centres, and touch the eight planes of the four pencil-pairs included in the associated system of reguli.

This theorem is likewise true, of course, for each of the four plane-and-point pairs of the system under consideration; whence it immediately follows that the latter, together with the plane-and-point pairs of the associated system, form two conjugate *quadruple groups*;* that is to say, all the sixteen singular points and planes of the configuration are included in them, and each of the two planes of every plane-and-point pair belonging to one of the two groups passes through one of the two points of every plane-and-point pair included in the other, and

* Caporali terms such a group of eight points (planes) an *ottupla* (*Reale Accademia de Lincei*, Vol. ii., 1878).

Corresponding Directrices.		Pencil-Pair.		System.	Pencil-Pair.		Corresponding Directrices.	
$\overline{A_1'A_1''}$	$\overline{B_1'B_1'''}$	$(A_1''\alpha_2)$,	$(B_1'''\beta_3)$	(i.)	$(A_1\alpha)$,	$(A_1'\alpha_1)$	$(\overline{CDA_1''A_1'''A_1'})$	$(\overline{CD}, \overline{B_1''B_1'''})$
$\overline{A_1'A_1'''}$	$\overline{B_1'B_1''}$	$(A_1'''\alpha_3)$,	$(B_1''\beta_2)$		$(B_1\beta)$,	$(B_1'\beta_1)$	$(\overline{CD}, \overline{A_1''A_1'''})$	$(\overline{CDB_1''B_1'''B_1'})$
$\overline{A_1'C}$	$\overline{B_1'C}$	$(C\gamma)$,	$(C_1\gamma_1)$		$(C_2\gamma_2)$,	$(D_3\delta_3)$	$(\overline{CA_1''}, \overline{DA_1'''})$	$(\overline{CB_1''}, \overline{DB_1'''})$
$\overline{A_1'D}$	$\overline{B_1'D}$	$(D\delta)$,	$(D_1\delta_1)$		$(C_3\gamma_3)$,	$(D_2\delta_2)$	$(\overline{CA_1'''}, \overline{DA_1''})$	$(\overline{CB_1'''}, \overline{DB_1''})$
$\overline{A_1''A_1'''}$	$\overline{B_1''B_1'}$	$(A_1'''\alpha_3)$,	$(B_1'\beta_1)$	(ii.)	$(A_1\alpha)$,	$(A_1''\alpha_2)$	$(\overline{CDA_1'''A_1'A_1''})$	$(\overline{CD}, \overline{B_1'''B_1'})$
$\overline{A_1''A_1'}$	$\overline{B_1''B_1'''}$	$(A_1'\alpha_1)$,	$(B_1'''\beta_3)$		$(B_1\beta)$,	$(B_1''\beta_2)$	$(\overline{CD}, \overline{A_1'''A_1'})$	$(\overline{CDB_1'''B_1'B_1''})$
$\overline{A_1''C}$	$\overline{B_1''C}$	$(C\gamma)$,	$(C_2\gamma_2)$		$(C_3\gamma_3)$,	$(D_1\delta_1)$	$(\overline{CA_1'''}, \overline{DA_1'})$	$(\overline{CB_1'''}, \overline{DB_1'})$
$\overline{A_1''D}$	$\overline{B_1''D}$	$(D\delta)$,	$(D_2\delta_2)$		$(C_1\gamma_1)$,	$(D_3\delta_3)$	$(\overline{CA_1'}, \overline{DA_1'''})$	$(\overline{CB_1'}, \overline{DB_1'''})$
$\overline{A_1'''A_1'}$	$\overline{B_1'''B_1'}$	$(A_1'\alpha_1)$,	$(B_1''\beta_2)$	(iii.)	$(A_1\alpha)$,	$(A_1'''\alpha_3)$	$(\overline{CDA_1'A_1''A_1'''})$	$(\overline{CD}, \overline{B_1'B_1''})$
$\overline{A_1'''A_1''}$	$\overline{B_1'''B_1''}$	$(A_1''\alpha_2)$,	$(B_1'\beta_1)$		$(B_1\beta)$,	$(B_1'''\beta_3)$	$(\overline{CD}, \overline{A_1'A_1''})$	$(\overline{CDB_1'B_1''B_1'''})$
$\overline{A_1'''C}$	$B_1'''C$	$(C\gamma)$,	$(C_3\gamma_3)$		$(C_1\gamma_1)$,	$(D_2\delta_2)$	$(\overline{CA_1'}, \overline{DA_1''})$	$(\overline{CB_1'}, \overline{DB_1''})$
$\overline{A_1'''D}$	$\overline{B_1'''D}$	$(D\delta)$,	$(D_3\delta_3)$		$(C_2\gamma_2)$,	$(D_1\delta_1)$	$(\overline{CA_1''}, \overline{DA_1'})$	$(\overline{CB_1''}, \overline{DB_1'})$
$(\overline{CA_1'A_1''A_1'''D})$	$\overline{CD}$	$(A_1\alpha)$,	$(D\delta)$	(iv.)	$(C\gamma)$,	$(B_1\beta)$	$\overline{DC}$	$(\overline{DB_1'B_1''B_1'''C})$
$(\overline{CA_1'}, \overline{A_1''A_1'''})$	$\overline{CB'}$	$(C_1\gamma_1)$,	$(B_1'\beta_1)$		$(D_1\delta_1)$,	$(A_1'\alpha_1)$	$\overline{DA_1'}$	$(\overline{DB_1'}, \overline{B_1''B_1'''})$
$(\overline{CA_1''}, \overline{A_1'''A'})$	$\overline{CB_1''}$	$(C_2\gamma_2)$,	$(B_1''\beta_2)$		$(D_2\delta_2)$,	$(A_1''\alpha_2)$	$\overline{DA_1''}$	$(\overline{DB_1''}, \overline{B_1'''B_1'})$
$(\overline{CA_1'''}, \overline{A_1'A_1''})$	$\overline{CB_1'''}$	$(C_3\gamma_3)$,	$(B_1'''\beta_3)$		$(D_3\delta_3)$,	$(A_1'''\alpha_3)$	$\overline{DA_1'''}$	$(\overline{DB_1'''}, \overline{B_1'B_1''})$
$\overline{CD}$	$(\overline{CB_1'B_1''B_1'''D})$	$(B_1\beta)$,	$(D\delta)$	(v.)	$(A_1\alpha)$,	$(C\gamma)$	$(\overline{DA_1'A_1''A_1'''C})$	$\overline{DC}$
$\overline{CA_1'}$	$(\overline{CB_1'}, \overline{B_1''B_1'''})$	$(C_1\gamma_1)$,	$(A_1'\alpha_1)$		$(D_1\delta_1)$,	$(B_1'\beta_1)$	$(\overline{DA_1'}, \overline{A_1''A_1'''})$	$\overline{DB_1'}$
$\overline{CA_1''}$	$(\overline{CB_1''}, \overline{B_1'''B_1'})$	$(C_2\gamma_2)$,	$(A_1''\alpha_2)$		$(D_2\delta_2)$,	$(B_1''\beta_2)$	$(\overline{DA_1''}, \overline{A_1'''A_1'})$	$\overline{DB_1''}$
$\overline{CA_1'''}$	$(\overline{CB_1'''}, \overline{B_1'B_1''})$	$(C_3\gamma_3)$,	$(A_1'''\alpha_3)$		$(D_3\delta_3)$,	$(B_1'''\beta_3)$	$(\overline{DA_1'''}, \overline{A_1'A_1''})$	$\overline{DB_1'''}$

correlatively each of the two points of the former lies in one of the two planes of every point-and-plane pair included in the latter.

It should be added, that here, as before, two reguli belonging to different and non-associated systems have, in general, one, and only one, generator in common; while two reguli belonging to the same system have no common generator (Art. 45).

57. The reguli conjugate to those included in any one of the preceding systems constitute, in the aggregate, another congruence (2, 2), having the same singular points and planes—in fact, the same focal surface F_4^4 as **C** itself (Art. 36). In passing from any system of reguli to the conjugate system, the centres and planes are simply interchanged, not only of the four pencil-pairs which each such system includes (Art. 36g), but also of the four pencil-pairs determined by the quadrics, on which the reguli of the system are situated, around the four pairs of points through which they all pass, and in the four pairs of planes which they all touch (Art. 56). From this it readily follows that the two congruences are identical which are formed by the reguli conjugate, respectively, to those of any two associated systems (Art. 55). From the congruence **C**, therefore, five, and only five, other congruences can be derived by the method under consideration. In the Tables of Art. 59 these are indicated by $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, $\mathbf{C}_4$, $\mathbf{C}_5$, respectively, and in the column appropriated to each will be found, in Table I., the centres of the pencils situated in the several singular planes, and, in Table II., the planes of the pencils which have the several singular points of the configuration for their centres.

58. Here, as in Arts. 40 and 46, it should be observed that any one of the above-mentioned five congruences may be taken for the primitive one, and that from it five others may be derived in precisely the same manner as that above described.* These five, however, are always identical with the five remaining congruences of the original six. Hence we may at once infer that the rays of any two of the six congruences (2, 2) which have a common focal surface group themselves, and that *in two different ways* (Art. 57),† into conjugate reguli

---

* If α and β be retained as generating planes, however, the correspondence between them will be a cubic one when $\mathbf{C}_4$ or $\mathbf{C}_5$ is taken as the primitive congruence (Art. 53), and, in resolving it into associated systems of quadric reguli, obvious modifications will, of course, have to be introduced (see Art. 65).

† The quartic regulus formed by all rays, of any given congruence (2, 2), which are incident with an arbitrary line (directrix), and on which the latter is one of two double lines (Art. 54), breaks up, therefore, into two quadric reguli when the directrix in question is a ray of any one of the five congruences having, with the given one, a common focal surface. These two quadric reguli belong to one of the pairs of associated systems into which the given congruence can be resolved (Art. 55).

situated on a system of quadrics which include, as degenerate forms, the elements of a quadruple group (Art. 56), and every surface of which passes through the eight points, and touches the eight planes of the conjugate quadruple group.

The total number of different systems of quadrics enveloping the focal surface F_4^4 of a given congruence (2, 2) is obviously 30; and this is also the number of different quadruple groups (15 conjugate pairs) which can be formed from the sixteen singular points and sixteen singular planes of the configuration.*

59. The following are the two Tables to which reference has already been made in Arts. 52 and 57. Their construction is substantially the same as that of the Table given in Art. 49, the main difference being that the singular planes now contain congruence-pencils solely, and that perfect reciprocity everywhere prevails.

Table I.

	C	**C**$_1$	**C**$_2$	**C**$_3$	**C**$_4$	**C**$_5$
α	A_1	A_1'	A_1''	A_1'''	D	C
β	B_1	B_1'	B_1''	B_1'''	C	D
α_1	A_1'	A_1	B_1'''	B_1''	D_1	C_1
β_1	B_1'	B_1	A_1'''	A_1''	C_1	D_1
α_2	A_1''	B_1'''	A_1	B_1'	D_2	C_2
β_2	B_1''	A_1'''	B_1	A_1'	C_2	D_2
α_3	A_1'''	B_1''	B_1'	A_1	D_3	C_3
β_3	B_1'''	A_1''	A_1'	B_1	C_3	D_3
γ_3	C_3	D_2	D_1	C	B_1'''	A_1'''
δ_3	D_3	C_2	C_1	D	A_1'''	B_1'''
γ_2	C_2	D_3	C	D_1	B_1''	A_1''
δ_2	D_2	C_3	D	C_1	A_1''	B_1''
γ_1	C_1	C	D_3	D_2	B_1'	A_1
δ_1	D_1	D	C_3	C_2	A_1'	B_1'
γ	C	C_1	C_2	C_3	B_1	A_1
δ	D	D_1	D_2	D_3	A_1	B_1

Table II.

	C	**C**$_1$	**C**$_2$	**C**$_3$	**C**$_4$	**C**$_5$
A_1	α	α_1	α_2	α_3	δ	γ
B_1	β	β_1	β_2	β_3	γ	δ
A_1'	α_1	α	β_3	β_2	δ_1	γ_1
B_1'	β_1	β	α_3	α_2	γ_1	δ_1
A_1''	α_2	β_3	α	β_1	δ_2	γ_2
B_1''	β_2	α_3	β	α_1	γ_2	δ_2
A_1'''	α_3	β_2	β_1	α	δ_3	γ_3
B_1'''	β_3	α_2	α_1	β	γ_3	δ_3
C_3	γ_3	δ_2	δ_1	γ	β_3	α_3
D_3	δ_3	γ_2	γ_1	δ	α_3	β_3
C_2	γ_2	δ_3	γ	δ_1	β_2	α_2
D_2	δ_2	γ_3	δ	γ_1	α_2	β_2
C_1	γ_1	γ	δ_3	δ_2	β_1	α_1
D_1	δ_1	δ	γ_3	γ_2	α_1	β_1
C	γ	γ_1	γ_2	γ_3	β	α
D	δ	δ_1	δ_2	δ_3	α	β

60. The nature of the correspondence, determined by any one of the six congruences (2, 2), between any two given planes may be thus

* Caporali, *loc. cit.*

recognised in Table I. If the points in the column devoted to the congruence in question, and on the lines appropriated to the two planes, are those which are common to the latter (Art. 53), the correspondence is cubic; if not, quadric. In the first case, the common points reappear in another column, on exchanged lines, and there indicate the principal double points of the planes to which those lines are respectively appropriated. In each of the remaining four columns a pair of associated principal single points of the two planes will be found. In the second case, each of two of the remaining five columns contains the self-corresponding points of the two planes, and each of the other three a pair of associated principal single points of the quadric correspondence existing between them.

61. Two conjugate quadruple groups (Art. 56) are determined by any one pair of planes (or points) belonging to either of them. In the one case, Table I. (in the other, Table II.) enables us readily to detect all their remaining elements. The methods of doing so being precisely similar in both cases, we will, for the sake of illustration, confine our attention to Table I., where, in each of two easily detected columns, both the points appear which are common to the two given planes. In each of the remaining four columns, and on the lines appropriated to those planes, will be found a pair of points of the quadruple group conjugate to that to which the given pair of planes belongs. The points of each of the four pairs just determined reappear in each of the above-mentioned two columns, and that on the lines appropriated to the two planes which pass through them. From the latter, finally, the quadruple group to which the given pair of planes belong is completed in a precisely similar manner.

62. If the two columns above referred to had been given, the conjugate quadruple groups could have been determined with equal readiness; and thus we should have solved, by inspection, the problem on which depends the complete definition of each of the two associated systems of quadrics whose conjugate reguli constitute, in the aggregate, the two congruences indicated by the given columns (Art. 58).

63. Little need be added with respect to the singularities of the focal surface F_4^4 common to the six congruences (2, 2) indicated in the preceding Tables. Each of the sixteen singular planes of the configuration is a double tangent-plane of the surface, and touches it along the conic which passes through the six singular points of the plane. This conic is the locus of the intersection of corresponding rays (Art. 36 d) of the two congruence-pencils determined by the quadrics which touch that plane, and at the same time belong to one or other of the systems

above defined. Each of the sixteen singular points, again, is a double point of F_4^4; the quadric cone of contact thereat being the envelope of the plane of corresponding rays (Art. 36c) of the two congruence-pencils determined by the quadrics, belonging to one of the systems already referred to, which pass through that point. This quadric cone manifestly touches each of the six singular planes of the configuration which pass through the singular point in question.

Congruences of the Second Order.

64. All Cremonian Congruences of the second order may be generated by the method given in Art. 22. The formulæ of that Article become, for such congruences,

$$\sum_1^{n-1} i\bar{a}_i = n-1,$$

$$\sum_1^{n-1} \bar{a}_i = n-2,$$

where n denotes the degree of the correspondence between the two generating planes, and $\bar{a}_i$ the number of principal points, belonging to either of them, which are of the order i and are situated on their intersection $\overline{a\beta}$. From these formulæ we may at once deduce

$$\sum_1^{n-1} (i-1)\,\bar{a}_i = 1,$$

and thence infer, first that no principal point whose order of multiplicity is higher than 2 can lie on $\overline{a\beta}$, secondly that one and only one principal double point of each plane must be so situated, and lastly that $n-3$ principal single points of each plane must also lie on $\overline{a\beta}$.

Kummer having shown that the class, $n-1$, of no congruence of the second order can exceed 7,* it follows readily from the above, and from Cremona's tabulated results,† that the only correspondences available for the generation of congruences of the second order, by the method under consideration, are of the degrees 3, 4, 5, and 6, respectively; and, further, that only one correspondence of each of these degrees can be so employed. The generated congruences are, of course, of the classes 2, 3, 4, and 5, respectively. The general properties of the first three of these are simply correlative to those already studied in Arts. 37 to 63; it will suffice, therefore, to direct attention to the special features revealed by the present mode of generation. The congruence (2, 5), on the other hand, will require a somewhat fuller treatment.

65. Of the cubic correspondence between a and β, let C and D be

* *Algebraische Strahlensysteme*, § 6, XXXV.
† *Bulletin*, Arts. 19 to 23.

the principal double points, the former being supposed to belong to β, the latter to α, and both situated on $\overline{\alpha\beta}$. Moreover, let A_1, B_1; A_1', B_1'; A_1'', B_1''; and A_1''', B_1''' be the four pairs of associated principal single points of the correspondence. From Art. 22, and well-known properties, it follows that the principal conics of the two planes intersect one another in C and D, and that each passes through the four principal single points of its plane.*

The principal double point of each plane is the centre of a congruence-pencil situated in the other plane, and each principal single point is the centre of a congruence-pencil whose plane passes through that point and its corresponding principal line. To each of the six lines, moreover, which pass through two principal single points of either plane corresponds, point by point, the line passing through the two non-associated principal single points of the other plane; and, since two such corresponding lines are necessarily incident in a self-corresponding point, the lines joining their remaining pairs of corresponding points necessarily form a congruence-pencil. The sixteen such pencils are thus all accounted for, and the congruence (2, 2) under consideration is clearly identical with that denoted by $\mathbf{C}_5$ in the Tables of Art. 59.

The five pairs of associated systems of quadric reguli into which $\mathbf{C}_5$ may be resolved (Art. 56) proceed from the following pencils of corresponding directrices, and in each system four degenerate reguli (pencil-pairs) are included:

α	β		α	β
(A_1)	$(CB_1'B_1''B_1''')$	$\mathbf{C}$	$(DA_1'A_1''A_1''')$	(B_1)
(A_1')	$(CB_1B_1''B_1''')$	$\mathbf{C}_1$	$(DA_1A_1''A_1''')$	(B_1')
(A_1'')	$(CB_1B_1'B_1''')$	$\mathbf{C}_2$	$(DA_1A_1'A_1''')$	(B_1'')
(A_1''')	$(CB_1B_1'B_1'')$	$\mathbf{C}_3$	$(DA_1A_1'A_1'')$	(B_1''')
(D)	(C)	$\mathbf{C}_4$	$(A_1A_1'A_1''A_1''')$	$(B_1B_1'B_1''B_1''')$

The reguli conjugate to those of any one of the above ten systems (five associated pairs) constitute, in the aggregate, the congruence indicated in the central column on the line upon which stand the pencils of corresponding directrices from which the system in question proceeds.

66. To generate the congruence (2, 3), by the present method, the

* A_1, B_1; A_1', B_1'; A_1'', B_1''; A_1''', B_1'''; C, D; D, C form, in fact, six pairs of corresponding points of two projective ranges on these conics. (See Note to Art. 51.)

quartic correspondence is employed which has three principal double, and three principal single points in each of the planes α and β.* To any principal single point in either of these planes corresponds an indeterminate point in a principal line passing through two principal double points of the other plane. The remaining principal double point of the latter plane will be termed the *associate* of the principal single point in the former, and the principal points of the planes α and β, thus associated, will be indicated by A_2, B_1; A_2', B_1'; A_2'', B_1''; A_1, B_2; A_1', B_2'; A_1'', B_2'', respectively, where the multiplicity of each point is indicated by a suffix.

If A_2 and A_1 be the two principal points of α, which, by Art. 64, must, in the present case, be situated on $\overline{\alpha\beta}$, it is easy to see that their associates B_1 and B_2 will likewise be so situated. Moreover, A_2 and B_2 will lie on the principal conics $(B_2B_2'B_2''B_1'B_1'')$ and $(A_2A_2'A_2''A_1'A_1'')$ which respectively correspond to them, and A_1 and B_1 on their corresponding principal lines $\overline{B_2'B_2''}$ and $\overline{A_2'A_2''}$.

From this it follows at once that, in the congruence **C** now generated, A_2 and B_2 are centres of congruence-pencils situated in β and α, respectively, and that each of the points A_1', A_1'', B_1', B_1'' is the centre of a congruence-pencil in the plane which connects it with its corresponding principal line. These four planes will be denoted by $\alpha_1, \alpha_2, \beta_1, \beta_2$, respectively.

The remaining four congruence-pencils which **C** possesses proceed from the pairs of corresponding lines $\overline{A_2'A_1'}$ and $\overline{B_2'B_1'}$, $\overline{A_2''A_1''}$ and $\overline{B_2''B_1''}$, $\overline{A_2'A_1''}$ and $\overline{B_2''B_1'}$, and $\overline{A_2''A_1'}$ and $\overline{B_2'B_1''}$. The lines of each pair being incident at a self-corresponding point, the congruence-lines connecting their remaining corresponding points form a pencil. These pencils will be denoted by $(C_1\gamma_1)$, $(C_2\gamma_2)$, $(D_1\delta_1)$, and $(D_2\delta_2)$, respectively. The lines $\overline{C_1C_2}$ and $\overline{\gamma_1\gamma_2}$, as well as $\overline{D_1D_2}$ and $\overline{\delta_1\delta_2}$, will obviously be identical.

The only five remaining singular points of **C** are vertices of quadric congruence-cones. Four of these vertices are at the principal points A_2', A_2'', B_2', B_2'', and the cones to which they belong pass through the principal conics corresponding to these points. The fifth congruence-cone belongs to one of the systems of quadric reguli into which **C** may be resolved (see Art. 68). It intersects α and β in a pair of corresponding conics of the pencils $(A_2'A_2''A_1'A_1'')$ and $(B_2'B_2''B_1'B_1'')$, and its vertex C lies in each of the four singular planes $\alpha_1, \alpha_2, \beta_1, \beta_2$.

67. Each of the ten singular planes of the congruence **C** under consideration passes through six of its fifteen singular points. The latter, which are readily recognised by methods previously employed (Art. 43),

---

* *Bulletin*, Art. 20.

are indicated on the line appropriated to that plane in the Table of Art. 69. Two of these six points are always vertices of congruence-cones, and the remaining four of congruence-pencils in **C**. The constituents of each of the forty-five *pairs* of planes that can be formed from the above ten planes have two singular points in common. In each of fifteen pairs, these common singular points are centres of congruence-pencils of **C** in the respective planes; but, in each of the remaining thirty, one is the centre of a congruence-pencil of **C** in a plane not co-incident with either of those forming the pair, while the other is the vertex of a congruence-cone.

The nature of the correspondence, determined by **C**, between the planes of a pair depends upon the above circumstances. Between the planes of each of the fifteen pairs, the correspondence is a quartic one of precisely the same kind as that from which **C** was generated. Between the planes of each of the thirty pairs, however, the correspondence is a cubic one of a special nature. Two of its four pairs of associated principal single points coincide at the vertex of the congruence-cone, situated on the intersection of the planes of the pair. The other singular point of **C** on this intersection is simply a self-corresponding point of the pair of planes. The principal double points of the cubic correspondence between them are at the remaining two vertices of congruence-cones which the planes contain. Of the three remaining singular points of **C** situated in each of them, two are principal single points of the cubic correspondence, and the third is the point to which all rays converge which connect the several points of the intersection of the pair of planes with their correspondents. This intersection, in fact, must be regarded as a line passing through one, only, of the two principal single points of its plane which are coincident with, or rather consecutive to, one another, on the principal conic of the plane. To it, therefore, will correspond a cubic which breaks up into a principal line, and a conic corresponding, point by point, to the intersection in question, and cutting it in two self-corresponding points. The convergence, at a point on this conic, of the congruence-rays joining its several points to their corresponding ones on the intersection is an immediate consequence (Art. 51).

68. Returning to the quartic correspondence between α and β, by means of which the congruence **C** was generated (Art. 66), I observe that the resolution of the latter into a system of quadric reguli may be effected by the aid of any one of the following five pencils of corresponding directrices:

$$(A_2') \text{ and } (B_2B_2'B_2''B_1'),$$
$$(A_2'') \text{ and } (B_2B_2'B_2''B_1''),$$
$$(A_2A_2'A_2''A_1') \text{ and } (B_2'),$$

$$(A_3 A_2' A_2'' A_1'') \text{ and } (B_2''),$$

and $$(A_2' A_2'' A_1' A_1'') \text{ and } (B_2' B_2'' B_1' B_1'').$$

The reguli conjugate to those of each system form, in the aggregate, another congruence (2, 3) having the same focal surface as **C**. They are indicated in the Table (Art. 69) by $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, $\mathbf{C}_4$, $\mathbf{C}_5$, respectively. The rays of any two of the above six congruences group themselves, in fact, into conjugate reguli on a system of quadric surfaces enveloping the common focal surface, passing, individually, through eight fixed points, and touching four fixed planes. Of such systems of quadrics there are, of course, $\frac{6 \times 5}{2} = 15$; each includes four degenerate forms, viz., three point-and-plane pairs, and one cone. The vertex of the latter appears, on one of the last five lines of the Table, in each of the columns appropriated to the two congruences whence the system of quadrics proceeds. The remaining eight points in these two columns, and on the last five lines, are those through which every quadric of the system passes. The three point-and-plane pairs of the system are recognised in the Table, by the fact that the points of the pair appear twice in each of the two columns,—viz., once on each of the two lines indicative of the planes of the pair. The remaining four of the first ten lines of the Table are those which are appropriated to the planes touched by every quadric of the system under consideration. They all pass, of course, through the vertex of the quadric cone included in that system.

69. The following Table exhibits, conveniently, the singularities of the six congruences (2, 3) which have the same focal surface. To each congruence, as before, a column is appropriated, in which, on the first ten lines, stand the centres of the congruence-pencils situated in the several planes to which those lines are appropriated. On the last five lines are the vertices of the quadric cones whose generators are rays of the respective congruences.

	C	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_3$	$\mathbf{C}_4$	$\mathbf{C}_5$
α	B_2	A'_2	A''_2	A''_1	A'_1	A_2
β	A_2	B''_1	B'_1	B'_2	B''_2	B_2
α_1	A'_1	C_1	D_2	C	B_2	B''_2
β_1	B'_1	C	A_2	C_1	D_1	A''_2
α_2	A''_1	D_1	C_2	B_2	C	B'_2
β_2	B''_1	A_2	C	D_2	C_2	A'_2
γ_1	C_1	A'_1	B'_2	B'_1	A'_2	C_2
γ_2	C_2	B''_2	A''_1	A''_2	B''_1	C_1
δ_1	D_1	A''_1	B''_2	A'_2	B'_1	D_2
δ_2	D_2	B'_2	A'_1	B''_1	A''_2	D_1
	A''_2	A''_2	B_2	C_2	D_2	B'_1
	A'_2	B_2	A'_2	D_1	C_1	B''_1
	B''_2	C_2	D_1	B''_2	A_2	A'_1
	B	D_2	C_1	A_2	B'_2	A''_1
	C	B'_1	B''_1	A'_1	A''_1	C

70. To generate the congruence (2, 4), the quintic correspondence is employed which has, in each of the planes α and β, a principal triple point, as well as three double, and three single principal points.* To each of the latter corresponds a principal line passing through the triple point and one of the former,—a circumstance which at once associates the double and single principal points of the correspondence, and suggests the notation A_2, B_1; A'_2, B'_1; A''_2, B''_1; A_1, B_2; A'_1, B'_2; A''_1, B''_2. The principal triple points are denoted by A_3 and B_3.

The conditions of Art. 64 will be satisfied by supposing A_2, A'_1, A''_1, B_2, B'_1, B''_1 to be all situated on the intersection $\overline{\alpha\beta}$; whence it will follow, as before, that A_2 and B_2 become centres of congruence-pencils situated in the planes β and α, respectively, and A_1 and B_1 centres of pencils situated in the planes α_1 and β_1, which connect these points with the principal lines respectively corresponding thereto. Moreover, since the lines $\overline{A_1A_3}$ and $\overline{B_1B_3}$ correspond, point by point, to $B'_2B''_2$ and $A'_2A''_2$, respectively, and each is incident with its corresponding line in a self-corresponding point, we have obviously two other congruence-pencils which may be denoted, respectively, by (C, γ) and (D, δ).

In addition to the above singularities, the congruence **C** now under

* *Bulletin*, Art. 21.

consideration possesses six quadric congruence-cones. Four of these have the vertices A_2', A_2'', B_2', B_2'', and pass, respectively, through the principal conics corresponding to these points. Each of the remaining two intersects a and β in a pair of corresponding conics of the pencils $(A_3A_2'A_2''A_1)$ and $(B_3B_2'B_2''B_1)$. In fact, the several pairs of corresponding conics of these pencils are, as will be seen in Art. 72, the directrices of one of the systems of quadric reguli into which the congruence **C** may be resolved. The generators of those reguli which pass through A_1 and B_1 are corresponding rays of two projective pencils (A_1a_1) and $(B_1\beta_1)$; and the double points, C' and C'', of the projective rows which these pencils determine on $\overline{a_1\beta_1}$, are obviously the vertices of the two congruence-cones above referred to.

Lastly, **C** contains two cubic congruence-cones; their vertices are at A_3 and B_3, and their traces on β and a, respectively, are the principal cubics corresponding to these vertices. $\overline{A_3B_3}$ is a double generator of each of these cones, and likewise a double ray of the congruence.

71. Each of the six singular planes of **C** contains six of its fourteen singular points, and to each *pair* of singular planes two singular points are common (see Table of Art. 72). In six of the fifteen different pairs formed by these planes the common points are centres of congruence-pencils situated in those planes; in three others they are both vertices of quadric congruence-cones; and in the six remaining pairs of planes one of the common points is the vertex of a cubic congruence-cone, and the other the centre of a congruence-pencil situated in a plane not coincident with either of those forming the pair.

Between the planes of each of the first six pairs, the congruence **C** determines a quintic correspondence of precisely the same type as that employed for its generation. The principal points thereof are easily recognised in the Table of Art. 72.

Between the planes of each of the above three pairs, however, the correspondence determined by **C** is quartic, and of the Jonquierian type (Art. 29). The principal triple points thereof are, of course, the vertices of the two cubic congruence-cones; and, of the six principal single points of each plane, two are coincident, with one another and with their associates, at each of the two singular points of **C** which are common to the pair of planes under consideration. Of the three remaining singular points of **C** situated in each of these planes, two are principal single points of the quartic correspondence between them, and the third is the point towards which converge all lines joining the points of the intersection of the planes to their several corresponding points. The latter, together with the point of convergence and the principal points of the plane, are all situated on a conic, seeing that the above intersection must be regarded as a line passing through two principal single points only; the points respectively coincident with these being, in reality, consecutive points thereto, situated on the principal cubic of the plane to which the intersection is supposed to belong.

Finally, between the planes of each of the remaining six pairs the congruence **C** determines a quartic correspondence of the type considered in Art. 66; but of a special form. Of the two singular points of **C** common to the planes of the pair, that which is the centre of a congruence-pencil is simply a self-corresponding point of this quartic correspondence, and with that which is the vertex of a cubic congruence-cone, one of the principal double points of each plane coincides. The other two principal double points situated in each plane are vertices of quadric congruence-cones in **C**, and their associated principal single points (Art. 66) in the quartic correspondence, are both coincident with the two coincident principal double points. The principal conic corresponding to each of the latter breaks up into the pair of principal lines which there intersect, and themselves correspond to the principal single points coincident with the principal double point in question.

Of the remaining two singular points of **C** situated in each of the two planes, one is the principal single point, of the quartic correspondence, which is associated with one of the two coincident double points, and the other (centre of a congruence-pencil of **C** situated in the plane under consideration) is the point of convergence of all congruence-lines joining points of the intersection of the pair of planes with their correspondents, situated on a conic.

72. Returning once more to the quintic correspondence between α and β by means of which the congruence **C** of Art. 70 was generated, I observe that the latter may be resolved, in three different ways, into a system of quadric reguli. Their respective directrices are the corresponding elements of the following pencils:—

$$(A_3) \quad \text{and} \quad (B_3 B_2 B_2' B_2''),$$
$$(A_3 A_2 A_2' A_2'') \quad \text{and} \quad (B_3),$$
$$(A_3 A_2' A_2'' A_1) \quad \text{and} \quad (B_3 B_2' B_2'' B_1).$$

The reguli conjugate to those included in each of the above three systems form, in the aggregate, another congruence (2, 4) having, with **C**, the same focal surface. In the following Table the latter are denoted by $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, respectively. The rays of any two of the four congruences group themselves into conjugate reguli situated on a system of quadric surfaces enveloping the common focal surface, passing through eight fixed points, and touching two fixed planes. Amongst the quadrics of each of the six systems, above referred to, there are always two point-and-plane pairs, and two cones. A glance at the columns of the Table appropriated to the two congruences under consideration is sufficient to recognise the points of the former (Art. 68), and on each of the two lines not appropriated by any of their planes will be found, in the last two columns, the vertices of the latter. The planes to which the two last-mentioned lines refer, are those which are touched by every quadric of the system, and the eight points through which all these surfaces pass are indicated on the two lines appropriated to the planes of each of the two point-and-plane pairs.

It should be observed that the two quadric cones common to any two of the four congruences, though they have the same vertices, are different from the two cones common to the remaining two congruences.

The vertices of the two cubic cones of any one of the four congruences will be found in the appropriate column, on the last two lines of the Table. Each of these cones passes twice through the vertex of the other, once through each of the vertices of the six quadric cones of the congruence, and once through each of the centres of three of its pencils—the three, in fact, whose planes pass through the vertex of the cubic cone under consideration.

	C	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_3$		
α	B_2	A_1	A_3	A_2	A'_2	A''_2
β	A_2	B_3	B_1	B_2	B'_2	B''_2
α_1	A_1	B_2	C	B_3	C'	C''
β_1	B_1	D	A_2	A_3	C'	C''
γ	C	A_3	A_1	D	B'_2	B''_2
δ	D	B_1	B_3	C	A'_2	A''_2
	A_3	A_2	B_2	A_1		
	B_3	C	D	B_1		

73. From the fact that the sextic correspondence by means of which the congruence (2, 5) is to be generated (Art. 64), must have one principal double point and three principal single ones, of each plane, situated on their intersection $\overline{\alpha\beta}$, we at once recognise the correspondence to be the remarkable one whose principal points are not of the same nature in both planes.* In one of these, say α, there are, in fact, three triple, one double, and four single points, whilst in the other, β, one of the principal points is quadruple, four are double, and three single.

* *Bulletin*, Art. 22.

The association of these points is readily effected. To each single point of α corresponds the principal line joining its associated double point in β to the quadruple point; whilst to each single point of the latter plane corresponds the principal line passing through its two non-associated triple points in α.

In conformity with previous practice, therefore, we may denote the several pairs of associated principal points thus:

$$A_1, B_3;\ A_1', B_2';\ A_1'', B_2'';\ A_1''', B_2''';\ A_3', B_1';\ A_3'', B_1'';\ A_3''', B_1''';$$

and the remaining two principal points of the correspondence by A_2 and B_4.

The principal points situated on $\overline{\alpha\beta}$ being $A_2, A_1', A_1'', A_1''', B_3, B_1', B_1'', B_1'''$—each situated, of course, on its corresponding principal conic or line—it is at once manifest that A_2 and B_3 will be centres of congruence-pencils situated, respectively, in β and α, and that A_1 will be the centre of a third such pencil situated in the plane $(A_1B_3B_4)$, which we will denote by γ. Moreover, B_4 is the vertex of a quartic congruence-cone which traces, on α, the principal curve corresponding to its vertex; viz., a quartic having double points at A_3', A_3'', A_3''', and passing likewise through A_2, A_1, A_1', A_1'', and A_1'''. The points A_3', A_3'', A_3''' are vertices of cubic congruence-cones, each of which traces, on β, the principal curve corresponding to its vertex;—that is to say, a cubic passing twice through B_4, and once through all the other principal points of the plane, the single one excepted which is associated with the vertex. In like manner, B_2', B_2'', B_2''' are vertices of quadric congruence-cones; the principal conic which each traces, on α, passes through the principal single point associated with its vertex, as well as through the double point A_2, and the triple points A_3', A_3'', A_3'''.

Besides the above three quadric cones, however, the congruence **C** now under consideration possesses three others. Their vertices C', C'', C''' are in the plane γ, and they belong to one of the two systems of quadric reguli into which **C** may be resolved (see Art. 75), and whose directrices are corresponding conics of the pencils

$$(A_3'A_3''A_3'''A_1) \text{ and } (B_4B_2'B_2''B_2''').$$

In fact, the generator, passing through A_1, of every such regulus must lie in the plane γ of that point and its corresponding line $\overline{B_3B_4}$, and the generator of it which passes through B_4 must belong to the quartic congruence-cone above described, which has this point for vertex. But the regulus will, obviously, become a cone when these two generators are incident with each other, and this will happen when the one last mentioned coincides with any one of the intersections, exclusive of $\overline{B_4A_1}$, of the above quartic congruence-cone and the plane γ. It can readily be shown that the vertices C_1', C_1'', C_1''' lie, with the principal points A_1, B_3, B_4, on a conic.

74. Six of the thirteen singular points of **C** lie in each of its three singular planes; one, B_3, being common to all three, and three others, A_2, B_4, A_1, to the pairs of planes α, β; β, γ; and γ, α, respectively. Between the planes, γ and α, of the last pair, **C** determines a sextic correspondence precisely like that from which it was generated. Between the planes β and γ, however, **C** determines a quintic correspondence of the type described in Art. 70, but of a very special kind. The principal triple points thereof both coincide, in fact, with B_4; the principal double points are at B_2', B_2'', B_2''' and C', C'', C'''; but the principal single points respectively associated with these are all coincident with the two coincident triple points at B_4. One consequence of this very special disposition of principal points is, that each of the principal cubics of the correspondence breaks up into three right lines, coincident with the principal lines of its plane. The point B_3, centre of a congruence-pencil in α, is now a self-corresponding point, and to $\overline{\beta\gamma}$, regarded as a line passing through the triple point of either plane, corresponds, point by point, a conic which passes through the three double points of the other, as well as through the self-corresponding point B_3, and the triple point at B_4. The latter point being likewise self-correspondent, the congruence-lines joining the several points of $\overline{\beta\gamma}$ with their corresponding ones all converge to a point on the above conic (Art. 51);—in fact, to A_2 in β, and to A_1 in γ.

75. Returning now to the sextic correspondence of Art. 73, I observe that the congruence **C** may be resolved, in two different ways, into a system of quadric reguli. By the one way, the directrices of these reguli are the several pairs of corresponding conics of the pencils

$$(A_3'A_3''A_3'''A_1) \text{ and } (B_4B_2'B_2''B_2'''),$$

already alluded to in the last Article; by the other, they are corresponding elements of the pencils

$$(A_3'A_3''A_3'''A_2) \text{ and } (B_4).$$

Each system of reguli includes three cones and one pencil-pair. In the first system the cones have their vertices at C', C'', C'''; in the second at B_2', B_2'', B_2'''. The pencil-pair, in the first system, is $(A_2\beta)$, $(B_3\alpha)$, and proceeds from the corresponding conics $(A_3'A_3''A_3'''A_1A_2)$ and $(B_4B_2'B_2''B_2'''B_3)$. In the second system it is $(A_1\gamma)$, $(B_3\alpha)$, proceeding from the corresponding elements $(A_3'A_3''A_3'''A_2A_1)$ and $\overline{B_4B_3}$ of the generating pencils.

The reguli conjugate to those included in each of the above systems form, in the aggregate, another congruence (2, 5), having the same focal surface as **C**. These congruences, in the following Table, are denoted by $\mathbf{C}_1$ and $\mathbf{C}_2$. The rays of any two of the three associated congruences group themselves, in fact, into conjugate reguli, situated on a system of quadric surfaces passing, individually, through eight fixed points, and touching one fixed plane. Each system, moreover, includes one point-and-plane pair, and three quadric cones.

Given any two of the three congrueuces, the elements of the point-and-plane pair are at once recognised in the Table in the usual manner (Art. 68); the remaining eight points indicated on the two lines appropriated to the planes of this point-and-plane pair, are those through which every quadric of the system of quadrics to which it belongs passes. The singular plane touched by every one of these quadrics, is that to which the remaining line of the Table is appropriated; on it, in the right-hand margin, will be found the vertices of the three quadric cones, whose generators are common rays of the two congruences under consideration.

In this same margin of the Table, and on each of the remaining two lines thereof, are the three singular points which are vertices of cubic cones in one of the two congruences (that indicated by the column which intersects the line in question in B_2), and of quadric cones in the other (Art. 36**c**). Of the three singular points, lastly, which stand in the margin at the bottom of the Table, each is the vertex of a quartic cone in the congruence indicated by the column in which it stands.

	C	$\mathbf{C}_1$	$\mathbf{C}_2$			
α	B_3	A_2	A_1	A_3'	A_3''	A_3'''
β	A_2	B_3	B_4	B_2'	B_2''	B_2'''
γ	A_1	B_4	B_2	C'	C''	C'''
	B_4	A_1	A_2			

On the Mutual Potential of Two Lines in Space.

By Horace Lamb, M.A.

[*Read June* 14*th*, 1883.]

By the mutual potential M of two lines C, C' is meant, in Electromagnetism, that part of the energy of the field which is due to the simultaneous presence of two electric currents of unit intensity flowing along these lines in the positive directions, the positive direction along each line being in the first instance chosen arbitrarily.

When both lines are *closed*, the value of M is known to be

$$M = \iint \frac{\cos \epsilon}{r}\, ds\, ds' \qquad \text{(1)},$$