On a Class of Integrable Reciprocants. By Mr. J. HAMMOND.

[*Read January* 14*th*, 1886.]

1. The notation, and nomenclature, of the present paper is that of Prof. Sylvester's Inaugural Lecture (*Nature*, Jan. 7th, 1886); in which $\frac{dy}{dx}$ is denoted by the single letter t, and $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots$ by $a, b, c, \ldots$ respectively, and a Reciprocant is a function of $t, a, b, c, \ldots$ which, to a factor *près*, remains unaltered when the variables x and y are interchanged.

Or, if $$\frac{dx}{dy} = \tau, \quad \frac{d^2x}{dy^2} = \alpha, \quad \frac{d^3x}{dy^3} = \beta, \quad \frac{d^4x}{dy^4} = \gamma, \; \ldots\ldots$$

and $\phi(\tau, \alpha, \beta, \gamma, \ldots) = \phi(t, a, b, c, \ldots)$ to a factor *près*,

the function ϕ is a reciprocant.

The class of reciprocants referred to in the title is characterised by having an integral of the form

$$a = F(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

i.e., this is an integral of the differential equation

$$\phi(t, a, b, c, \ldots) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

obtained by equating the reciprocant ϕ to zero.

The integration of (1) may be performed by well-known and simple methods which show that, in the case considered, the complete primitive of (2) is the equation of a curve whose Cartesian coordinates are

$$\left.\begin{aligned} x &= \int \frac{dt}{F(t)} + \text{const.} \\ y &= \int \frac{t\,dt}{F(t)} + \text{const.} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

or whose intrinsic equation is

$$s = \int \frac{\sec^3\psi\, d\psi}{F(\tan\psi)} + \text{const.}$$

2. The first reciprocant of this class that came under the author's notice was Prof. Sylvester's orthogonal reciprocant, which is the

left-hand member of the equation,*

$$(1+t^2)\,c-10abt+15a^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

This may be solved by assuming its first integral to be of the form

$$Pa^2+Qb = \text{const.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

where P and Q are unknown functions of t in which neither $a, b, c, \ldots$ nor the variables x, y enter.

Then, since $$\frac{d}{dx} = a\delta_t+b\delta_a+c\delta_b+\ldots,$$

the differentiation of (5) gives

$$\frac{d}{dx}(Pa^2+Qb) = a^3\frac{dP}{dt}+ab\left(2P+\frac{dQ}{dt}\right)+cQ = 0.$$

Comparing this with (4), we see that

$$Q : 2P+\frac{dQ}{dt} : \frac{dP}{dt} = 1+t^2 : -10t : 15 \quad \ldots\ldots\ldots\ldots (6),$$

whence P and Q may be determined.

But it will shorten the work to assume

$$\left.\begin{aligned} P &= \frac{d^4u}{dt^4} \\ Q &= \frac{1}{15}(1+t^2)\frac{d^5u}{dt^5} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

when one of the equations (6) is satisfied, and the other becomes

$$(1+t^2)\frac{d^6u}{dt^6}+12t\frac{d^5u}{dt^5}+30\frac{d^4u}{dt^4} = 0,$$

which, written in the form

$$\frac{d^6}{dt^6}\{(1+t^2)\,u\} = 0,$$

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* Captain MacMahon has transformed (4) into

$$\frac{d^3\psi}{ds^3}+18\left(\frac{d\psi}{ds}\right)^3 = 0,$$

which is the differential equation of the curve whose intrinsic equation is

$$\sin 3\psi = \frac{1}{\sqrt{2}}\,\text{sn}\left(3m\sqrt{2}s,\ \frac{1}{\sqrt{2}}\right).$$

gives
$$u = \frac{A+Bt}{1+t^2}+C+Dt+Et^2+Ft^3.$$

It follows, from (7), that we may take

$$\left.\begin{aligned} P_1 &= \frac{d^4}{dt^4}\left(\frac{1}{1+t^2}\right) \\ Q_1 &= \frac{1}{15}(1+t^2)\frac{d^5}{dt^5}\left(\frac{1}{1+t^2}\right) \\ P_2 &= \frac{d^4}{dt^4}\left(\frac{t}{1+t^2}\right) \\ Q_2 &= \frac{1}{15}(1+t^2)\frac{d^5}{dt^5}\left(\frac{t}{1+t^2}\right) \end{aligned}\right\} \quad \text{.....................(8)},$$

and thus obtain two first integrals of the form (5),

$$P_1 a^2 + Q_1 b = C_1,$$
$$P_2 a^2 + Q_2 b = C_2,$$

from which, by the elimination of b, a is found expressed as a function of t, thus

$$a = \sqrt{\frac{C_1 Q_2 - C_2 Q_1}{P_1 Q_2 - P_2 Q_1}} = F(t) \quad \text{.....................(9)},$$

and the complete primitive is of the same form as (3), containing four arbitrary constants; two of which are the C_1 and C_2 which appear in the function of t just found, the remaining two being the constants of integration in (3).

3. Writing
$$t = \tan\theta,$$
and performing the differentiations indicated in (8), we find, without difficulty,

$$P_1 = 24\cos^5\theta\cos 5\theta,$$
$$Q_1 = -8\cos^4\theta\sin 6\theta,$$
$$P_2 = 24\cos^5\theta\sin 5\theta,$$
$$Q_2 = 8\cos^4\theta\cos 6\theta,$$

whence
$$P_1 Q_2 - P_2 Q_1 = 192\cos^{10}\theta,$$

and these values, substituted in (9), will give

$$a = \sec^5\theta\sqrt{\kappa\cos 6\theta+\lambda\sin 6\theta} \text{.....................(10)},$$

in which κ and λ are mere numerical multiples of the former arbitrary constants C_1 and C_2.

Now (10) is the second differential equation of the curve represented by the complete primitive of (4), and may be written in the form
$$\rho^2 \cos 6(\theta - A) = B \qquad \text{(11)},$$
where $\rho = \dfrac{\sec^3\theta}{a}$ is the radius of curvature of this curve at any point, and θ is the inclination of the tangent at that point to the axis of x.

Thus, if we refer the curve to new axes, making an angle A with the old ones, and take for our unit of linear measurement the length of the radius of curvature which is parallel to the new axis of y, we may write
$$\rho^2 \cos 6\theta = 1,$$
whence we obtain
$$s = \int \frac{d\theta}{\sqrt{\cos 6\theta}} = \int \frac{d\theta}{\sqrt{1-2\sin^2 3\theta}},$$
leading to the intrinsic equation to the curve
$$\sin 3\theta = \frac{1}{\sqrt{2}}\,\text{sn}\left(3\sqrt{2}s,\ \frac{1}{\sqrt{2}}\right).$$
Comparing (10) with (11), we see that
$$\kappa = \frac{\cos 6A}{B}, \quad \lambda = \frac{\sin 6A}{B},$$
so that the corresponding simplification of (10) is effected by writing $\kappa = 1$ and $\lambda = 0$. The Cartesian coordinates of the curve are easily seen to be
$$x = \int \frac{\cos\theta\, d\theta}{\sqrt{\cos 6\theta}}, \quad y = \int \frac{\sin\theta\, d\theta}{\sqrt{\cos 6\theta}},$$
and the Cartesian equation of the curve is found by eliminating θ between these two in the following manner :—By means of the first $\cot^2\theta$ is expressible as an elliptic function of x, and by means of the second $\tan^2\theta$ is expressible as an elliptic function of y; the product of these two elliptic functions equated to unity is the Cartesian equation of the curve.

In fact, if we write
$$\tan^2\theta = \sin^2\phi + \frac{k'^2}{k^2}\cos^2\phi,$$
where
$$k = \sin 15^\circ \text{ and } k' = \sin 75^\circ,$$

after some easy reductions we shall find

$$my = \int \frac{d\phi}{\sqrt{1-k'^2 \sin^2\phi}},$$

where $$m^2 = 8\sqrt{3},$$

so that $$\tan^2\theta = \mathrm{sn}^2(my, k') + \frac{k'^2}{k^2}\mathrm{cn}^2(my, k'),$$

and similarly $$\cot^2\theta = \mathrm{sn}^2(mx, k) + \frac{k^2}{k'^2}\mathrm{cn}^2(mx, k).$$

Thus the curve is similar to the one whose equation is

$$1 = \left\{\mathrm{sn}^2(x, k) + \frac{k^2}{k'^2}\mathrm{cn}^2(x, k)\right\}\left\{\mathrm{sn}^2(y, k') + \frac{k'^2}{k^2}\mathrm{cn}^2(y, k')\right\},$$

which reduces to $$k'^2\,\mathrm{tn}^2(x, k) = k^2\,\mathrm{tn}^2(y, k').$$

The complete primitive of (4), with its full number of arbitrary constants, may be obtained from this equation by the *orthogonal* substitution of $$lx + my + n_1 \text{ for } x,$$

and $$mx - ly + n_2 \text{ for } y;$$

as may be verified by differentiating it four times in succession, after the substitution has been made, and eliminating the four arbitrary constants l, m, n_1, n_2.

For the results given in the present article I am indebted to Prof. Greenhill, who first pointed out the advantage of using $\tan\theta$ instead of t. The restoration of t in (10) will give

$$a^2 = \kappa(1-15t^2+15t^4-t^6) + \lambda(6t-20t^3+6t^5) \quad \ldots\ldots\ldots(12),$$

leading to the form of the complete primitive of (4), originally given in *Nature*, viz.,

$$\left.\begin{aligned} x &= \int \frac{dt}{\sqrt{\kappa(1-15t^2+15t^4-t^6)+\lambda(6t-20t^3+6t^5)}} + \mu \\ y &= \int \frac{t\,dt}{\sqrt{\kappa(1-15t^2+15t^4-t^6)+\lambda(6t-20t^3+6t^5)}} + \nu \end{aligned}\right\},$$

where the integrals which occur are reducible to elliptic, instead of being, as was stated without due consideration, hyper-elliptic integrals.

4. The form of (12) clearly indicates that it is an integral of a

reciprocant; for, on interchanging the dependent and independent variables, or, what is the same thing, writing

$$a = -\frac{\alpha}{\tau^3} \quad \text{and} \quad t = \frac{1}{\tau},$$

$$a^2 = \kappa\,(1-15t^2+15t^4-t^6)+\lambda\,(6t-20t^3+6t^5)$$

becomes $$\alpha^2 = \kappa'\,(1-15\tau^2+15\tau^4-\tau^6)+\lambda'\,(6\tau-20\tau^3+6\tau^5),*$$

and obviously the differential equation obtained by differentiating the latter twice with respect to y, and eliminating κ' and λ', will be precisely similar to that found by differentiating the former twice with respect to x, and eliminating λ and μ; *i.e.*, it will be precisely similar to the original equation (4).

More generally, if, when the variables are interchanged,

$$F\,(t,\ a,\ A,\ B,\ C,\ \ldots) = 0 \qquad (13)$$

becomes $$F\,(\tau,\ \alpha,\ A',\ B',\ C',\ \ldots) = 0,$$

the form of the function remaining unaltered, and only the values of the arbitrary constants A, B, C, ... suffering change; the same reasoning as before will show that (13) is an integral of a reciprocant.

And if the same permanence of form accompanies any linear substitution of the variables, say

$$\left.\begin{aligned} x &= lX+mY+n \\ y &= l'X+m'Y+n' \end{aligned}\right\},$$

(13) will be an integral of what, after Prof. Sylvester, we call a Pure Reciprocant. In this case τ and α are defined by

$$\tau = \frac{dY}{dX},\quad \alpha = \frac{d^2Y}{dX^2};$$

and, since $dx = (l+m\tau)\,dX$ and $dy = (l'+m'\tau)\,dX$,

we have $$t = \frac{l'+m'\tau}{l+m\tau} \qquad (14).$$

Now, writing $$a = \frac{dt}{dx} = \frac{1}{l+m\tau}\,\frac{dt}{dX},$$

* In the case considered, $\kappa' = -\kappa$ and $\lambda' = \lambda$; but the form of the relation between a and t is also permanent when subjected to any *orthogonal* transformation, in which case the relations between κ, λ and κ', λ' will differ from those given.

we obtain, from (14), $a = \frac{lm' - l'm}{(l+m\tau)^3}\,\alpha$ ....(15),

and obviously

$$(A,\ B,\ C,\ \dots \between 1,\ t)^\kappa = \frac{1}{(l+m\tau)^\kappa}\,(A',\ B',\ C',\ \dots \between 1,\ \tau)^\kappa,$$

so that the relation $a^{\frac{1}{3}\kappa} = (A,\ B,\ C,\ \dots \between 1,\ t)^\kappa$ ....(16)

possesses this permanence of form, and is consequently an integral of some pure reciprocant.

5. From (16), by making $\kappa = 1, 2, 3 \dots$ in succession, we derive, by a process of alternate differentiation with respect to x and division by a, a series of pure reciprocants, from which, as Protomorphs, all other pure reciprocants may be algebraically deduced. The degree of these is however, in general, greater than that of Prof. Sylvester's series of Protomorphs.

Thus, when $\kappa = 1$, we have

$$a^{\frac{1}{3}} = A + Bt,$$

whence, by differentiation with respect to x,

$$\tfrac{1}{3}a^{-\frac{2}{3}}b = Ba\,;$$

dividing by a and differentiating again with respect to x, we have

$$\frac{d}{dx}\,(a^{-\frac{5}{3}}b) = a^{-\frac{5}{3}}c - \tfrac{5}{3}a^{-\frac{8}{3}}b^2 = 0,$$

or,

$$3ac - 5b^2 = 0,$$

where the expression on the left is Prof. Sylvester's Parabolic Protomorph.

In exactly the same way, the Mongian

$$9a^2d - 45abc + 40b^3$$

is obtained from $a^{\frac{2}{3}} = (A,\ B,\ C \between 1,\ t)^2.$

But, when $\kappa = 3$,

$$a = (A,\ B,\ C,\ D \between 1,\ t)^3$$

gives the form $a^3e - 7a^2bd - 4a^2c^2 + 25ab^2c - 15b^4.$

Multiplying this by 5, and adding on 3 times the square of the Parabolic Protomorph, we have

$$a\,(5a^2e - 35abd + 7ac^2 + 35b^2c),$$

where the expression in brackets is Prof. Sylvester's Protomorph of weight 4.

6. The case of $\kappa = 6$ deserves special attention.

In it $$a^2 = (A, B, C, D, E, F, G \between 1, t)^6,$$

when treated by the process of the preceding article, yields the pure reciprocant

$$a^5h - 15a^4bg - 21a^4cf - 21a^4de + 105a^3b^2f + 231a^3bce + 105a^3bd^2 + 105a^3c^2d$$
$$-420a^2b^3e - 1050a^2b^2cd - 280a^2bc^3 + 945ab^4d + 1260ab^3c^2 - 945b^5c,$$

which, since the complete primitive of the differential equation (obtained by equating this to zero) is given by (3) in the form

$$\left.\begin{aligned} x &= \int \frac{dt}{\sqrt{(A, B, \ldots G \between 1, t)^6}} + \text{const.} \\ y &= \int \frac{tdt}{\sqrt{(A, B, \ldots G \between 1, t)^6}} + \text{const.} \end{aligned}\right\},$$

may be called the Hyper-Elliptic Pure Reciprocant.

When the sextic function of t has two equal roots; *i.e.*, when

$$\frac{a^2}{(t+F)^2} = (A, B, C, D, E \between 1, t)^4,$$

the reasoning of Art. 4 shows that the form of this relation is permanent when we substitute for t and a their values in terms of τ and a, given by equations (14) and (15). Hence, if we eliminate the constants A, B, C, D, E, F, we shall arrive at the Pure Reciprocant whose complete primitive is

$$\left.\begin{aligned} x &= \int \frac{dt}{(t+F)\sqrt{(A, B, C, D, E \between 1, t)^4}} + \text{const.} \\ y &= \int \frac{tdt}{(t+F)\sqrt{(A, B, C, D, E \between 1, t)^4}} + \text{const.} \end{aligned}\right\},$$

where the second integral may be replaced by

$$y + Fx = \int \frac{dt}{\sqrt{(A, B, C, D, E \between 1, t)^4}} + \text{const.}$$

All the constants except F may be eliminated by the process of the preceding article; which $\left(\text{since } a = \frac{dt}{dx} \text{ so that } \frac{1}{a} \cdot \frac{d}{dx} = \frac{d}{dt}\right)$ is equivalent to continued differentiation with respect to t, and gives

$$\frac{d^5}{dt^5}\left\{\frac{a^2}{(t+F)^2}\right\}=0.$$

After performing the differentiation and multiplying by $(t+F)^7$, to clear the equation of fractions, we shall obtain the following quintic in $t+F$,

$$\left\{(t+F)^5\frac{d^5}{dt^5}-10(t+F)^4\frac{d^4}{dt^4}+60(t+F)^3\frac{d^3}{dt^3}\right.$$
$$\left.-240(t+F)^2\frac{d^2}{dt^2}+600(t+F)\frac{d}{dt}-720\right\}a^2=0.$$

A final differentiation will give another quintic in $t+F$, and the resultant of these two quintics will be the Pure Reciprocant in question. Its value, expressed in terms of $a, b, c, \ldots$, appears to be too complicated to be of any use, and for this reason has not been calculated.

When the sextic in t has five equal roots, we may write

$$a^2=A(t+B)^5(t+C),$$

whence, by logarithmic differentiation,

$$\frac{2b}{a^2}=\frac{5}{t+B}+\frac{1}{t+C}.$$

Differentiating again with respect to x, and dividing by a,

$$\frac{2ac-4b^2}{a^4}=-\frac{5}{(t+B)^2}-\frac{1}{(t+C)^2}=-\frac{5}{(t+B)^2}-\left(\frac{2b}{a^2}-\frac{5}{t+B}\right)^2,$$

which reduces to $c(t+B)^2-10ab(t+B)+15a^3=0$ .... (17).

(The close resemblance of this to (4) may be noticed *en passant.*)

A final differentiation gives

$$d(t+B)^2-2(4ac+5b^2)(t+B)+35a^2b=0,$$

and the pure reciprocant we are in search of is obtained by eliminating B between this and (17); or, what is the same thing, it is the resultant of the two binary quadrics

$$(c,\ 5ab,\ 15a^3 \between X,\ Y)^2,$$

$$(d,\ 4ac+5b^2,\ 35a^2b \between X,\ Y)^2.$$

The discriminant of the first of these is $5a^2(3ac-5b^2)$, that of the second $\quad 35a^2bd-16a^2c^2-40ab^2c-25b^4,$

and their connective is $\quad 5a\,(3a^2d-abc-10b^3).$

Hence, rejecting the factor $5a^3$ from their resultant, we obtain

$$5\,(3a^2d-abc-10b^3)^2-4\,(3ac-5b^2)\,(35a^2bd-16a^2c^2-40ab^2c-25b^4),$$

which divides again by a, and gives

$$45a^3d^2-450a^2bcd+192a^2c^3+400ab^3d+165ab^2c^2-400b^4c,$$

or the "Quasi-Discriminant" whose evanescence serves to mark points of closest contact of a cubical parabola with any curve.

Equation (17) is the first integral of the differential equation to the general cubical parabola whose coordinates are

$$\left.\begin{aligned} x &= \int\frac{dt}{\sqrt{A\,(t+B)^5\,(t+C)}}+\text{const.} \\ y &= \int\frac{t\,dt}{\sqrt{A\,(t+B)^5\,(t+C)}}+\text{const.} \end{aligned}\right\}.$$

If, now, we differentiate (17) with respect to B, and eliminate B from it by means of the resulting equation; we see that the discriminant of (17) regarded as a quadric in B, or $3ac-5b^2=0$, is a singular first integral of the differential equation to the cubical parabola. The geometrical property indicated is, that at some points at least on any curve where the Quasi-Discriminant vanishes it is possible to draw a common parabola through six consecutive points of the curve.

7. The present seems to be a fitting opportunity for pointing out the form of algebraic relation that must subsist between a and t in order that the differential equation, freed from arbitrary constants, of the curve implied by this relation may be expressed by the evanescence of a reciprocant.

The reasoning employed in Art. 4 will show that the most general algebraic relation of this kind is

$$a^m\,(1,\,t)^n+a^{m-1}\,(1,\,t)^{n+3}+a^{m-2}\,(1,\,t)^{n+6}+\,\dots\dots=0\dots\dots\;(18),$$

and that the final differential equation obtained from it will be of the form $\qquad$ *Pure* Reciprocant $=0$;

provided only that the coefficients of all the quantics in t, which multiply the different powers of a, are either general or else connected by some *invariantive* condition; *e.g.*, $(1,\,t)^n$ may have two or more equal roots, and then its coefficients will be connected by an invariantive relation.

The second differential equation of any algebraic curve may, of course, be exhibited as an algebraic relation between a and t by eliminating x and y between the equation to the curve and the two other equations found by differentiating it twice with respect to x.

But (18) includes transcendental as well as algebraic curves.

As an easy example, the second differential equation of the conic

$$Ax^2+2Hxy+By^2+2Gx+2Fy+C=0,$$

expressed in this form, is $a^2\Delta = (A+2Ht+Bt^2)^3$,

where Δ is the discriminant.

When the curve is unicursal, let u, v, w denote rational integral functions of θ, then

$$x=\frac{u}{w}, \quad \text{and} \quad dx=\frac{u'w-uw'}{w^2}\,.\,d\theta,$$

$$y=\frac{v}{w}, \quad \text{and} \quad dy=\frac{v'w-vw'}{w^2}\,.\,d\theta;$$

whence
$$t=\frac{v'w-vw'}{u'w-uw'} \quad \text{.........................} \quad (19),$$

and
$$a=\frac{dt}{dx}=\frac{w^2}{u'w-uw'}\cdot\frac{dt}{d\theta},$$

or, after some easy reductions,

$$a=\frac{w^3}{(u'w-uw')^3}\begin{vmatrix} u & v & w \\ u' & v' & w' \\ u'' & v'' & w'' \end{vmatrix} \quad \text{..............} \quad (20),$$

and the elimination of θ between (19) and (20) gives the second differential equation of the curve in the form of an algebraic relation connecting a and t.

Thursday, February 11*th*, 1886.

J. W. L. GLAISHER, Esq., F.R.S., President, in the Chair.

Prof. P. H. Schoute, Ph.D., Professor of Mathematics at the Government University of Groningen, Netherlands, was elected a Member.

The following communications were made:—

On Perpetuant Reciprocants: Captain MacMahon, R.A.

Note on the Functions $Z(u)$, $\Theta(u)$, $\Pi(u, a)$: the President.

Note on a Z (*u*) Function: J. Griffiths, M.A.

On Polygons Circumscribed about a Conic and Inscribed in a Cubic: R. A. Roberts, M.A.

The following presents were received:—

Carte-de-Visite likeness of Mr. J. D. H. Dickson.

"Educational Times," for February.

"Journal of the Institute of Actuaries," Vol. xxv., Part III., CXXXVII., April, 1885.

"Proceedings of the Physical Society of London," Vol. VII., Part III., January, 1886.

"Bulletin des Sciences Mathématiques," for January and February, 1886.

"Annales de l'Ecole Polytechnique de Delft," Tome I., L. 3 and 4; Leide, 1885.

"Acta Mathematica," VII., 3; Stockholm, 1885.

"Atti della R. Accademia dei Lincei," Rendiconti, Vol. I., F. 28, 1885; Vol. II. F. 1; Roma, 1886.

"Beiblätter zu den Annalen der Physik und Chemie," B. x., St. 1; Leipzig, 1886.

"Sur le mouvement d'un corps pesant de révolution fixé par un point de son axe," par M. G. Darboux (from Journal de Mathématiques pures et appliquées).

"Appendix to Mathematical Questions and Solutions from Educational Times," Vol. XLIII.—"Solutions of some Old Questions," by Asûtosh Mukhopâdhyây, M.A., from the Author.

Perpetuant Reciprocants. *By* Captain MacMahon.

[*Read February* 11*th*, 1886.]

Reference is made to Prof. Sylvester's account of his discovery of Reciprocants, in *Nature* for January 7th, 1886; to several short articles on the same in recent numbers of the *Comptes Rendus*, and of the *Messenger of Mathematics.*

What is done, in this paper, is merely to present the numerical enumeration of the perpetuant reciprocants of the first six degrees, which is carried out on the same plan as that initiated by the author of their being for the allied Theory of Invariants, in Vol. v. of the *American Journal of Mathematics.*

The Theory of Invariants is, for the algebraist, concerned with the solutions of the lineo-linear partial differential equation

$$\lambda a\delta_b + \mu b\delta_c + \nu c\delta_d + \ldots = 0;$$

these are now termed *binariants;* and, were we to calculate any such general form, we would find that the coefficient of every term was partly numerical and partly composed of the letters λ, μ, ν, ..., and that, on putting $\lambda = \mu = \nu = \ldots = 1$, the binariant would become a