A Class of Algebraical Identities and Arithmetical Equalities.

By E. B. ELLIOTT. Read and received March 14th, 1901.

1. An arithmetical equality of universal application may or may not have as its basis a fact of algebraical identity. For instance, one which has is Gauss's

$$n = \Sigma\phi(d),$$

where the numbers d are the divisors of n, and, for an n whose distinct prime factors are $p, q, \ldots, t$, $\phi(n)$ denotes Euler's indicator $n\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\ldots\left(1-\frac{1}{t}\right)$, which is the expression for the number of numbers prime to and not exceeding n. This is an identity in $p, q, \ldots, t$ when n and the divisors d are given their expressions as products of primes—a different identity of course for one n from what it is for another of different algebraic form as a product of primes. It is just possible that the class of identities which follows, productive of a class of arithmetical equalities including the one above mentioned, may have hitherto escaped notice.

Let $p_1, p_2, p_3, \ldots$ be a set—either finite or infinite—of distinct symbols which obey the ordinary laws of algebraic combination, and let $p_r, p_s, p_t, \ldots$ be any chosen finite selection from them; and in a product such as

$$\Pi\left(1-\frac{1}{p}\right)$$

suppose that there is a factor corresponding to each of the whole set $p_1, p_2, p_3, \ldots$. Denote by

$$F_m(p_r^\rho p_s^\sigma p_t^\tau \ldots),$$

where $\rho, \sigma, \tau, \ldots$ are positive integers, the part of the direct expansion of

$$p_r^\rho p_s^\sigma p_t^\tau \ldots \Pi\left(1-\frac{1}{p}\right)^m$$

in descending positive zero and negative powers of the p's which involves no negative power of any p—the integral part, let us say. It is also of course the integral part of

$$p_r^\rho p_s^\sigma p_t^\tau \ldots \left(1-\frac{1}{p_r}\right)^m\left(1-\frac{1}{p_s}\right)^m\left(1-\frac{1}{p_t}\right)^m\ldots;$$

and for the values 0, 1, at any rate, of m is the whole of this product. The parameter m of an F_m is for the present unrestricted.

The following is an identity for any m:—

$$F_{m-1}(p_r^\rho p_s^\sigma p_t^\tau \dots) \equiv \Sigma F_m(p_r^{\rho'} p_s^{\sigma'} p_t^{\tau'}), \tag{A}$$

where the summation covers all combinations of integral and zero values of ρ', σ', τ', ... such that

$$0 \leqq \rho' \leqq \rho, \quad 0 \leqq \sigma' \leqq \sigma, \quad 0 \leqq \tau' \leqq \tau, \quad \dots .$$

We have in fact

$$p_r^\rho p_s^\sigma p_t^\tau \dots \Pi\left(1-\frac{1}{p}\right)^{-1} \equiv \Sigma(p_r^{\rho'} p_s^{\sigma'} p_t^{\tau'} \dots) + \text{fractional terms}; \tag{1}$$

and in this identity the integral parts of the two sides must be identical. This gives at once the case of (A)

$$F_{-1}(p_r^\rho p_s^\sigma p_t^\tau \dots) \equiv \Sigma F_0(p_r^{\rho'} p_s^{\sigma'} p_t^{\tau'} \dots).$$

Now multiply (1) by $\Pi\left(1-\frac{1}{p}\right)^m$, thus obviously getting

$$p_r^\rho p_s^\sigma p_t^\tau \dots \Pi\left(1-\frac{1}{p}\right)^{m-1} \equiv \Sigma\left\{p_r^{\rho'} p_s^{\sigma'} p_t^{\tau'} \dots \Pi\left(1-\frac{1}{p}\right)^m\right\}$$
$$+ \text{fractional terms}.$$

The identification of the integral part of the left in this with that of the right gives (A) in its generality.

2. It is also possible, for any m, to express F_{m+1} linearly in terms of F_m's. From

$$p_r^\rho p_s^\sigma p_t^\tau \dots \Pi\left(1-\frac{1}{p}\right) \equiv p_r^\rho p_s^\sigma p_t^\tau \dots \left\{1-\left(\frac{1}{p_r}+\frac{1}{p_s}+\dots\right)+\left(\frac{1}{p_r p_s}+\dots\right)\right.$$
$$\left. -\left(\frac{1}{p_r p_s p_t}+\dots\right)+\dots\right\}$$
$$+ \text{fractional terms}$$
$$\equiv \Sigma\mu(p_r^{\rho-\rho'} p_s^{\sigma-\sigma'} p_t^{\tau-\tau'} \dots) p_r^{\rho'} p_s^{\sigma'} p_t^{\tau'} \dots$$
$$+ \text{fractional terms}, \tag{2}$$

where $\mu(p_r^{\rho''} p_s^{\sigma''} p_t^{\tau''} \dots)$ stands for 0 if any one of ρ'', σ'', τ'', ... exceeds 1, and for 1 or -1 if none of them exceeds 1, according as the number of them equal to 1 is even (zero reckoned even) or odd, we obtain at once by identification of integral parts

$$F_1(p_r^\rho p_s^\sigma p_t^\tau \ldots) \equiv \Sigma\mu(p_r^{\rho-\rho'} p_s^{\sigma-\sigma'} p_t^{\tau-\tau'} \ldots) F_0(p_r^{\rho'} p_s^{\sigma'} p_t^{\tau'} \ldots).$$

Now multiply (2) by $\Pi\left(1-\frac{1}{p}\right)^m$, and then use the fact that the integral parts of the identical expansions of the two sides must be themselves identical. The result is that, for any m,

$$F_{m+1}(p_r^\rho p_s^\sigma p_t^\tau \ldots) \equiv \Sigma\mu(p_r^{\rho-\rho'} p_s^{\sigma-\sigma'} p_t^{\tau-\tau'} \ldots) F_m(p_r^{\rho'} p_s^{\sigma'} p_t^{\tau'} \ldots). \quad \text{(B)}$$

3. We can now apply arithmetically the identities (A) and (B). Let $p_1, p_2, p_3, \ldots$ be all the prime numbers $2, 3, 5, \ldots$, and let $p_r, p_s, p_t, \ldots$ be those of them which are factors of a number

$$n = p_r^\rho p_s^\sigma p_t^\tau \ldots.$$

The numbers $p_r^{\rho'} p_s^{\sigma'} p_t^{\tau'} \ldots$ are the divisors d, each once, of n. The central F, $F_0(n)$, of n is n itself. $F_1(n)$ is the indicator

$$\phi(n) = n\left(1-\frac{1}{p_r}\right)\left(1-\frac{1}{p_s}\right)\left(1-\frac{1}{p_t}\right)\ldots$$

of n. $F_{-1}(n)$ is the ordinary expression

$$\psi_1(n) = n\,\frac{1-\dfrac{1}{p_r^{\rho+1}}}{1-\dfrac{1}{p_r}}\,\frac{1-\dfrac{1}{p_s^{\sigma+1}}}{1-\dfrac{1}{p_s}}\ldots$$

for the sum of the divisors of n. The above has proved, by consideration of algebraical identities, a class of arithmetical equalities, of which one is the well known

$$n = \Sigma\phi(d),$$

and of which the general expression is

$$F_{m-1}(n) = \Sigma F_m(d), \quad \text{(C)}$$

and also the reversed class

$$F_{m+1}(n) = \Sigma\mu\left(\frac{n}{d}\right) F_m(d), \quad \text{(D)}$$

where $\mu(s)$ has its ordinary arithmetical definition as 0, or 1, or -1 according as s has a square factor, or an even (including zero), or an odd, number of unrepeated prime factors.

The law of formation of $F_m(n)$ needs no restatement. As examples of it we may write down, for instance,

$$F_3(p^3q^2r) = p^3q^2r\left(1-\frac{1}{p}\right)^3\left(1-\frac{1}{q}\right)^2\left(1-\frac{2}{r}\right),$$

$$F_{-2}(p^3q^2r) = p^3q^2r\left(1+\frac{2}{p}+\frac{3}{p^2}+\frac{4}{p^3}\right)\left(1+\frac{2}{q}+\frac{3}{q^2}\right)\left(1+\frac{2}{r}\right),$$

$$F_{-\frac{1}{2}}(p^3q^2r) = p^3q^2r\left(1+\frac{1}{2p}+\frac{3}{8p^2}+\frac{5}{16p^3}\right)\left(1+\frac{1}{2q}+\frac{3}{8q^2}\right)\left(1+\frac{1}{2r}\right).$$

4. We can readily write down sums for which the expressions are $F_m(n)$ for negative integral values of m. Whatever m be in (C), or say in

$$F_m(n) = \Sigma F_{m+1}(d),$$

we may replace, by the same law, each $F_{m+1}(d)$ by $\Sigma F_{m+2}(\delta)$ for all divisors δ of d. In the double sum obtained δ is in turn every divisor of n, and $F_{m+2}(\delta)$ occurs as many times as there are divisors d of n which δ divides, *i.e.*, $\psi_0\left(\frac{n}{\delta}\right)$ times, where ψ_0 means the number of divisors of its argument. Thus, changing notation,

$$F_m(n) = \Sigma F_{m+2}(d)\,\psi_0\left(\frac{n}{d}\right)$$

for all divisors d of n.

Let us now replace each $F_{m+2}(d)$ by the corresponding $\Sigma F_{m+3}(\delta)$. We deduce, writing d in place of δ, that

$$F_m(n) = \Sigma F_{m+3}(d)\left\{\psi_0\left(\frac{n}{d_1}\right)+\psi_0\left(\frac{n}{d_2}\right)+\dots\right\},$$

where d is in turn each divisor of n, and, for each d, the numbers $d_1, d_2, \dots$ are the $\psi_0\left(\frac{n}{d}\right)$ divisors of n which have d for a divisor, *i.e.*, $\frac{n}{d_1}, \frac{n}{d_2}, \dots$ are the $\psi_0\left(\frac{n}{d}\right)$ divisors of n whose conjugates have d for a divisor, *i.e.*, the $\psi_0\left(\frac{n}{d}\right)$ divisors of $\frac{n}{d}$. Write this

$$F_m(n) = \Sigma F_{m+3}(d)\,\psi_0^{(2)}\left(\frac{n}{d}\right),$$

where $\psi_0^{(2)}(n)$ is defined as $\Sigma\psi_0(d)$ for all divisors d of n.

Let us further define generally, for any positive integer N,

$$\psi_0^{(N)}(n) \quad \text{as} \quad \Sigma\psi_0^{(N-1)}(d),$$

with, to begin with,

$$\psi_0^{(0)}(n) = 1, \quad \psi_0^{(1)}(n) = \psi_0(n).$$

We obtain by continued repetition of the above process

$$F_m(n) = \Sigma F_{m+4}(d)\, \psi_0^{(3)}\left(\frac{n}{d}\right)$$

$$= \Sigma F_{m+5}(d)\, \psi_0^{(4)}\left(\frac{n}{d}\right)$$

$$= \dots$$

$$= \Sigma F_{m+N}(d)\, \psi_0^{(N-1)}\left(\frac{n}{d}\right). \qquad \text{(E)}$$

Here m is still unrestricted. But give it the negative integral value $-N$, and remember that $F(d) = d$, and there results the sum equal to $F_{-N}(n)$ of which we were in search, namely,

$$F_{-N}(n) = \Sigma\, d\psi_0^{(N-1)}\left(\frac{n}{d}\right), \qquad \text{(F)}$$

a formula of which the early cases are

$$F_{-1}(n) = \Sigma d = \psi_1(n),$$

$$F_{-2}(n) = \Sigma d\psi_0\left(\frac{n}{d}\right) = \Sigma \psi_1(d),$$

$$F_{-3}(n) = \Sigma d\psi_0^{(2)}\left(\frac{n}{d}\right),$$

&c., &c.

It remains to exhibit $\psi_0^{(N)}(n)$ numerically, for any positive integral n and N. If, expressed as a product of prime factors,

$$n = p_r^{\rho} p_s^{\sigma} p_t^{\tau} \dots,$$

we have $\psi_0^{(1)}(n) = \psi_0(n) = (\rho+1)(\sigma+1)(\tau+1) \dots .$

Consequently $\psi_0^{(2)}(n) = \Sigma\Sigma\Sigma \dots (\rho'+1)(\sigma'+1)(\tau'+1) \dots,$

for the ranges $0 \leqq \rho' \leqq \rho, \quad 0 \leqq \sigma' \leqq \sigma, \quad 0 \leqq \tau' \leqq \tau,$

$$= \Sigma(\rho'+1)\, \Sigma(\sigma'+1)\, \Sigma(\tau'+1)$$

$$= \frac{(\rho+1)(\rho+2)}{1.2} \frac{(\sigma+1)(\sigma+2)}{1.2} \frac{(\tau+1)(\tau+2)}{1.2} \dots .$$

Next, in like manner,

$$\psi_0^{(3)}(n) = \frac{(\rho+1)(\rho+2)(\rho+3)}{1.2.3}\,\frac{(\sigma+1)(\sigma+2)(\sigma+3)}{1.2.3}\ldots;$$

and generally

$$\psi_0^{(N)}(n) = \frac{(\rho+N)!}{\rho!\,N!}\,\frac{(\sigma+N)!}{\sigma!\,N!}\,\frac{(\tau+N)!}{\tau!\,N!}\ldots. \tag{G}$$

The sum (F) is then definitely given as a sum of multiples of divisors d for any n. As a simple example we may write down

$$\begin{aligned} F_{-3}(12) &= 1\psi_0^{(2)}(12)+2\psi_0^{(2)}(6)+3\psi_0^{(2)}(4)+4\psi_0^{(2)}(3)+6\psi_0^{(2)}(2) \\ &\qquad +12\psi_0^{(2)}(1) \\ &= 1\,\frac{3.4}{1.2}\,\frac{2.3}{1.2}+2\left(\frac{2.3}{1.2}\right)^2+3\,\frac{3.4}{1.2}+4\,\frac{2.3}{1.2}+6\,\frac{2.3}{1.2}+12.1 \\ &= 18 + 18 + 18 + 12 + 18 + 12 \\ &= 96, \end{aligned}$$

which is correctly equal to $12\left(1+\frac{3}{2}+\frac{6}{2^2}\right)\left(1+\frac{3}{3}\right)$.

5. We still desire the summation for which $F_N(n)$, for a positive integral N, is the expression.

By (D) we have, for any m,

$$F_m(n) = \Sigma F_{m-1}(d)\,\mu\left(\frac{n}{d}\right),$$

where the definition of $\mu(s)$, for any number s, may be stated that it is the signed unit which is the coefficient of s in the expansion of the product

$$\Pi\,(1-p),$$

for all primes p, if the product equal to s actually occurs in the expansion, and is otherwise zero. Let us further define

$$\mu^{(N)}(s)$$

as the coefficient of s, if it actually occurs, and zero otherwise, in the expansion of the product

$$\Pi\,(1-p)^N.$$

We have, by a repetition of the reduction (D),

$$F_m(n) = \Sigma F_{m-2}(d)\left\{\mu\left(\frac{n}{d_1}\right)\mu\left(\frac{d_1}{d}\right)+\mu\left(\frac{n}{d_2}\right)\mu\left(\frac{d_2}{d}\right)+\ldots\right\},$$

for all divisors d of n, where $\frac{n}{d_1}$, $\frac{n}{d_2}$, ... are the divisors of $\frac{n}{d}$. Now the sum in brackets here is the coefficient of the product of primes $\frac{n}{d}$ as it occurs in the product of $\Pi(1-p)$ and $\Pi(1-p)$; in other words, it is $\mu^{(2)}\left(\frac{n}{d}\right)$, the coefficient of $\frac{n}{d}$ as and if it occurs as a product of primes or as 1 in the expansion of the product $\Pi(1-p)^2$. Thus

$$\begin{aligned}
F_m(n) &= \Sigma F_{m-2}(d)\,\mu^{(2)}\left(\frac{n}{d}\right) \\
&= \Sigma F_{m-3}(d)\left\{\mu^{(2)}\left(\frac{n}{d_1}\right)\mu\left(\frac{d_1}{d}\right)+\mu^{(2)}\left(\frac{n}{d_2}\right)\mu\left(\frac{d_2}{d}\right)+\ldots\right\} \\
&= \Sigma F_{m-3}(d)\left\{\text{coefficient of } \frac{n}{d} \text{ in product } \Pi(1-p)^2\Pi(1-p)\right\} \\
&= \Sigma F_{m-3}(d)\,\mu^{(3)}\left(\frac{n}{d}\right) \\
&= \ldots \\
&= \Sigma F_{m-N}(d)\,\mu^{(N)}\left(\frac{n}{d}\right). \qquad \text{(H)}
\end{aligned}$$

Now give to the unrestricted m the value of the positive integer N. This produces our desired result, namely,

$$F_N(n) = \Sigma d\mu^{(N)}\left(\frac{n}{d}\right). \qquad \text{(K)}$$

It should be noticed that all numbers $F_{-N}(n)$ are positive, but that this is not the case with all numbers $F_N(n)$.

6. A second method of reversing the equality (C) so as to obtain an expression linear in F_{m-1}'s for an F_m, which is at first sight different from (D), is the extension of one given by Glaisher (*Phil. Mag.*, 1884) and Hammond (*Messenger*, 1891) for expressing $\phi(n)$, *i.e.*, $F_1(n)$, linearly in terms of 1, 2, 3, ..., n, *i.e.*, in terms of $F_0(1)$, $F_0(2)$, $F_0(3)$, ..., $F_0(n)$. Write down the n equalities (C) for the values 1, 2, 3, ..., n of n. They furnish n linear equations for the determination of $F_m(1)$, $F_m(2)$, $F_m(3)$, ..., $F_m(n)$. $F_m(1)$ occurs with coefficient 1 in all the equations, $F_m(2)$ with coefficient 1 in the second, fourth, &c., $F_m(3)$ in the third, sixth, &c., and so on, and, lastly, $F_m(n)$ with coefficient 1 in the last only. Thus the determinant of the right-hand sides is unity, and $F_m(n)$ is equal to a

determinant of n^2 constituents, whose first $n-1$ columns—or say rather rows—are

$$\begin{matrix} 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & \ldots, \\ 0, & 1, & 0, & 1, & 0, & 1, & 0, & 1, & 0, & \ldots, \\ 0, & 0, & 1, & 0, & 0, & 1, & 0, & 0, & 1, & \ldots, \\ 0, & 0, & 0, & 1, & 0, & 0, & 0, & 1, & 0, & \ldots, \\ & & & & \&c., & & \&c., & & & \end{matrix}$$

and whose n-th row is

$$F_{m-1}(1),\ F_{m-1}(2),\ F_{m-1}(3),\ \ldots,\ F_{m-1}(n).$$

By means of (E) it is easy in like manner to write down a determinant expression for $F_m(n)$, with a last row consisting of

$$F_{m-N}(1),\ F_{m-N}(2),\ F_{m-N}(3),\ \ldots,\ F_{m-N}(n),$$

and first $n-1$ rows consisting of $\psi_0^{(N-1)}$s of numbers up to n, with zeroes in places as above.

7. Another easily proved determinant theorem includes the one known as H. J. S. Smith's (*Proc. Lond. Math. Soc.*, Vol. VII., p. 208), that

$$\phi(n)\,\phi(n-1)\,\phi(n-2)\,\ldots\,\phi(1),$$

i.e.,

$$F_1(n)\,F_1(n-1)\,F_1(n-2)\,\ldots\,F_1(1),$$

is equal to the determinant of n^2 constituents in which the constituent in the r-th row and s-th column, for each r and s, is g_{rs}, the G.C.M. of r and s. The more general fact as to our functions is that

$$F_m(n)\,F_m(n-1)\,F_m(n-2)\,\ldots\,F_m(1)$$

is equal to the result of replacing in Smith's determinant each g_{rs} by $F_{m-1}(g_{rs})$.

To prove it take, from (C),

$$F_{m-1}(g_{rn}) = \Sigma F_m(\delta),\ \text{for divisors } \delta \text{ of } g_{rn},$$
$$= \Sigma F(d),\ \text{for divisors } d \text{ of } n,$$

where $F(d)$ denotes $F_m(d)$ or 0 according as d does or does not divide r. Hence, by the Dedekind-Liouville theorem of reversion,

$$F(n) = \Sigma\mu\left(\frac{n}{d}\right)F_{m-1}(g_{rd}),\ \text{for divisors of } n;$$

i.e., the right-hand side is equal to $F_m(n)$ or 0 according as n does or does not divide r. Now take the determinant Δ_n in which the type constituent is $F_{m-1}(g_{rs})$. As

$$\mu\left(\frac{n}{n}\right) = \mu(1) = 1,$$

the value of Δ_n is not altered when we replace the n-th column by the sum of the multiples $\mu\left(\frac{n}{d}\right)$ of the various d-th columns, for divisors d of n. The last constituent in the column thus becomes $F_m(n)$, as n divides n, and the r-th constituent for any $r < n$ becomes 0, as n does not divide r. Thus

$$\Delta_n = F_m(n)\,\Delta_{n-1},$$

where the formation of Δ_{n-1}, of $(n-1)^2$ constituents, is according to the same law as that of Δ_n. Repeating this argument $n-1$ times, and noticing that

$$\Delta_1 = F_{m-1}(1) = 1 = F_m(1).$$

we have, as stated,

$$\Delta_n = F_m(n)\,F_m(n-1)\,F_m(n-2)\,\ldots\,F_m(1).$$

The proof is of course applicable when instead of F_m, F_{m-1} we have any two arithmetical functions χ, λ which are such that for every n

$$\lambda(n) = \Sigma\chi(d).$$

Probably the theorem in its generality ought to be regarded as known. A generalization of his theorem given by Smith himself (*loc. cit.*) produces it with the aid of Dedekind's reversion.*

8. Let us examine more closely the linear expression for an F_m in terms of F_{m-1}'s which is exhibited in determinant form in § 6. Though in form very unlike the expression (D), it may be seen to be really equivalent to it. As far as I know it has not been noticed even that the Glaisher-Hammond determinant expression for $\phi(n)$ is really the same as the expression $\Sigma\mu\left(\frac{n}{d}\right)d$. The statement of this is, in accordance with what we have seen, a case of the more general statement as to the expressions for an F_m in terms of F_{m-1}'s; and this again is a class of cases of the more general statement that, if we have two arithmetical functions $\lambda(n)$, $\chi(n)$ such that for every n

$$\lambda(n) = \Sigma\chi(d),$$

* [The theorem has been given explicitly by Cesaro, *Nouvelles Annales* (3), v., p. 44.]

the two apparently different reversions of this

$$\chi(n) = \Sigma\mu\left(\frac{n}{d}\right)\lambda(d),$$

$$\chi(n) = \begin{vmatrix} 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & \ldots, & 1 \\ 0, & 1, & 0, & 1, & 0, & 1, & 0, & 1, & \ldots & \\ 0, & 0, & 1, & 0, & 0, & 1, & 0, & 0, & \ldots & \\ 0, & 0, & 0, & 1, & 0, & 0, & 0, & 1, & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \ldots \\ \lambda(1), & \lambda(2), & \lambda(3), & \lambda(4), & \lambda(5), & \lambda(6), & \lambda(7), & \lambda(8), & \ldots, & \lambda(n) \end{vmatrix}$$

are in effect the same, the functions $\lambda(r)$ for r's which do not divide n being mere superfluities in the last row.

We will prove that $\Sigma\mu\left(\frac{n}{d}\right)\lambda(d)$ is a factor of the determinant, and that the other factor is unity. This will be shown, since

$$\mu(1) = 1,$$

if we can prove that, by adding to the last column multiples $\mu\left(\frac{n}{d}\right)$ respectively of the other d-th columns, we can reduce all the constituents but the last in the n-th column to zero. For, except in the last row, all constituents below and to the left of the principal diagonal vanish, and those in the principal diagonal are units.

Let ϕ_{rs} denote 0 or 1 according as r and s are unequal or equal. The constituent C_{rs} in the r-th row and s-th column ($r<n$) is 0 or 1 according as s is not or is a multiple of r. Thus

$$C_{rs} = \Sigma\phi_{r\delta}, \text{ for divisors } \delta \text{ of } s,$$

or, with changed notation, $C_{rn} = \Sigma\phi_{rd}$.

Now this necessitates that

$$\phi_{rn} = \Sigma\mu\left(\frac{n}{d}\right)C_{rd},$$

and consequently that the right-hand sum vanishes except for $r = n$. This proves what was required. It has incidentally found the values —some zero, some 1, and some -1—of the determinants of $n-1$ rows and columns which are obtained by omitting single columns from the first $n-1$ rows of the determinant $\chi(n)$.

Of course the whole determinant need not be written as one of n, but only as one of $\psi_0(n)$, rows and columns. Of the numbers

1, 2, 3, ..., n let 1, d_2, d_3, ..., n be those, in order, which are divisors of n, and e_1, e_2, e_3, ... those which are not. The numbers of d's and e's are $\psi_0(n)$ and $n-\psi_0(n)$ respectively. In the various e-th rows the constituents in the various d-th columns are all zero—for no e divides any d. Accordingly the determinant is the product of two determinants of orders $\psi_0(n)$ and $n-\psi_0(n)$, the former containing the constituents at intersections of the various d-th rows and the various d-th columns, and the latter those at intersections of e-th rows and e-th columns. The latter determinant is equal to unity—for the constituents in its principal diagonal are units, while those below and to the left of that diagonal are all zero. The former is a determinant of order $\psi_0(n)$, which has for its last row

$$\lambda(1),\ \lambda(d_2),\ \lambda(d_3),\ \ldots,\ \lambda(n).$$

What the multipliers of these are, in the expansion of the determinant, has been completely seen above.

With regard to the particular application to arithmetical functions F_m, F_{m-1}, we are then assured that the Glaisher-Hammond determinant method adds no essentially new information to that afforded by the algebraical identities which we dealt with at the outset.

9. A known fact of some generality (*cf.* Bachmann, *Encyclopädie*, Band I., p. 650) is that, if λ_1, χ_1, λ_2, χ_2 are four arithmetical functions such that, for all numbers n,

$$\lambda_1(n) = \Sigma\chi_1(d) \quad \text{and} \quad \lambda_2(n) = \Sigma\chi_2(d),$$

then

$$\Sigma\chi_1\left(\frac{n}{d}\right)\lambda_2(d) = \Sigma\chi_2\left(\frac{n}{d}\right)\lambda_1(d).*$$

This has an interesting application to our functions F_m. Since

$$F_{m-1}(n) = \Sigma F_m(d) \quad \text{and} \quad F_{\mu-1}(n) = \Sigma F_\mu(d),$$

it gives that

$$\Sigma F_m\left(\frac{n}{d}\right)F_{\mu-1}(d) = \Sigma F_\mu\left(\frac{n}{d}\right)F_{m-1}(d).$$

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* A very convenient way of rendering visible such transformations of summations is to arrange elements in a square array and equate their sum taken by rows to their sum taken by columns. Thus, if we take a row and a column for each divisor of n, and place, in each d row, $\chi_1(d)\,\chi_2\left(\frac{n}{\delta}\right)$ or 0 in the δ-column according as d is or is not a divisor of δ, the two different ways of adding elements give the equality here before us. Most familiar arithmetical equalities can be made clear in this sort of way.

In particular $\Sigma F_m\left(\frac{n}{d}\right) F_{-m}(d) = \Sigma F_{1-m}\left(\frac{n}{d}\right) F_{m-1}(d)$

$$= \Sigma F_{m-1}\left(\frac{n}{d}\right) F_{1-m}(d).$$

Taking m integral in this, we get

$$\Sigma F_N\left(\frac{n}{d}\right) F_{-N}(d) = \Sigma F_{N-1}\left(\frac{n}{d}\right) F_{1-N}(d) = \Sigma F_{N-2}\left(\frac{n}{d}\right) F_{2-N}(d)$$

$$= \dots$$

$$= \Sigma F_0\left(\frac{n}{d}\right) F_0(d)$$

$$= \Sigma\left(\frac{n}{d} d\right) = n\psi_0(n).$$

So, too,
$$\Sigma F_{N+1}\left(\frac{n}{d}\right) F_{-N}(d) = \Sigma F_1\left(\frac{n}{d}\right) F_0(d)$$

$$= \Sigma d\phi\left(\frac{n}{d}\right) = n\Sigma\frac{\phi(d)}{d};$$

$$\Sigma F_{N-1}\left(\frac{n}{d}\right) F_{-N}(d) = \Sigma F_0\left(\frac{n}{d}\right) F_{-1}(d)$$

$$= \Sigma \frac{n}{d}\psi_1(d) = n\Sigma\frac{\psi_1(d)}{d};$$

and more generally

$$\Sigma F_{N+r}\left(\frac{n}{d}\right) F_{-N}(d) = n\Sigma\frac{F_r(d)}{d},$$

for any positive or negative r.

These equalities can be at once stated in other terms by means of (F) and (K).

10. In conclusion the remark may be made that examples may with ease be written down of other identities than those in §§ 1, 2 which yield expressions for simple arithmetical sums. For instance, take

$$p_r^{2\rho} p_s^{2\sigma} p_t^{2\tau} \dots \Pi\left(1-\frac{1}{p^2}\right)^{-1} \equiv \Sigma\,(p_r^{2\rho'} p_s^{2\sigma'} p_t^{2\tau'} \dots) + \text{fractional terms},$$

where $0 \leqq \rho' \leqq \rho, \quad 0 \leqq \sigma' \leqq \sigma, \quad 0 \leqq \tau' \leqq \tau$, &c.

The non-fractional part on the right is the sum of the squares of the algebraic divisors of $p_r^\rho p_s^\sigma p_t^\tau \ldots$. Upon multiplication by $\Pi\left(1-\frac{1}{p}\right)$ this yields an identity which gives

$$\text{integral part of expansion of } p_r^{2\rho} p_s^{2\sigma} p_t^{2\tau} \ldots \Pi\left(1+\frac{1}{p}\right)^{-1}$$

$$\equiv \Sigma\left\{\text{integral part of expansion of } p_r^{2\rho'} p_s^{2\sigma'} p_t^{2\tau'} \ldots \Pi\left(1-\frac{1}{p}\right)\right\}.$$

Now take the prime numbers for the p's, and any number

$$n = p_r^\rho p_s^\sigma p_t^\tau \ldots,$$

thus getting

part of $n^2\Pi\left(1+\frac{1}{p}\right)^{-1}$ integral in p's, when n is expressed as a product of p's,

$$= \Sigma\phi\,(d^2), \text{ for all divisors } d \text{ of } n;$$

a result which may also be written

$$S - S' = \Sigma\phi\,(d^2) = \Sigma d\phi\,(d) = 2\Sigma\phi_1\,(d) - 1,$$

where $\phi_1(d)$ denotes the sum of the numbers prime to and not exceeding d, and S, S' denote the sums of the divisors of n^2 which are and are not respectively products of even (including zero) numbers of unequal or equal prime factors.

In like manner the integral parts, when n is replaced by its expression as a product of primes, of the expansions of

$$n^3\Pi\left(1+\frac{1}{p}+\frac{1}{p^2}\right)^{-1},$$

$$n^4\Pi\left(1+\frac{1}{p}+\frac{1}{p^3}+\frac{1}{p^3}\right)^{-1},$$

&c., &c.,

are

$$\Sigma\phi\,(d^3) = \Sigma d^2\phi\,(d),$$

$$\Sigma\phi\,(d^4) = \Sigma d^3\phi\,(d),$$

&c., &c.