NOTE IN ADDITION TO A FORMER PAPER ON CONDITIONALLY CONVERGENT MULTIPLE SERIES

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IN a paper which appeared recently in these *Proceedings** I proved the convergence of a general class of n-ple series, of which

$$\Sigma \frac{\frac{\cos}{\sin}(i_1\theta_1+i_2\theta_2+\ldots+i_n\theta_n)}{(i_1a_1+i_2a_2+\ldots+i_na_n)^\rho}$$

is typical. Here $a_1, a_2, \ldots, a_n, \rho$ are all real and positive, and no one of $\theta_1, \ldots, \theta_n$ is a multiple of 2π. In that paper I was concerned entirely with *proper multiple series*; series of the type which, according to the notation developed by Prof. Bromwich and myself in the preceding paper, would be denoted by $\underset{(1, 2, \ldots, n)}{\Sigma}$.

I wish in this note to point out that all these series are convergent also when summed according to the type $\underset{(1, 2, \ldots, p)(p+1, \ldots, q)\ldots(r+1, \ldots, n)}{\Sigma}$ or $\underset{(\alpha)(\beta)\ldots(\mu)}{\Sigma}$. This follows at once from the following lemma, which is an obvious extension of a lemma proved by Pringsheim for double series.

LEMMA.—*The quantity* $\overline{\lim_{(1, 2, \ldots, n)}} s_{i_1, i_2, \ldots, i_n}$

is not increased, and the quantity

$$\underline{\lim_{(1, 2, \ldots, n)}} s_{i_1, i_2, \ldots, i_n}$$

is not decreased, by replacing the single bracket $(1, 2, \ldots, n)$ *by any system of brackets* $(\alpha)(\beta)\ldots(\mu)$.

To prove this it is evidently enough to prove that

$$\overline{\lim_{(1, 2, \ldots, n)}} s \geqslant \overline{\lim_{(1, 2, \ldots, p)}}\ \overline{\lim_{(p+1, \ldots, n)}} s \quad \text{or} \quad \geqslant \overline{\lim_{(\alpha)}}\ \overline{\lim_{(\beta)}} s,$$

say. Denote the quantity on the left by L; then, however small be σ, we can determine I so that if $i > I$ then $s < L+\sigma$.

* "On the Convergence of Certain Multiple Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 1, p. 124.

Making (β) tend to infinity, we deduce $\lim\limits_{\overline{(\beta)}} s \leqslant L+\sigma$ for $(\alpha) > I$, and so $\lim\limits_{\overline{(\alpha)}\overline{(\beta)}} s \leqslant L$.

The lemma is therefore proved.

Now let a, u* be two systems of quantities satisfying the conditions of § 4 of my former paper. I proved there that $\sum\limits_{(1,\, \ldots,\, n)} au$ is convergent, and the same argument shows that $\sum\limits_{(\beta)} au$ is convergent. Now

$$\sum_{\overline{(1,\, 2,\, \ldots,\, n)}} au = \sum_{\underline{(1,\, 2,\, \ldots,\, n)}} au = S,$$

say. Hence, by the lemma, $\sum\limits_{\overline{(\alpha)}(\beta)} au \leqslant S$ and also $\sum\limits_{\underline{(\alpha)}(\beta)} au \geqslant S$. That is to say, $\sum\limits_{(\alpha)(\beta)} au$ is convergent and $= S$. Similarly, we can show (by repeating this argument a finite number of times) that, if we divide the indices into any number of groups $(\alpha)\,(\beta)\ldots(\mu)$, the resulting series is convergent. The most interesting special case of this theorem is that the series

$$\sum_1^\infty \sum_1^\infty \ldots \sum_1^\infty \frac{\genfrac{}{}{0pt}{}{\cos}{\sin}\,(i_1\theta_1 + \ldots + i_n\theta_n)}{(i_1a_1 + \ldots + i_na_n)^\rho}$$

is convergent when the summations are carried out *successively*.

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* I write a for what was α in the former paper.