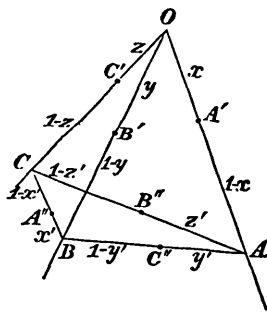


*Quaternion Proof of Mr. Samuel Roberts' Theorem of Four  
Co-intersecting Spheres. By J. J. WALKER, M.A.*

[Read May 12th, 1881.]

If, taking  $O$  as origin, three vectors  $OA$  ( $\alpha$ ),  $OB$  ( $\beta$ ),  $OC$  ( $\gamma$ ) determine a tetrahedron, and if on these three arbitrary points  $A', B', C'$  are taken, the four points  $O, A', B', C'$  determine a sphere ( $S_1$ ). If on  $BC, CA, AB$ , also, arbitrary points  $A'', B'', C''$  are taken, then  $A, A'', B'', C''$  determine a second sphere ( $S_2$ );  $B, A'', B'', C''$  a third sphere ( $S_3$ ); and  $C, A'', B'', C''$  a fourth sphere ( $S_4$ ). The theorem to be proved is, that these four spheres have a common point of intersection.



Let  $\overline{OA}, \overline{OB}, \overline{OC}$  be  $x\alpha, y\beta, z\gamma$  respectively; then the vector of any point on ( $S_1$ ) is  $\rho = lx\alpha + my\beta + nz\gamma$ , where  $l, m, n$  satisfy

$$l(l-1)x^2\alpha^2 + \dots + 2mnyzS\beta\gamma + \dots = 0 \dots\dots\dots(1).$$

Similarly the vectors to any points on ( $S_2$ ), ( $S_3$ ) respectively are (if  $\overline{AB''} = z'\overline{AC}, \overline{AB''} = y'\overline{AB}, \overline{BA''} = x'\overline{BC}$ ),

$$\alpha - \rho = l'(1-x)\alpha + m'y'(\alpha - \beta) + n'z'(\alpha - \gamma),$$

or  $\rho = (lx - l' + 1 - m'y' - n'z')\alpha + m'y'\beta + n'z'\gamma,$

with  $l'(l'-1)(1-x)^2\alpha^2 + m'(m'-1)y'^2(\alpha - \beta)^2 + \dots$   
 $+ 2m'n'S(\alpha - \beta)(\alpha - \gamma) + \dots = 0 \dots\dots(2);$

$$\beta - \rho = l''(1-y')(\beta - \alpha) + m''(1-y)\beta + n''x'(\beta - \gamma),$$

or  $\rho = l''(1-y')\alpha + (l''y' - l'' + m''y - m'' - n''x' + 1)\beta + n''x'\gamma,$

with  $l''(l''-1)(1-y')^2(\beta - \alpha)^2 + m''(m''-1)(1-y)^2\beta^2$   
 $+ n''(n''-1)x'^2(\beta - \gamma)^2 + 2m''n''x'(1-y)S\beta(\beta - \gamma)$   
 $+ 2n''l''x'(1-y')S(\beta - \alpha)(\beta - \gamma)$   
 $+ 2l''m''(1-y')(1-y)S\beta(\beta - \alpha) = 0 \dots\dots\dots(3).$

Identifying the  $\rho$  of  $S_1$  with that of  $S_2$ ,  $l' = \frac{lx + my + nz - 1}{x - 1}$ ,  $m' = \frac{my}{y}$ ,  $n' = \frac{nz}{z}$ ; and these values, substituted in (2), give

$$\begin{aligned} & [(lx + my + nz - 1)\{(l-1)x + my(my-1-y') + nz(nz-1-z')\} + 2mnyz]\alpha^2 \\ & + my(my-y')\beta^2 + nz(nz-z')\gamma^2 + 2mnyzS\beta\gamma \\ & + 2\{nlzx + nz(z'-1)\}S\gamma\alpha + 2\{lmny + my(y'-1)\}S\alpha\beta = 0; \end{aligned}$$

or, in virtue of (1),

$$\{x(1-l) + (1-x)(my+nz) - myy' - nzz'\} \alpha^3 + my(y-y') \beta^3 + nz(z-z') \gamma^3 + 2nz(x'-1) S\gamma\alpha + 2my(y'-1) Sa\beta = 0 \dots \dots \dots (4).$$

Similarly, by identifying the  $\rho$  of  $S_1$  with that of  $S_3$ , and substituting the values of  $l'', m'', n''$  in (3),

$$lx(x-1+y') \alpha^3 + \{y(1-m) - y(nz+lx) + n(1-x')z + ly'x\} \beta^3 + nz(z-x') \gamma^3 + 2nz(x'-1) S\beta\gamma - 2lx'y'Sa\beta = 0 \dots \dots \dots (5).$$

Multiplying (4) by  $lx$  and (5) by  $my$ , and adding, in virtue of (1), there results

$$nz[\{(1-x-z') \alpha^3 + (z-x') \gamma^3 + 2z'S\gamma\alpha\} lx + \{(1-x'-y) \beta^3 + (z-x') \gamma^3 + 2x'S\beta\gamma\} my + \gamma^3(n-1)z] = 0 \dots (6).$$

(4), (5), (6) determine two sets of values of  $l, m, n$ ; in one of which  $n = 0$ ; verifying the otherwise known result, that one intersection of  $S_1, S_3, S_4$  lies in the plane  $OAB$ . The other set are found from the second factor of (6), together with (4), (5), which may be arranged as

$$\alpha^3(l-1)x + \{(-1+x+y') \alpha^3 + (y'-y) \beta^3 + 2(1-y') Sa\beta\} my + \{(x+z'-1) \alpha^3 + (z'-z) \gamma^3 + 2(1-x') S\gamma\alpha\} nz = 0 \dots (7),$$

$$\{(-x-y'+1) \alpha^3 + (y-y') \beta^3 + 2y'Sa\beta\} lx + \beta^3(m-1)y + \{(x'+y-1) \beta^3 + (x'-z) \gamma^3 + 2(1-x') S\beta\gamma\} nz = 0 \dots (8).$$

Now the points of intersection of  $S_1, S_3, S_4$  would evidently be determined by changing  $\beta$  into  $\gamma$ ,  $y$  into  $z$ ,  $m$  into  $n$ ,  $y'$  into  $z'$ ,  $x'$  into  $(1-x')$ , and *vice versa*, in the above equations; but, by these interchanges, (6) becomes actually  $my[(8)]$ , (7) is unchanged, and (8) becomes the second factor of (6). Thus, that intersection ( $P$ ) of  $S_1, S_3, S_4$  which does not lie in the plane  $OCA$ , is shown to coincide with the intersection of  $S_1, S_3, S_4$ , which does not lie in the plane  $OAB$ . Similarly, by interchanging  $\alpha, \gamma$ ;  $l, n$ ;  $x, z$ ;  $y', 1-x'$ ;  $z', 1-z'$ , we obtain from (6)  $lx[(7)]$ , from (7) the second factor of (6), while (8) remains unchanged. But by these interchanges the triad  $S_1, S_3, S_4$  is changed into  $S_1, S_3, S_4$ ; and thus  $S_1, S_4$  are shown to have one point of intersection with  $S_3$  coinciding with  $P$ .

### *Some Solutions of Kirkman's 15-school-girl Problem.*

By ERNEST CARPMAEL, M.A.

[Read May 12th, 1881.]

Several communications with reference to the above problem have, from time to time, appeared in various Mathematical journals, without,