Hence $\rho = \frac{1}{2}(a+b)$. Also, at the boundary of the given ellipse, we have $c = \frac{1}{2}(a_1+b_1)$. The current function for the motion about the ellipse is therefore $\psi = -\frac{1}{2}V\left(a+b-\frac{(a_1+b_1)^2}{a+b}\right)\frac{b'}{h}$.

Here (a, b) are the elliptic coordinates of P, so that the Cartesian coordinates (x, y) of P are

$$x = \frac{aa'}{h},\ y = \frac{bb'}{h},\ \text{and } h^2 = a^2-b^2 = a'^2+b'^2.$$

We may also write this current function in the form

$$\psi = -V\left(b-\frac{(a_1+b_1)\,b_1}{a+b}\right)\frac{b'}{h}.$$

Note on Abel's Theorem. By THOMAS CRAIG, United States Coast and Geodetic Survey.

[*Read February* 10*th*, 1881.]

Denoting by s and z two variables, let

$$F(s, z) = 0$$

express an algebraic relation connecting them and one which is incapable of reduction to any simpler form. The solution of this equation gives s as a many-valued function of z, and one for the single-valued spread of which we require a Riemann's, say $2p+1$-fold, surface, which we may denote by R. Denoting then by f a rational function of s and z, and considering, as usual, only discontinuities of a polar nature, let

$$p_1, p_2 \dots p_n$$

denote the poles of f, and $\quad q_1, q_2 \dots q_n$

the zero points of f. Draw, in the usual manner, the systems of cuts ordinarily denoted by a, b, which have the effect of reducing the surface R to a simply-connected surface, say R_1. Draw also the non-intersecting lines

$$p_1 \dots\dots q_1,$$
$$p_2 \dots\dots q_2,$$
$$\dots \quad \dots \quad \dots$$
$$p_n \dots\dots q_n,$$

upon R_1. If we should conceive R_1 cut through along these lines (pq) we should obviously have a new surface, part of whose boundaries would be the two sides of these cuts; without actually cutting the surface, however, we may still consider the lines (pq) as the partial boundaries of the new surface, which we may denote by R_{11}. With the ordinary conventions for the positive bounding of a surface, we may say at once, without entering into elementary explanation, that whatever value $\log f$ may have upon the negative side of one of the lines (pq), its value on the positive side will be greater than on the negative by $2\pi i$. Also, the value of $\log f$ on the positive side of a will be greater than on the negative side by $2m\pi i$, and on the positive of b greater than on the negative side by $2n\pi i$; m and n being integers which in general depend upon the form of the function f and the position of the cuts a, b. Let z_k denote a particular point on the surface R where, in general, g sheets hang together. Calling the indefinitely small area immediately surrounding a point the *region* (Gebiet) of that point, we know that the region of any point z_k is formed by a line which goes round the point g times $(g = 1, 2, 3 \ldots p)$. In order to prove the theorem in view, it is necessary to find a function, say w, which will give the integral

$$\int \log f dw = 0$$

over the boundary of a modification of the surface R_{11}; *i.e.*, we must find a function w such that

$$\log f \frac{dw}{dz}$$

shall be single-valued over the modified form of R_{11}. Denoting then by

$$\frac{dw}{dz}$$

a rational function of s and z, we can develop this function in the region of a point z_k (the points z_k are supposed not to coincide with the points p, q) in ascending powers of $(z-z_k)^{\frac{1}{g}}$, *i.e.*,

$$\frac{dw}{dz} = (z-z_k)^{\frac{\mu}{g}} \{E_\mu + (z-z_k) E_{\mu+1} + (z-z_k)^2 E_{\mu+2} + \ldots\}.$$

For the poles of $\frac{dw}{dz}$ we need only consider the negative values of μ, that is, we need this development only in the regions of points z_k where μ is negative. For an infinitely distant point we, of course, have

$$\frac{dw}{dz} = z^{-\frac{\nu}{g}} \{E'_\nu + z^{-1} E'_{\nu+1} + z^{-2} E'_{\nu+2} + \ldots\}.$$

Around each point z_k draw a small circle on R_{11}, which denote by c_k; c_k in general goes round the point c_k g times. Similarly, draw a circle c_t around the origin as centre, enclosing *all* the points z_k, and excluding points z_∞; let now R_{11} be cut through along these circles, and a new surface R_{111} will be formed, which is the modification of R_{11} referred to above. The boundaries of R_{111} are formed by the cuts a, b, the lines (pq), and the circular cuts c_k and c_t. The function

$$\frac{dw}{dz} \log f$$

is now, as we know by the elementary properties of the theory of functions, a single-valued and continuous function all over R_{111}, and we therefore have

$$\int_{R_{111}} \log f \, dw = 0.$$

Denoting now by A and B the moduli of periodicity of the function w at the cuts a and b, we have, by separation of this integral into its several parts,

$$2\pi i \sum_i \int_{p_i}^{q_i} dw + 2\pi i \sum_j n_j B_j - 2\pi i \sum_j m_j A_j + \sum_k \int_{c_k} \log f \, dw + \sum_t \int_{c_t} \log f \, dw = 0.$$

For the last two integrals we need only, as before, to develop $\log f$ in the region of the points z_k; thus

$$\log f = G_0 + G_1 (z - z_k)^{\frac{1}{g}} + G_2 (z - z_k)^{\frac{2}{g}} + \ldots,$$

and outside the circle c_t

$$\log f = G_0' + G_1' z^{-\frac{1}{g}} + G_2' z^{-\frac{2}{g}} + \ldots.$$

For the quantities E, G, E', G', we have obviously

$$E_n = \frac{1}{\underline{n-\mu|}} \frac{d^{n-\mu}}{d\zeta^{n-\mu}} \left[\zeta^{-\mu} \frac{dw}{dz}\right] \quad \begin{matrix} z = z_k \\ s = s_k \end{matrix}$$

$$E_n' = \frac{1}{\underline{n-\nu|}} \frac{d^{n-\nu}}{d\eta^{n-\nu}} \left[\eta^{-\nu} \frac{dw}{dz}\right] \quad \begin{matrix} z = \infty \\ s = s_t \end{matrix}$$

in which

$$\zeta = (z - z_k)^{\frac{1}{g}},$$

$$\eta = z^{-\frac{1}{g}},$$

s_t being the value of s for $z = \infty$. Also, we have

$$G_n = \frac{1}{\underline{n|}} \left[\frac{d^n \log f}{d\zeta^n}\right] \quad \begin{matrix} z = z_k \\ s = s_k \end{matrix}$$

$$G_n' = \frac{1}{\underline{n|}} \left[\frac{d^n \log f}{d\eta^n}\right] \quad \begin{matrix} z = \infty \\ s = s_t \end{matrix}.$$

Substitution of these in the values of $\log f$, given above, conducts in a very simple manner to

$$\int_{c_k} \log f \, dw = 2\pi i g \{G_0 E_{-g} + G_1 E_{-g-1} + \ldots\},$$

$$\int_{c_t} \log f \, dw = 2\pi i g \{G_0' E_g' + G_1' E_{g-1}' + \ldots\}.$$

We have, therefore, the required theorem in the form

$$\sum_i \int_{p_i}^{q_i} dw + \sum_j m_j B_j - \sum_j n_j A_j$$
$$+ \sum_k g \{ G_0 E_{-g} + G_1 E_{-g-1} + \dots G_{-\mu-g} E_\mu \}$$
$$+ \Sigma g \{ G'_0 E_g + G'_1 E_{g-1} + \dots G'_{g-\nu} E'_\nu \} = 0,$$

$2\pi i$ being a common factor, and so disappearing.

On some Definite Integrals expressible in terms of the First Complete Elliptic Integral and of Gamma Functions. By J. W. L. Glaisher, M.A., F.R.S.

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1. *Theorem.*—If the value of the integral

$$\int_0^\infty \phi(t^2)\, dt$$

be denoted by A, then

$$\int_0^\infty \int_0^\infty \phi(x^4 + 2x^2y^2 \cos 2\gamma + y^4)\, dx\, dy = \tfrac{1}{2} A \,.\, F^1(\sin \gamma);$$

where $F^1(\sin a)$ denotes the complete elliptic integral of the first kind whose modulus is $\sin \gamma$.

To prove this, substitute $\sqrt{c}\,.\,t$ for t in the equation

$$A = \int_0^\infty \phi(t^2)\, dt,$$

and we have
$$\frac{A}{\sqrt{c}} = \int_0^\infty \phi(ct^2)\, dt;$$

whence, putting $c = 1 - k^2 \sin^2 \theta$,

$$\frac{A}{\sqrt{(1-k^2 \sin^2 \theta)}} = \int_0^\infty \phi(t^2 - k^2t^2 \sin^2 \theta)\, dt;$$

and therefore, integrating with regard to θ between the limits 0 and π,

$$A \int_0^\pi \frac{d\theta}{\sqrt{(1-k^2 \sin^2 \theta)}} = \int_0^\infty \int_0^\pi \phi(t^2 - k^2t^2 \sin^2 \theta)\, dt\, d\theta$$
$$= \int_0^\infty \int_0^\pi \phi(t^2 - 4k^2t^2 \sin^2 \tfrac{1}{2}\theta \cos^2 \tfrac{1}{2}\theta)\, dt\, d\theta.$$

Transforming the double integral by putting $r^2 = t$ and replacing $\frac{1}{2}\theta$ by